MAE143 B - Linear Control - Spring 2015
Final, June 8th

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your notes and the reader. You may use a calculator with no communication capabilities
2. You have 170 minutes
3. Do not get stuck! Move on and come back later if you have time
4. Write answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor
6. Initial conditions are zero unless otherwise noted
7. The exam has 7 question with for a total of 62 points and 9 bonus points

Questions:

1. **The Device.**

   The diagram in Fig. 1 represents a tracking device that uses two sensors separated by a distance $\ell > 0$ at positions x and y to locate a target at position z. The device is stationary but can rotate about its center of mass, the point o. The equation:

   $$J\,\ddot{\psi} + b\,\dot{\psi} = u, \tag{1}$$

   where u is a torque applied by a motor connected to the device's center of mass, J is the device's moment of inertia, and b is the device's viscous damping coefficient, is a simplified model for the dynamics of the tracking device.

   (a) (3 points) Draw a block-diagram relating the signals ψ, $\dot{\psi}$, and u without using differentiators.

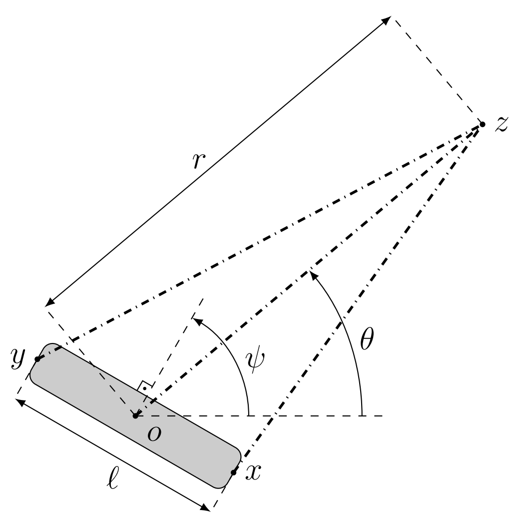
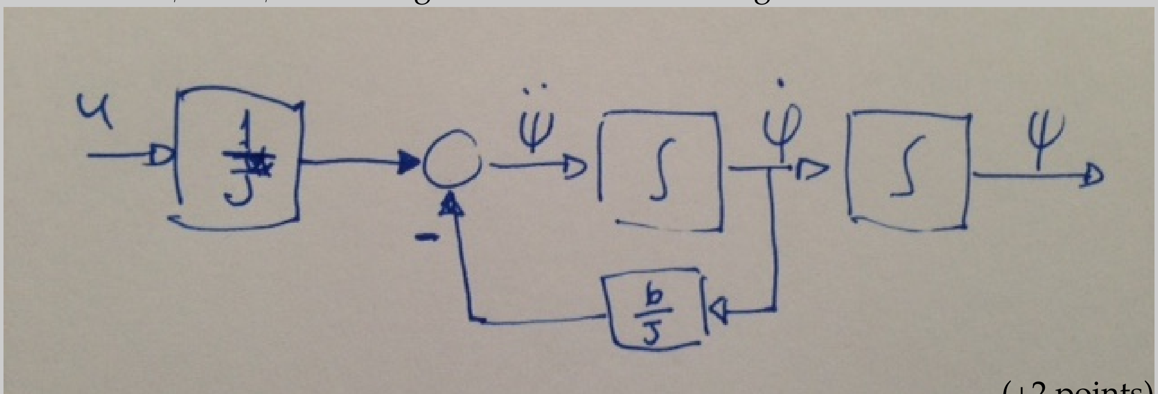


Figure 1: Locating a target z with sensors at x and y

Solution: Isolate the highest derivative:

$$\ddot{\psi} = \frac{1}{J}u - \frac{b}{J}\dot{\psi} \qquad \text{(+1 point)}$$

and realize ψ and $\dot{\psi}$ after integration to obtain the diagram:



(+2 points)

(b) (3 points) Calculate the transfer-function from u to ψ.

Solution: Assume zero initial conditions and apply the Laplace transform to obtain:

$$(s^2 J + bs)\,\Psi(s) = U(s) \qquad \text{(+2 points)}$$

and calculate the transfer-function

$$\frac{\Psi(s)}{U(s)} = \frac{1/J}{s^2 + (b/J)s} \qquad \text{(+1 point)}$$

2. **The Sensors.**

Using the law of cosines, the distance from the sensors to the target can be shown to be:

$$r_x(\psi,\theta,r) = \sqrt{r^2 + (\ell/2)^2 - r\,\ell\sin(\psi-\theta)}, \quad r_y(\psi,\theta,r) = \sqrt{r^2 + (\ell/2)^2 + r\,\ell\sin(\psi-\theta)},$$

where θ is the angle and r is the distance to the target measured from o with respect to an inertial frame as shown in Fig. 1.

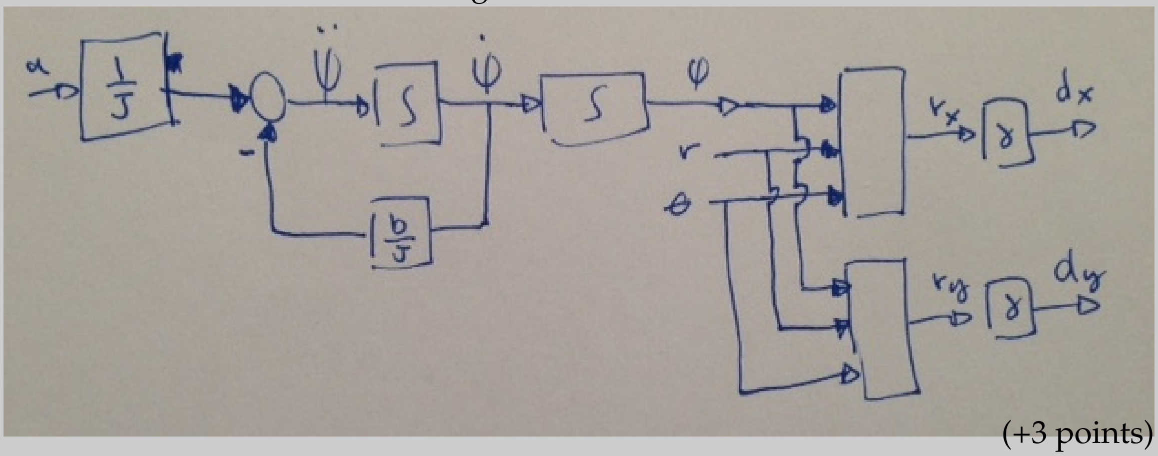
Many types of sensors can produce signals that are proportional to the distance to the target, for example, sonar, radar or even imaging, depending on the application. Identical sensors at points x and y produce the signals:

$$d_x = \gamma\, r_x, \qquad d_y = \gamma\, r_y,$$

where γ is the sensor gain.

(a) (3 points) Assume that the tracking device is modeled by (1). Draw a block-diagram relating the signals d_x, d_y, ψ, $\dot{\psi}$, r, θ, and u. Use a single block to produce r_x from ψ, θ and r. Do the same for r_y.

Solution: One solution is the diagram:



(+3 points)

NOTE TO GRADERS: Any reasonable correct solution is fine.

3. **Control Part I.**

   One approach to tracking is to use the control law:

$$u = -K \frac{d_y^2 - d_x^2}{2\,\gamma\,\ell} = -K\,\gamma\,r \sin(\psi - \theta). \tag{2}$$

   (a) (3 points) Show that the connection of the tracking device with the control law (2) can be modeled by the nonlinear dynamic equations:

$$J\,\ddot{e} + b\,\dot{e} + K\gamma\,r \sin(e) = -J\,\ddot{\theta} - b\,\dot{\theta}, \qquad e = \psi - \theta. \tag{3}$$

Solution:
Substitute (2) into (1) to obtain:

$$J\ddot{\psi} + b\dot{\psi} = -K\gamma\,r\,\sin(\psi - \theta). \qquad \text{(+1 point)}$$

Note that because $e = \psi - \theta$ that

$$\ddot{\psi} = \ddot{e} + \ddot{\theta}, \qquad \dot{\psi} = \dot{e} + \dot{\theta}, \qquad \text{(+1 point)}$$

from which

$$J\,(\ddot{e} + \ddot{\theta}) + b\,(\dot{e} + \dot{\theta}) = -K\gamma\,r\,\sin(e). \qquad \text{(+1 point)}$$

which produces (3) after rearranging the terms.

   (b) (2 points (bonus)) Show that

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 - \dfrac{b}{J} x_1 \\ -\dfrac{K\gamma\,r}{J} \sin(x_1 - \theta) \end{pmatrix}, \qquad e = x_1 - \theta$$

   is a state-space realization for the differential equation (3). *Hint: differentiate 'e' twice.*

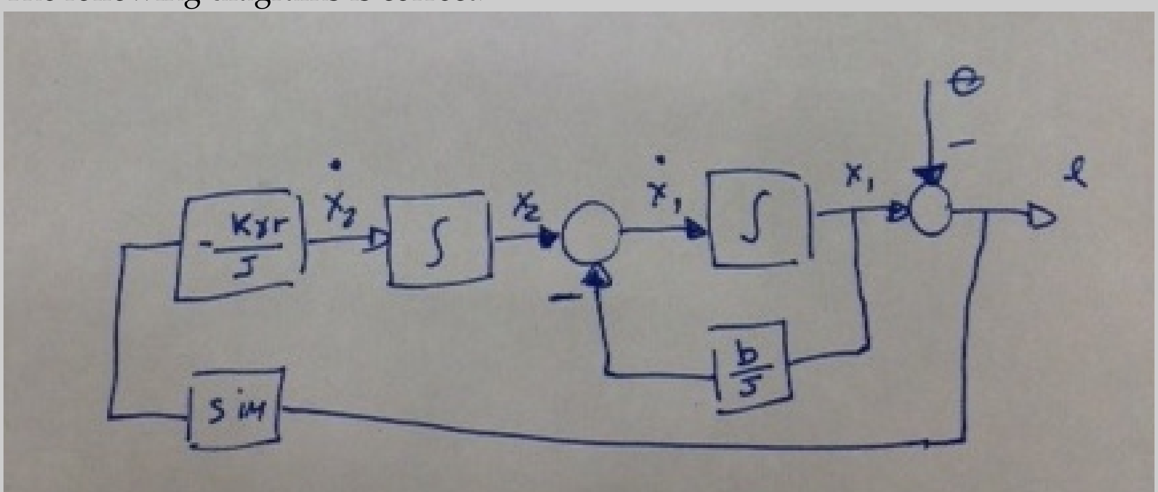
Solution:
Differentiate e twice to obtain:

$$\begin{aligned}
\ddot{e} &= \ddot{x}_1 - \ddot{\theta} \\
&= \dot{x}_2 - \frac{b}{J}\dot{x}_1 - \ddot{\theta} && \text{(+1 point)} \\
&= -\frac{K\gamma r}{J}\sin(e) - \frac{b}{J}(\dot{e} + \dot{\theta}) - \ddot{\theta} && \text{(+1 point)}
\end{aligned}$$

which is (3).

(c) (3 points) Draw the block-diagram associated with the state-space realization in part (b) without using differentiators.

Solution:
The following diagrams is correct:



(+3 points)

NOTE TO GRADER: There are other solutions.

(d) (2 points) Assume that $\theta = \bar{\theta}$ and $r = \bar{r}$ are constant and show that all possible equilibrium points for the closed-loop tracking system are such that $e = \bar{e} = \kappa\pi$, $\kappa \in \mathbb{Z}$. Interpret the result.

Solution:
Equation (3) is of the form

$$\dot{x} = f(x, \theta), \qquad e = g(x, \theta)$$

where

$$f(x,\theta) = \begin{pmatrix} x_2 - \frac{b}{J}x_1 \\ -\frac{K\gamma r}{J}\sin(x_1 - \theta) \end{pmatrix}, \qquad g(x,\theta) = x_1 - \theta. \qquad \text{(+1/2 point)}$$

For equilibrium $f(\bar{x}, \bar{\theta}) = 0$ which means

$$\bar{x}_2 = \frac{b}{J}\bar{x}_1, \qquad \frac{K\gamma r}{J}\sin(\bar{x}_1 - \bar{\theta}) = 0 \qquad \text{(+1/2 point)}$$

or

$$\bar{e} = \bar{x}_1 - \bar{\theta} = \kappa\pi, \quad \kappa \in \mathbb{Z}. \qquad \text{(+1/2 point)}$$

It means that the device is equilibrium when it is pointing at the target, κ is even, or when pointing away to the target, κ is odd. (+1/2 point)

(e) (2 points (bonus)) Linearize about the equilibrium points to show that the transfer-function of the linearized system between $\tilde{\theta} = \theta - \bar{\theta}$ and $\tilde{e} = e - \bar{e}$ is:

$$S(s) = \frac{\tilde{E}(s)}{\tilde{\Theta}(s)} = -\frac{s^2 + (b/J)s}{s^2 + (b/J)s + \delta K\,\gamma\,\bar{r}/J},$$

where $\delta = 1$ when κ is even and $\delta = -1$ when κ is odd.

Solution:

The linearized system is:

$$\begin{aligned} \dot{\tilde{x}}_1 &= \begin{bmatrix} -\dfrac{b}{J} & 1 \\ -\dfrac{K\gamma\,\bar{r}}{J}\cos(\bar{e}) & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ \dfrac{K\gamma\,\bar{r}}{J}\cos(\bar{e}) \end{bmatrix} \tilde{\theta}, \qquad \text{(+1 point)} \\ \tilde{e} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} -1 \end{bmatrix} \tilde{\theta} \end{aligned}$$

When κ is even $\cos(\bar{e}) = \delta = 1$ and when κ is odd $\cos(\bar{e}) = \delta = -1$. The associated transfer-functions is:

$$\frac{\tilde{E}(s)}{\tilde{\Theta}(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$$

where by inspection:

$$\begin{aligned} a_1 &= \frac{b}{J}, & a_2 &= \frac{\delta K\gamma\,\bar{r}}{J}, & & \qquad \text{(+1 point)} \\ b_0 &= -1, & b_2 &= \frac{\delta K\gamma\,\bar{r}}{J} + a_2 b_0 = 0, & b_1 &= a_1 b_0 = -\frac{b}{J} \end{aligned}$$

which matches the one in the statement.

(f) (2 points) Assume that all constants are positive and classify the corresponding equilibria as asymptotically stable or unstable.

Solution:

Stability of the linearized system can be infered from the poles of the linearized transfer-function $S(s)$.

When κ is even $\delta = 1$ and all coefficients of the denominator are positive, hence the poles have negative real part and the corresponding equilibrium point is asymptotically stable. (+1 point)

When κ is even $\delta = -1$ and one coefficient of the denominator is positive, hence one of the poles have positive real part and the corresponding equilibrium point is unstable. (+1 point)

This is good news: the equilibrium when pointing away from the target being unstable will eventually steer the system to the only stable equilibrium which is pointing to the target.

(g) (2 points) Can the closed-loop system asymptotically track a target that has a constant angle $\theta(t) = \underline{\theta} \neq \bar{\theta}$, $t \geq 0$? Explain.

Solution:

The value of $\bar{\theta}$ does not affect the nature of the equilibrium point. Hence one can think of $\underline{\theta}$ as the equilibrium point and $\bar{\theta}$ as an initial condition and repeat the analysis of equilibrium to conclude that in a neighborhood of $\underline{\theta}$ the system will asymptotically converge to $\underline{\theta}$. (+2 points)

Alternatively one can look at $S(s)$ and see that it has one zero at 0 and the response to a constant θ it will converge to zero. (+2 alternative points)

(h) (1 point (bonus)) What is the impact of a variable r on the performance of the closed-loop tracking system?

Solution:

The radius r affects the closed-loop gain in that a larger constant r means a larger gain and a smaller constant r means a smaller gain. It will affect the performance by changing the poles of the closed-loop depending on r. (+1 point)

(i) (1 point (bonus)) One way to mitigate the dependence on r is to use the control law:

$$u = -K \frac{d_y - d_x}{\gamma\, \ell}. \tag{4}$$

Show that $u \approx -K(\psi - \theta)$ when $\psi - \theta$ is small and $r \gg \ell$.

Solution:

Note that

$$\frac{d_y - d_x}{\gamma\, \ell} = f(e), \quad f(e) = \frac{\sqrt{r\,\ell}\left(\sqrt{\xi + \sin(e)} - \sqrt{\xi - \sin(e)}\right)}{\ell}, \quad \xi = \frac{r^2 + (\ell/2)^2}{r\,\ell}$$

and that for small e

$$f(e) \approx f(0) + f'(0)\, e$$

where $f(0) = 0$ and

$$f'(e) = \frac{\sqrt{r\,\ell}}{2\,\ell}\left(\frac{\cos(e)}{\sqrt{\xi + \sin(e)}} + \frac{\cos(e)}{\sqrt{\xi - \sin(e)}}\right), \quad \implies \quad f'(0) = \frac{\sqrt{r\,\ell}}{\ell\sqrt{\xi}} = \frac{r}{\sqrt{r^2 + (\ell/2)^2}}.$$

and that

$$f'(0) = \frac{r}{\sqrt{r^2 + (\ell/2)^2}} \approx 1$$

when $r \gg \ell$, hence $u = -K(d_y - d_x)/(\gamma\ell) = -Kf(e) \approx -Ke = -K(\psi - \theta)$. (+1 point)

4. **Control Part II.**

   Consider the more sophisticated control law:

$$u = -Ke, \qquad e = \arcsin\left(\frac{d_y^2 - d_x^2}{\gamma\,\ell\sqrt{2}\sqrt{d_y^2 + d_x^2 - \gamma^2\ell^2/2}}\right). \tag{5}$$

   (a) (1 point (bonus)) Show that an exact implementation of the control law (5) implies:

$$u = -Ke, \qquad e = \psi - \theta. \tag{6}$$

   Solution:

   Note that

$$d_y^2 - d_x^2 = 2\gamma^2 r\ell\sin(\psi - \theta)$$
$$d_y^2 + d_x^2 - \gamma^2\ell^2/2 = 2\gamma^2\left(r^2 + (\ell/2)^2\right) - \gamma^2\ell^2/2 = 2\gamma^2 r^2$$

   and that

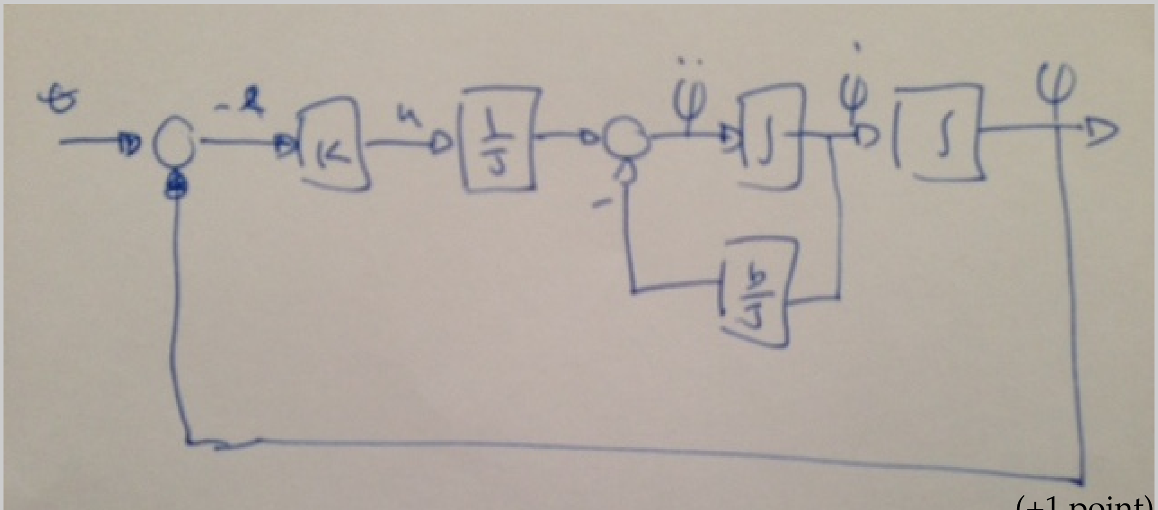
$$h = \frac{d_y^2 - d_x^2}{\gamma\,\ell\sqrt{2}\sqrt{d_y^2 + d_x^2 - \gamma^2\ell^2/2}} = \frac{2\gamma^2 r\ell\sin(\psi - \theta)}{\gamma\ell\sqrt{2}\sqrt{2\gamma^2 r^2}} = \sin(\psi - \theta)$$

   so that $e = \arcsin(h) = \psi - \theta$. (+1 point)

   Note that because of the use of $\arcsin$ one should be carefull to map h in a proper interval. We do it here to generate a linear control problem for you to work on. In practice one might simply use h and end up with a nonlinear system as in 3. Similar properties will hold because $h \approx e$ when e is small.

   (b) (1 point) Draw a block-diagram representing the connection of the linear control law (6) with the linear model (1).

   Solution:



(+1 point)

(c) (2 points) Can the closed-loop system asymptotically track a target that has a constant angle $\theta(t) = \bar{\theta}$, $t \geq 0$? Explain.

Solution:
The closed-loop system in the diagram of part (b) is standard with θ acting as a reference input. It has sensitivity transfer-function:

$$S(s) = \frac{E(s)}{\Theta(s)} = \frac{1}{1 + KG(s)}, \qquad G(s) = \frac{1/J}{s(s + b/J)}. \qquad \text{(+1 point)}$$

The plant model, $G(s)$, has a pole at zero and there are no pole-zero cancellations therefore $S(s)$ has a zero at zero and the closed-loop system achieves asymptotic tracking of constant inputs $\theta(t)$. (+1 point)

(d) (2 points) Explain why the closed-loop system cannot track a target that has an angle that depends affinely on time, that is $\theta(t) = \alpha + \beta\, t$, $t \geq 0$.

Solution:
An angle $\theta(t) = \alpha + \beta t$ has Laplace transform:

$$\Theta(s) = \frac{\alpha}{s} + \frac{\beta}{s^2} = \frac{\alpha s + \beta}{s^2}. \qquad \text{(+1 point)}$$

For asymptotic tracking one would need that $S(s)$ have two zeros at $s = 0$. However, since $G(s)$ has one pole at 0 and $K(s) = K$ has none, $S(s)$ will have only one zero at 0 and the closed-loop system will not achieve asymptotic tracking.
(+1 point)

(e) (2 points) What are the minimum requirements for a controller that uses the signal $e = \psi - \theta$ to asymptotically track a target $\theta(t) = \alpha + \beta\, t$, $t \geq 0$?
You do not need to design a controller. You will do that in the next sections!

Solution:
Since the closed-loop is standard, one would need that $K(s)$ have at least one pole

at $s = 0$ *and* be such that $S(s)$, i.e. the closed-loop system, is asymptotically stable. (+2 points)

Let $L(s)$ be the transfer-function from the input u to ψ calculated in 1(b).
For the rest of the exam consider the data:

$$J = 1, \qquad b = 1$$

In the next questions you will design linear controllers that can stabilize the linear model (1) in closed-loop using the control law (5)-(6).

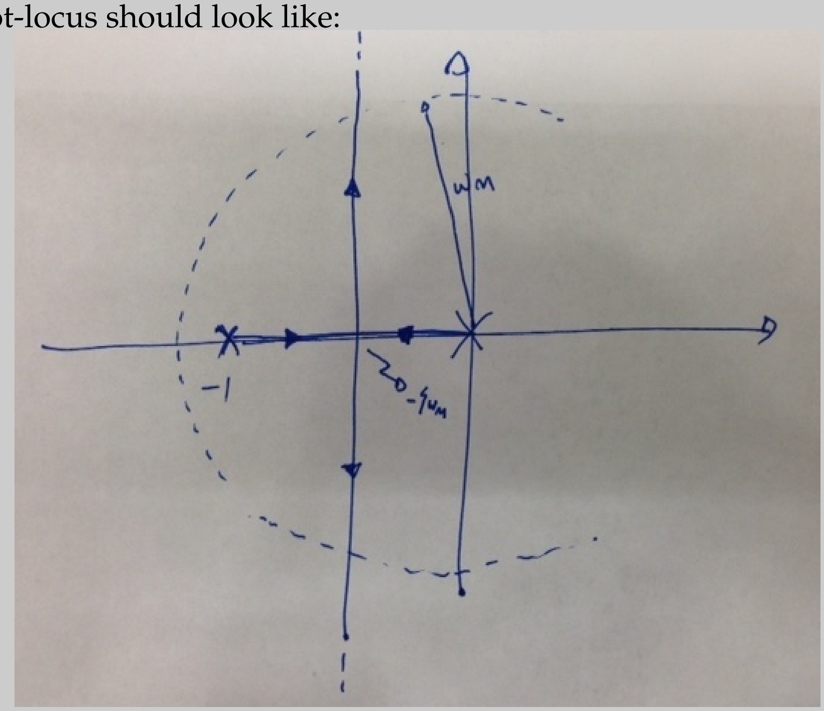
5. **Root-Locus Design.**

(a) (3 points) Sketch the root-locus diagram of $L(s)$. Is it possible to select a gain such that the proportional control law (6) stabilizes the closed-loop? Explain.

Solution:
For the given data

$$L(s) = G(s) = \frac{1}{s(s+1)}$$

and the root-locus should look like:



(+2 points)

Selecting a point in the diagram corresponds to selecting the zeros of $1 + \alpha L(s) = 1 + KG(s)$, $\alpha > 0$, that is the closed-loop poles. In this case any branch has all poles asymptotically stable, for example, the point marked in the sketch is one such point. (+1 point)

(b) (2 points) Describe how could you select a gain on the root-locus diagram to set the closed-loop natural frequency to a desired $\omega_n > 0$.

Solution:

Roots that have the same ω_n lie on the circle of radius $\omega_n > 0$. (+1 point)

Simply select the gain at the intersection of the root-locus with a circle of radius ω_n to achieve the desired goal. (+1 point)

(c) (1 point) Can you arbitrarily set both the closed-loop natural frequency, $\omega_n > 0$, *and* the damping ratio, ζ? Explain using the root-locus diagram from (a).

Solution:

No because the intersection of the root-locus with the circle of radius ω_n may not have the desired ζ. (+1 point)

For a given ω_n, complex roots that have the same $0 < \zeta < 1$ lie on the vertical line $s = -\zeta\omega_n$.

(d) (6 points) Introduce a derivative term (a zero) in the controller, modify $L(s)$ accordingly, and repeat parts (a)–(c). *You do not need to introduce an extra pole!*

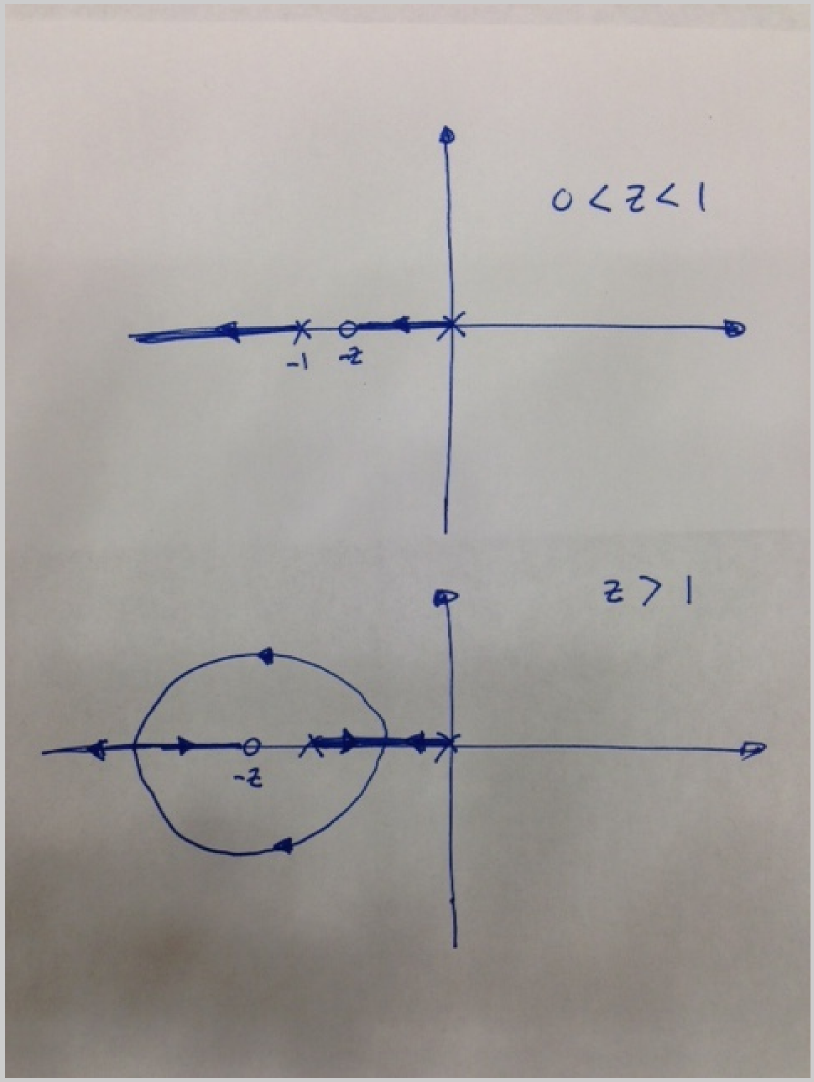
Solution:

(a)

For the given data

$$L(s) = (s + z)G(s) = \frac{s + z}{s(s + 1)}$$

and for any $z > 0$ the root-locus should look like:



depending on the location of the zero. (+2 points)

Selecting a point in the diagram corresponds to selecting the zeros of $1 + \alpha L(s) = 1 + KG(s)$, $\alpha > 0$, that is the closed-loop poles. In this case any branch has all poles asymptotically stable, for example, the point marked in the sketch is one such point. (+1 point)

(b)

Select the gain at the intersection of the root-locus with a circle of radius ω_n to achieve the desired goal. Depending on z the intersection may lead to a real closed-loop pole. (+1 point)

(c)

For a given ω_n, complex roots that have the same $0 < \zeta < 1$ lie on the vertical line $s = -\zeta\omega_n$. (+1 point)

Now it is possible to select z so that the intersection of the root-locus with the circle of radius ω_n have the desired ζ. (+1 point)

In fact, in closed loop

$$1 + \alpha L(s) = \frac{s^2 + (1+\alpha)s + \alpha z}{s(s+1)}$$

from which one can see that any $\omega_n > 0$ and $\zeta > 0$ are possible adjusting both $\alpha > 0$ and the location of the zero $z > 0$.

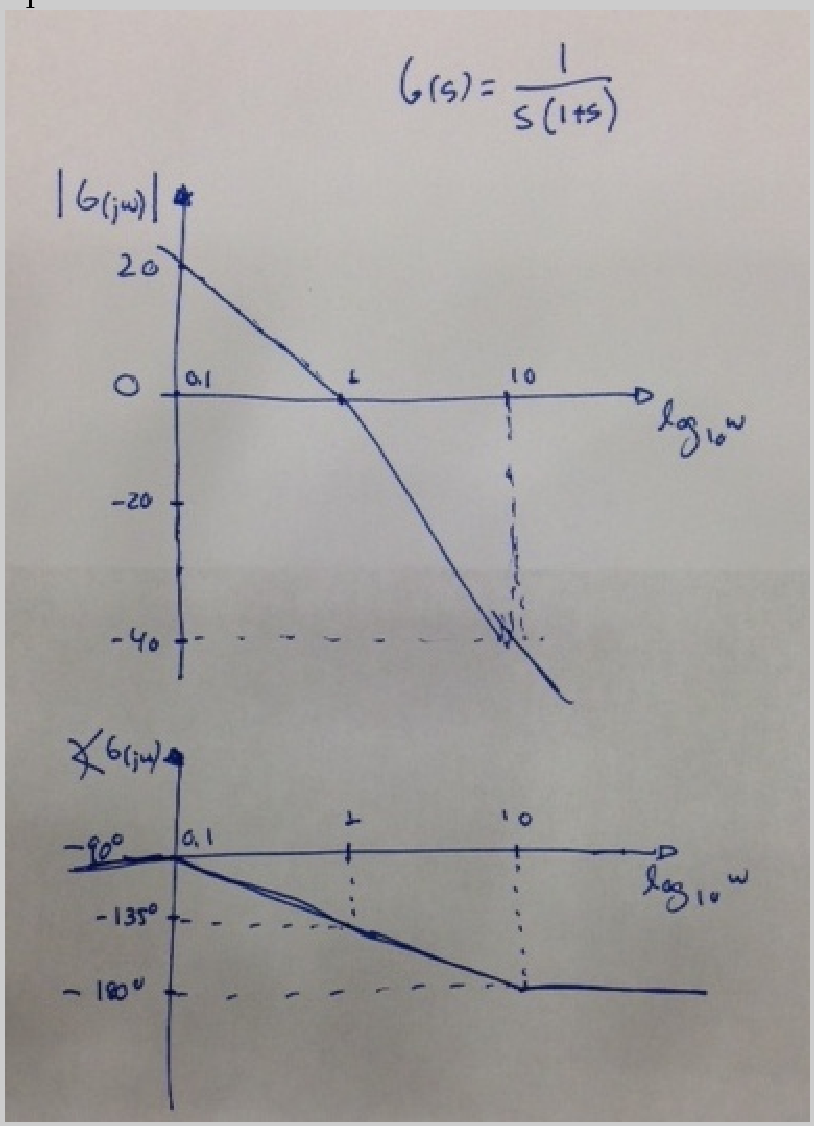
6. **Frequency Domain Design.**

   (a) (2 points) Sketch the magnitude and phase of the Bode plot of $L(j\omega)$.

Solution:
For the given data

$$L(s) = G(s) = \frac{1}{s(s+1)}$$

and the Bode plot should look like:

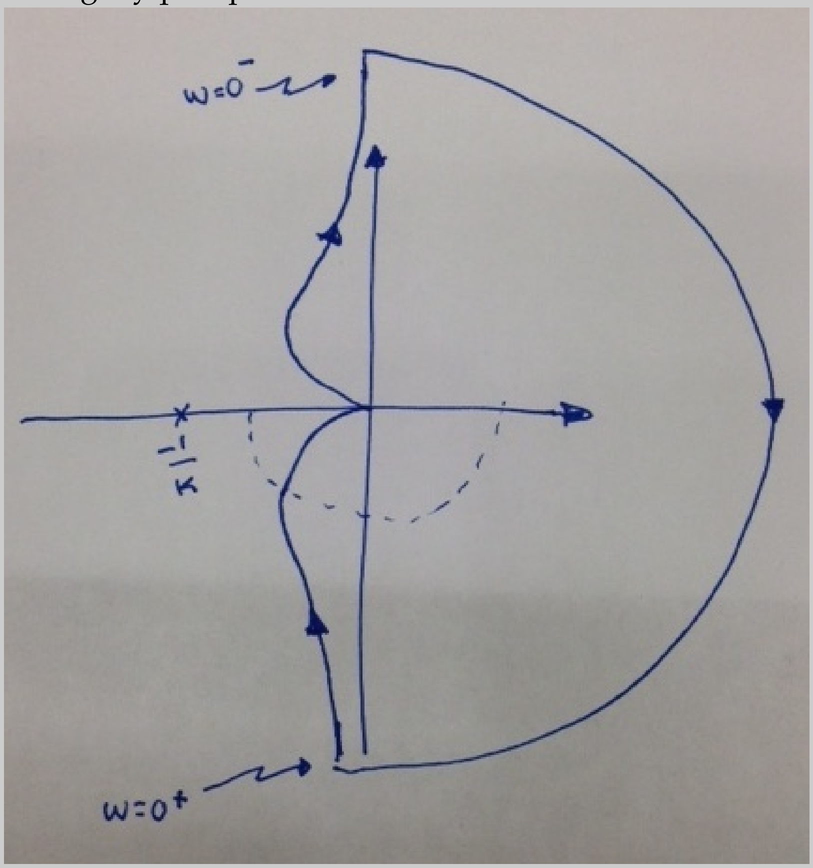


depending on the location of the zero. (+2 points)

(b) (2 points) Sketch the Nyquist plot of $L(j\omega)$.

Solution:

The corresponding Nyquist plot is:



(+2 points)

(c) (2 points) Explain how can you use the plot from part (b) to design a proportional control law (6) that stabilizes the closed-loop.

Solution:

For closed-loop asymptotic stability the Nyquist plot should encircle the point $-1/K$, $K > 0$, P_Γ times on the counter-clockwise direction. (+1 point)

Because $G(s)$ has no poles with positive real part $P_\Gamma = 0$. Because the plot of part (b) never encircles points $-1/K$, $K > 0$, the closed-loop is asymptotically stable for any $K > 0$. (+1 point)

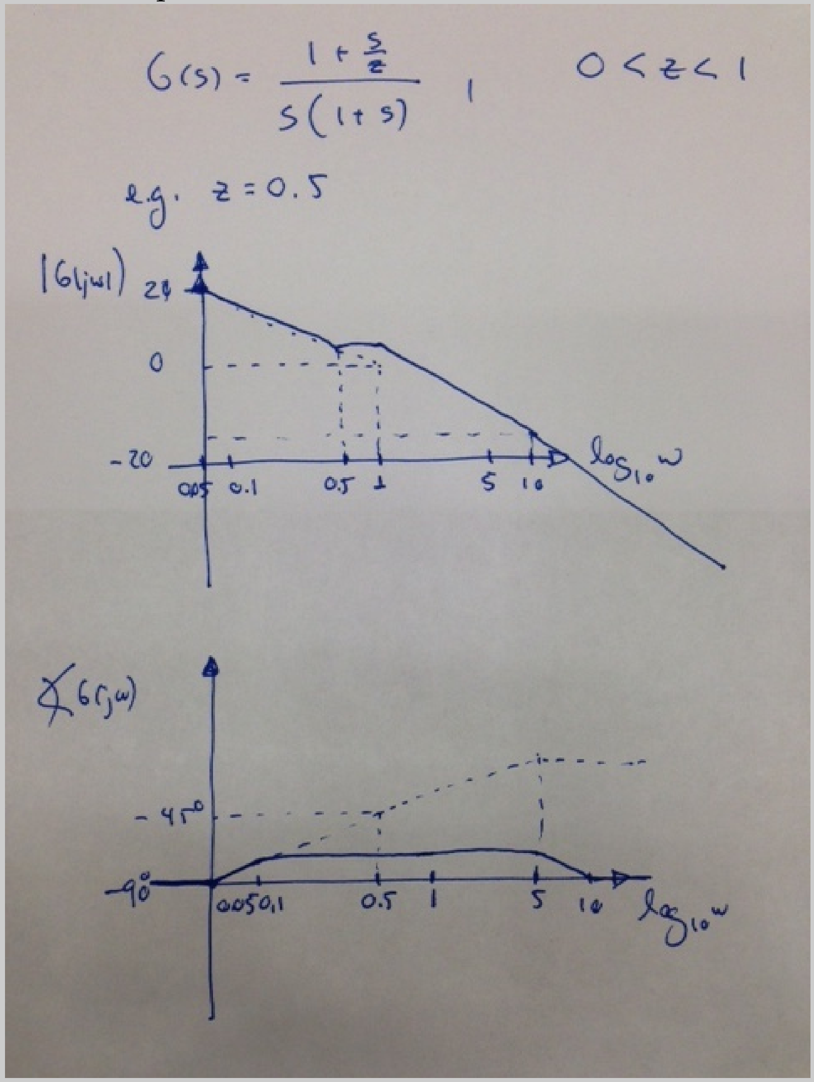
(d) (6 points) Introduce a derivative term (a zero) in the controller, modify $L(s)$ accordingly, and repeat parts (a)–(c). *You do not need to introduce an extra pole!*

Solution:

For the given data

$$L(s) = (1 + s/z)G(s) = \frac{1 + s/z}{s(s+1)}$$

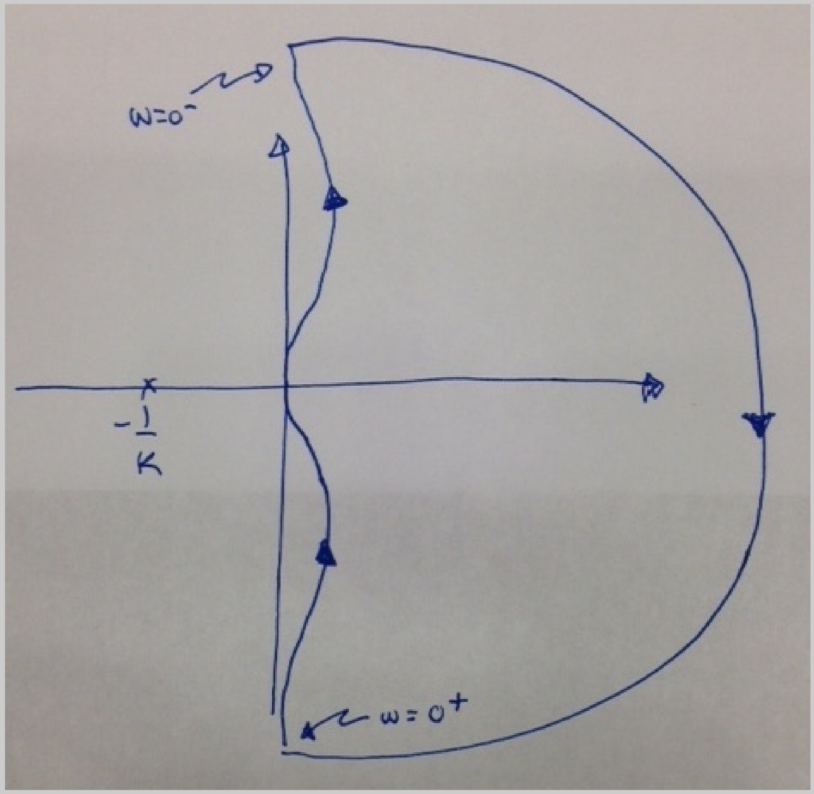
and for $z > 0$ the Bode plot should look like:



(+2 points)

(b)

The corresponding Nyquist plot is:



(+2 points)

(c)

For closed-loop asymptotic stability the Nyquist plot should encircle the point $-1/K$, $K > 0$, P_Γ times on the counter-clockwise direction.

Because $G(s)$ has no poles with positive real part $P_\Gamma = 0$. Because the plot of part (b) never encircles points $-1/K$, $K > 0$, the closed-loop is asymptotically stable for any $K > 0$. (+2 points)

7. **Integral Control.**

   (a) (3 points) Explain using the root-locus diagram why the integral controller:

$$K(s) = \frac{K}{s}, \qquad K > 0,$$

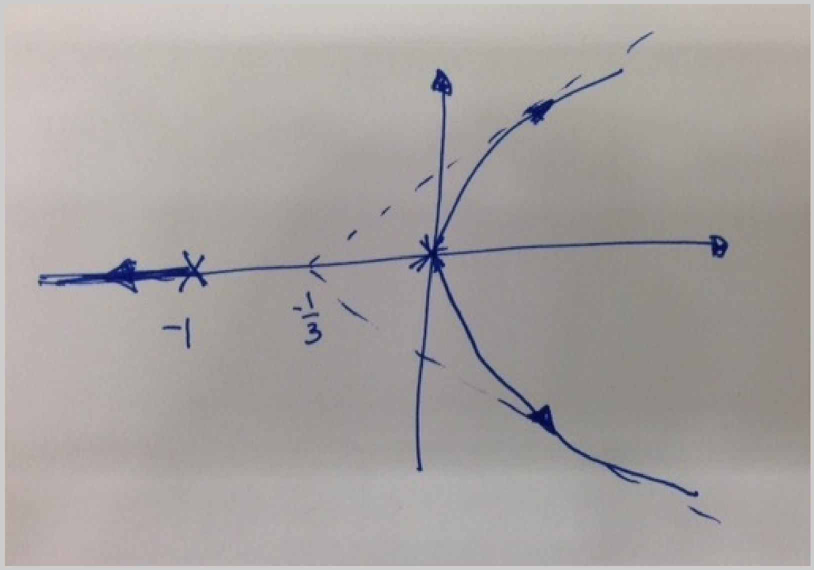
cannot stabilize model (1) in closed-loop.

Solution:

For the given data

$$L(s) = G(s) = \frac{1}{s^2(s+1)}$$

and the root-locus should look like:



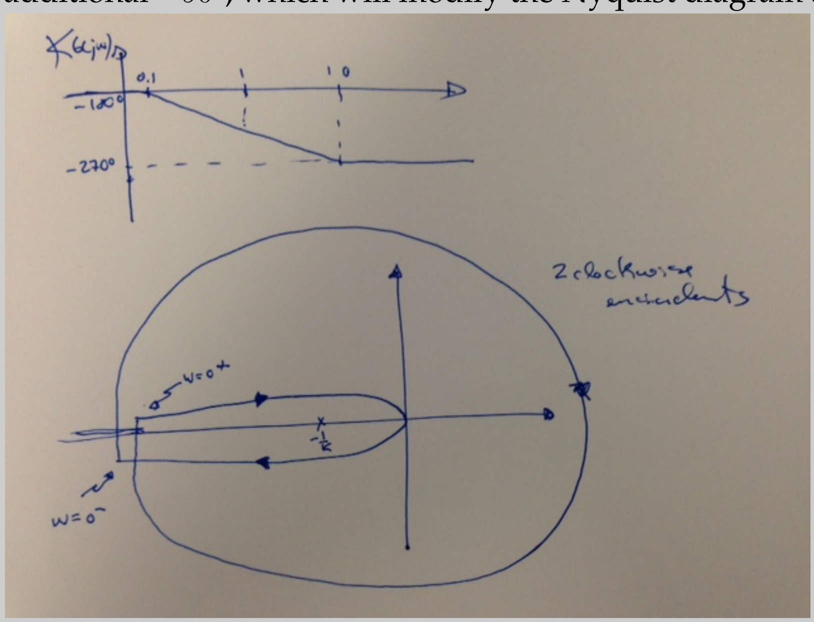
(+2 points)

Since there is always a branch that goes to the right-hand side it is not possible to stabilize the closed-loop with a choice of $K > 0$. (+1 point)

(b) (1 point (bonus)) Repeat part (a) using a Nyquist plot instead of the root-locus.

Solution:

The only change in the magnitude is a change in slope. The corresponding Bode magnitude plot remains essentially the same, with a higher slope. The phase is offset by an additional $-90°$, which will modify the Nyquist diagram as follows:



Because of the two clockwise encirclements there is no value of $-1/K$, $K > 0$, for which the closed-loop system is asymptotically stable.

(c) (3 points) Explain using the root-locus diagram how the PI controller:

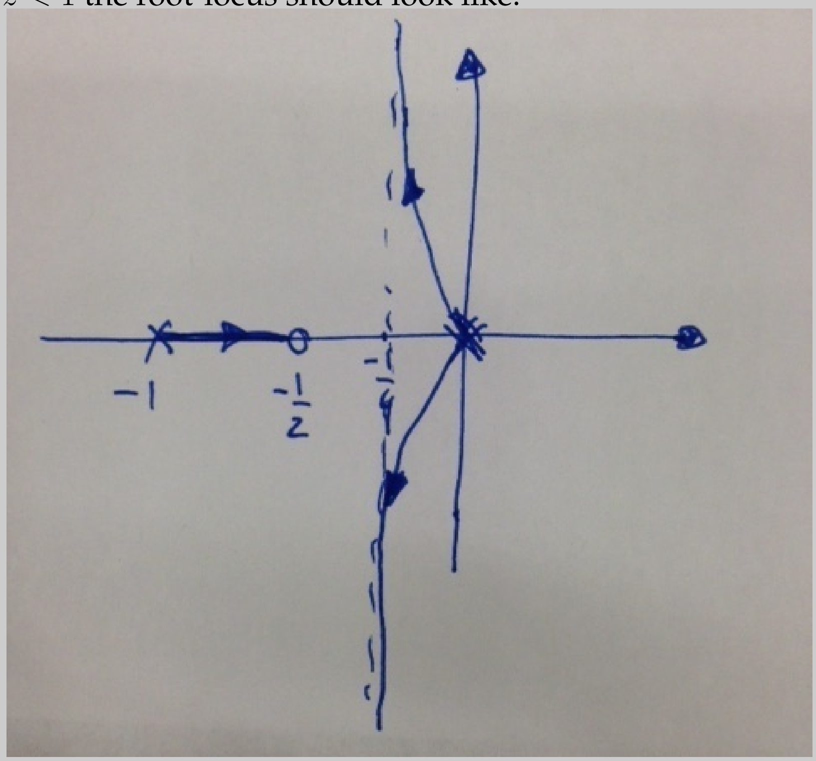
$$K(s) = \frac{K(s+z)}{s}, \qquad K > 0,$$

can stabilize model (1) in closed-loop with z positive and real.

Solution:
For the given data

$$L(s) = G(s) = \frac{s+z}{s^2(s+1)}$$

and for $0 < z < 1$ the root-locus should look like:

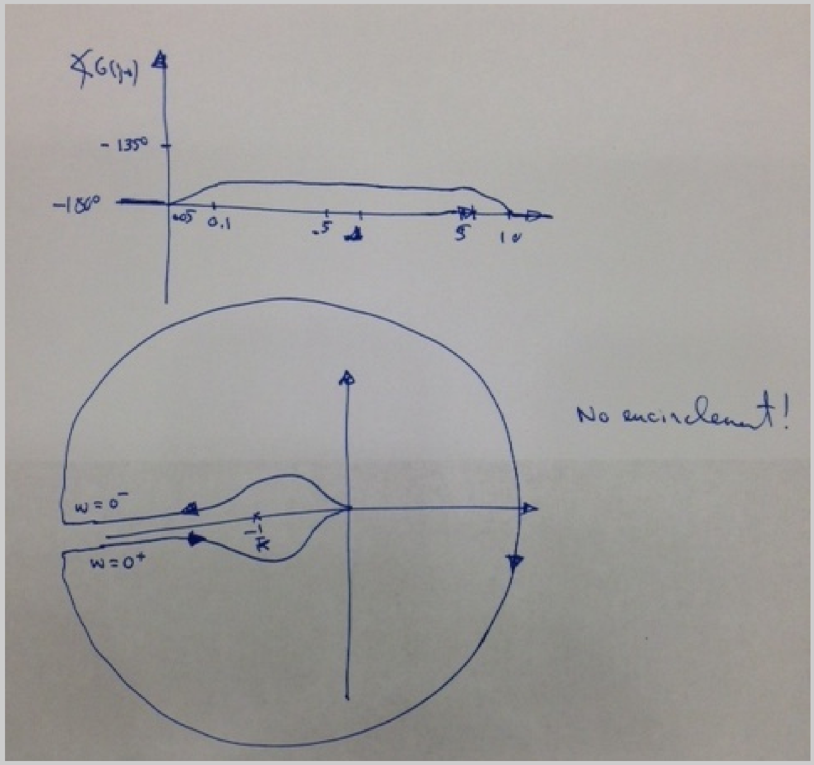


(+2 points)

Note the asymptote at $(-1+z)/2$ which for $0 < z < 1$ is on the left hand side of the complex plane. It is now possible to stabilize the closed-loop with a choice of $K > 0$. (+1 point)

(d) (1 point (bonus)) Repeat part (c) using a Nyquist plot instead of the root-locus.

Solution:
The only change in the magnitude is a change in slope. The corresponding Bode magnitude plot remains essentially the same, with a higher slope. The phase is offset by an additional $-90°$, which will modify the Nyquist diagram as follows:



Because there is never an encirclement, all values of $-1/K$, $K > 0$, make the closed-loop system asymptotically stable.

(e) (2 points) What type of target can the controller from part (c) track asymptotically?

Solution:
Because $L(s)$ has two poles at the origin, it can track a target that has an affine dependence of t, such as a ramp or the target considered in 3(d-e). (+2 points)

(f) (2 points) What type of torque disturbance can the controller from part (c) reject asymptotically?

Solution:
The controller has a single pole at the origin so the closed-loop system can reject constant, i.e. step, torque disturbances. (+2 points)