MAE 143B - Homework 2 Solutions

P2.3

From P2.1, the ODE is

$$\dot{v}(t) + \frac{b}{m}v(t) = g.$$

The steady state velocity is $\tilde{v} = \frac{mg}{b} = 1m/s$. The solution is

$$v(t) = \tilde{v} + [v(0) - \tilde{v}]e^{-\frac{b}{m}t} = 1 + [v(0) - 1]e^{-10t}.$$

Integrate the velocity over t to find the position

$$x(t) = \int v(t) = t - \frac{1}{10}e^{-10t} + C,$$

where C is found by the initial condition $x(0) = 0$, that is $C = \frac{1}{10}[v(0) - 1]$. And so

$$x(t) = t - \frac{1}{10}[v(0) - 1](e^{-10t} - 1).$$

Figure 1 is the plot of $x(t)$ with various values of $v(0)$, using the below `matlab` code:

```
v0 = [0 1 -1];
t = 0:.01:1;
n = length(t);
x = zeros(3,n);

for i = 1:3
    x(i,:) = t - 1/10*(v0(i) - 1).*(exp(-10*t) - 1);
end

figure(1)
plot(t,x(1,:),t,x(2,:),t,x(3,:))
xlabel('Time (s)');ylabel('Position (m)')
legend('v(0) = 0m/s','v(0) = 1m/s','v(0) = -1m/s')
```

P2.4

First convert the velocities to m/s: $200km/h = 200 \times 1000m/3600s = 55.6m/s$, $20km/h = 5.6m/s$, $29km/h = 8.1m/s$. For the free-fall phase, the ODE is

$$\dot{v}_f + \frac{b_f}{m}v_f = g$$

whose solution is

$$v_f(t) = \tilde{v}_f + \beta_f e^{-\frac{b_f}{m}t}.$$

Using the given data, $b_f = \frac{mg}{\tilde{v}_f} = 12.5kg/s$. Assuming $v_f(0) = 0$, i.e the diver jumps down with no initial velocity, $\beta_f = v(0) - \tilde{v}_f = -55.6$. And so

$$v_f(t) = 55.6 - 55.6e^{-0.18t}.$$

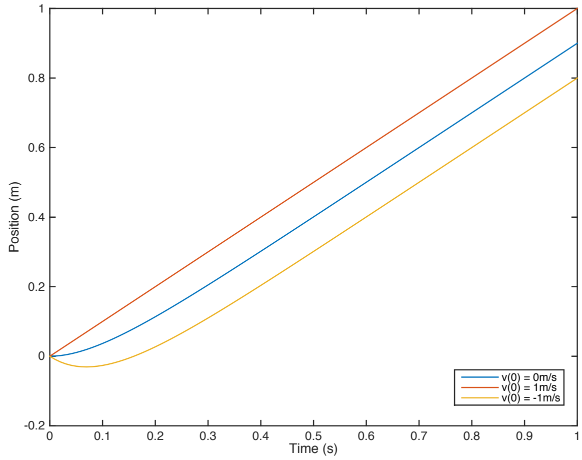

Figure 1: Position $x(t)$.

Similarly, the solution to the ODE describing the parachute phase can be found to be

$$v_p(t) = 5.6 + 50e^{-1.8t},$$

using $\tilde{v}_p = 5.6m/s$, $b_p = \frac{mg}{\tilde{v}_p} = 125kg/s$, $v_p(0) = 55.6m/s$.

The time constants are $\lambda_f = 1/0.18 = 5.6s$, $\lambda_p = 1/1.8 = 0.6s$.

Using the parachute ODE and setting the velocity to 29km/h (8m/s)

$$5.6 + 50e^{-1.8t} \leq 8,$$

and solving for t, getting $t \geq 1.69s$. This means the diver should open the parachute at least 1.69s before hitting the ground, at a height of

$$h \geq \int_0^{1.69} v_p(t)dt = 35.6m.$$

For the last part, the distance traveled during the free-fall phase is

$$x_f = \int_0^{60} v_f(t)dt = 3027m.$$

And so the distance traveled during the parachute phase is $x_p = 4000 - 3025 = 973m$. The time t_p it takes to hit the ground satisfies

$$x_p = 975 = \int_0^{t_p} v_p(t)dt = 5.6t_p - \frac{50}{1.8}(e^{-1.8t_p} - 1).$$

Using `matlab`, $t_p = 169s$. The total traveling time is $t = 60 + t_p = 229s$.

P2.7

Since the belt is inextensible, the velocity of any two points on the belt must be the same. Considering a point on J_1 and the other on J_2, we get $\omega_1 r_1 = \omega_2 r_2$, and so $\dot{\omega}_2 = \frac{r_1}{r_2}\dot{\omega}_1$. Substitute this into the ODE for J_2 and the ODE describing the whole system can be obtained as

$$(J_1 r_2^2 + J_2 r_1^2)\dot{\omega}_1 = r_2^2\tau.$$

P2.16

From the equation describing the circuit

$$i_a = \frac{v_a - K_e\omega}{R_a}.$$

The torque is then

$$\tau = K_t\frac{v_a - K_e\omega}{R_a}.$$

Substitute the above into the ODE

$$\dot{\omega} + \frac{b + \frac{K_tK_e}{R_a}}{J}\omega = \frac{K_t}{R_aJ}v_a.$$

When v_a is constant, the solution to the above ODE takes the form

$$\omega(t) = \tilde{\omega} + \beta\omega_H(t).$$

The steady-state angular velocity is

$$\tilde{\omega} = \frac{(K_t/R_a)v_a}{b + \frac{K_tK_e}{R_a}}.$$

Supposing the initial condition $\omega(0) = 0$,

$$\beta = -\frac{(K_t/R_a)v_a}{b + \frac{K_tK_e}{R_a}}.$$

The homogenous solution is $\omega_H(t) = e^{\lambda t}$ where

$$\lambda = -\frac{b + \frac{K_tK_e}{R_a}}{J}.$$

Thus the solution is given by

$$\omega(t) = \frac{(K_t/R_a)v_a}{b + \frac{K_tK_e}{R_a}} - \frac{(K_t/R_a)v_a}{b + \frac{K_tK_e}{R_a}}e^{-\frac{b+\frac{K_tK_e}{R_a}}{J}t}.$$

If $\tau = 0.1s$ we can find $\lambda = -1/\tau = -10$. Plus we know $\tilde{\omega} = 5000rpm$. Subsequently the solution is

$$\omega(t) = 5000 + 5000e^{-10t}.$$

This means that even though we do not have enough information to estimate all the parameters in the system, the ODE of the model can be properly described and its solution obtained.

P2.17

Knowing $\tilde{\omega}$, R_a and v_a we can estimate $K_t = 833J$ from the equation for $\tilde{\omega}$. To estimate the other parameters, one can vary the value of R_a by means of a potentiometer and J by adding weights to the motor and find the parameters by the data obtained during the experiment.

P2.18

Using $V_a(t) = K(\bar{\omega} - \omega(t))$, the ODE of the closed-loop system is found to be

$$\dot{\omega} + \frac{b + \frac{K_tK_e}{R_a} + \frac{KK_t}{R_a}}{J}\omega = \frac{KK_t}{JR_a}\bar{\omega},$$

whose solution, according to Section 2.5, is

$$\omega(t) = \tilde{\omega}(1 - e^{\lambda t}) + \omega_0 e^{\lambda t},$$

where we can assume $\omega_0 = 0$ and the steady-state angular velocity is

$$\tilde{\omega} = \frac{\frac{KK_t}{R_a}}{b + \frac{K_t K_e}{R_a} + \frac{KK_t}{R_a}} \bar{\omega},$$

and

$$\lambda = -\frac{b + \frac{K_t K_e}{R_a} + \frac{KK_t}{R_a}}{J}.$$

From P2.16, using the open-loop time constant of 0.1s, we find that

$$b + \frac{K_t K_e}{R_a} = 10J,$$

and then using the steady-state angular velocity $\tilde{\omega} = \frac{(K_t/R_a)v_a}{b + \frac{K_t K_e}{R_a}} = 5000RPM$ and $v_a = 12V$, we find that

$$\frac{K_t}{R_a} = 4166.7J.$$

Let $\tilde{\omega} = H(0)\bar{\omega}$ where

$$H(0) = \frac{\frac{KK_t}{R_a}}{b + \frac{K_t K_e}{R_a} + \frac{KK_t}{R_a}}.$$

The sensitivity is $S(0) = 1 - H(0)$. The steady-state tracking error is $e_{ss} = \lim_{t\to\infty} e(t) = S(0)\bar{\omega}$. Setting $S(0) = 10\%$ and using the above information, the proportional gain can be found to be $K = 0.02$.

Using the above gain, $\lambda = -93.3$ and the time constant $\tau = -1/\lambda = 0.01s$

If $\bar{\omega} = 4000$ and $\omega(0) = 0$ then

$$v_a(0) = K(\bar{\omega} - \omega(0)) = 80V.$$

For $v_a(0) < 12V$ we have $K\bar{\omega} < 12$, or $K < 0.003$.