MAE 143B - Homework 3

Remark

All functions of time t considered in this homework are defined only for $t \geq 0$.

3.1

f)

$$\mathcal{L}\{\sin(t) - e^{-t}\cos(2t)\} = \mathcal{L}\{\sin(t)\} - \mathcal{L}\{e^{-t}\cos(2t)\} = \frac{1}{s^2+1} - \frac{s+1}{(s+1)^2+4}$$

j)

$$\mathcal{L}\{\delta(t) + te^{-t}\sin(t)\} = \mathcal{L}\{\delta(t)\} + \mathcal{L}\{te^{-t}\sin(t)\} = 1 - \frac{d}{ds}\frac{1}{(s+1)^2+1} = 1 + \frac{2s+2}{((s+1)^2+1)^2}$$

3.2

e)

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \operatorname*{Res}_{s=1}\left\{\frac{1}{s^2-1}e^{st}\right\} + \operatorname*{Res}_{s=-1}\left\{\frac{1}{s^2-1}e^{st}\right\} = \left.\frac{1}{s+1}e^{st}\right|_{s=1} + \left.\frac{1}{s-1}e^{st}\right|_{s=-1} = \frac{1}{2}\left(e^{t} - e^{-t}\right)$$

g)

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = \operatorname*{Res}_{s=-1+j}\left\{\frac{1}{(s+1)^2+1}e^{st}\right\} + \operatorname*{Res}_{s=-1-j}\left\{\frac{1}{(s+1)^2+1}e^{st}\right\} =$$
$$\left.\frac{1}{s+1+j}e^{st}\right|_{s=-1+j} + \left.\frac{1}{s+1-j}e^{st}\right|_{s=-1-j} = \frac{1}{2j}\left(e^{tj} - e^{-tj}\right)e^{-t} = \sin(t)e^{-t}$$

3.12

d)

$$F(s) = \mathcal{L}\left\{\mathcal{L}^{-1}\{F(s)\right\} =$$
$$\mathcal{L}\left\{\operatorname*{Res}_{s=0}\left\{\frac{1}{s(s+1)(s+2)}e^{st}\right\} + \operatorname*{Res}_{s=-1}\left\{\frac{1}{s(s+1)(s+2)}e^{st}\right\} + \operatorname*{Res}_{s=-2}\left\{\frac{1}{s(s+1)(s+2)}e^{st}\right\}\right\} =$$
$$\mathcal{L}\left\{\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right\} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}$$

e)

$$F(s) = \mathcal{L}\left\{\mathcal{L}^{-1}\{F(s)\right\} = \mathcal{L}\left\{\operatorname*{Res}_{s=0}\left\{\frac{1}{s^2(s+1)}e^{st}\right\} + \operatorname*{Res}_{s=-1}\left\{\frac{1}{s^2(s+1)}e^{st}\right\}\right\} =$$
$$\mathcal{L}\left\{\frac{d}{ds}\frac{1}{s+1}e^{st}\bigg|_{s=0} + \frac{1}{s^2}e^{st}\bigg|_{s=-1}\right\} = \mathcal{L}\left\{t - 1 + e^{-t}\right\} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

3.15

d)

Taking Laplace transforms of the ODE with $y(0) = 0$ gives

$$Y(s) = \frac{1}{s(s-1)},$$

such that

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{s-1}e^{st}\bigg|_{s=0} + \frac{1}{s}e^{st}\bigg|_{s=1} = -1 + e^t.$$

h)

Taking Laplace transforms of the ODE with $y(0) = \dot{y}(0) = 0$ gives

$$Y(s) = \frac{s}{(s^2+1)^2} = \frac{s}{(s+j)^2(s-j)^2},$$

such that

$$y(t) = \operatorname*{Res}_{s=j}\left\{Y(s)e^{st}\right\} + \operatorname*{Res}_{s=-j}\left\{Y(s)e^{st}\right\} = \frac{d}{ds}\frac{s}{(s+j)^2}e^{st}\bigg|_{s=j} + \frac{d}{ds}\frac{s}{(s-j)^2}e^{st}\bigg|_{s=-j} =$$
$$t\frac{s}{(s+j)^2}e^{st}\bigg|_{s=j} + \frac{-s+j}{(s+j)^3}e^{st}\bigg|_{s=j} + t\frac{s}{(s-j)^2}e^{st}\bigg|_{s=-j} + \frac{-s-j}{(s-j)^3}e^{st}\bigg|_{s=-j} = t\frac{1}{4j}\left(e^{jt} - e^{-jt}\right) = \frac{t}{2}\sin(t).$$

3.27

Taking Laplace transforms of the ODE with zero initial conditions yields the transfer function

$$H(s) = \frac{\Omega(s)}{V_a(s)} = \frac{K_t/JR_a}{s + (b + \frac{K_tK_e}{R_a})/J},$$

which has a single pole at

$$s = -\frac{b + \frac{K_t K_e}{R_a}}{J} = -\frac{bR_a + K_t K_e}{JR_a}.$$

Given that all constants are positive, this pole has strictly negative real part and the transfer function is asymptotically stable.

3.28

For $v_a(t) = \tilde{v}_a$, $t \geq 0$, we have $V_a(s) = \tilde{v}_a/s$ and

$$\omega(t) = \mathcal{L}^{-1}\{H(s)V_a(s)\} = \mathcal{L}^{-1}\left\{\frac{K_t/JR_a}{s + (b + \frac{K_t K_e}{R_a})/J} \cdot \frac{\tilde{v}_a}{s}\right\} = \frac{K_t \tilde{v}_a}{bR_a + K_t K_e}\left(1 - e^{-\frac{bR_a + K_t K_e}{JR_a}t}\right).$$

This input results in the steady-state response

$$\lim_{t\to\infty} \omega(t) = \frac{K_t \tilde{v}_a}{bR_a + K_t K_e}.$$

For $v_a(t) = \tilde{v}_a \cos(wt)$, $t \geq 0$, we have $V_a(s) = \tilde{v}_a s/(s^2 + w^2)$ and

$$\omega(t) = \mathcal{L}^{-1}\left\{\frac{K_t/JR_a}{s + (b + \frac{K_t K_e}{R_a})/J} \cdot \frac{\tilde{v}_a s}{s^2 + w^2}\right\} =$$
$$\frac{K_t \tilde{v}_a}{(bR_a + K_t K_e)^2 + (JR_a w)^2}\left(JR_a w \sin(wt) + (bR_a + K_t K_e)\left(\cos(wt) - e^{-\frac{bR_a + K_t K_e}{JR_a}t}\right)\right).$$

For large times, the exponential term diminishes, resulting in the sinusoidal steady-state solution

$$\frac{K_t \tilde{v}_a}{(bR_a + K_t K_e)^2 + (JR_a w)^2}\left(JR_a w \sin(wt) + (bR_a + K_t K_e)\cos(wt)\right).$$

3.29

As we have seen in Section 3.8, the steady-state response of a linear time-invariant system like in this problem to an of the form input $u(t) = A\cos(wt + \phi)$ is

$$y_{ss}(t) = A|G(jw)|\cos(wt + \phi + \angle G(jw)),$$

where in our case $y(t) = w(t)$ and $G(jw) = H(s = jw)$. For the first input signal, we have $w = \phi = 0$ and $A = \tilde{v}_a$, such that

$$\omega_{ss}(t) = \tilde{v}_a|H(jw)|\cos(\angle H(jw))|_{w=0} = \tilde{v}_a \frac{K_t}{bR_a + K_t K_e}\cos(0) = \frac{K_t \tilde{v}_a}{bR_a + K_t K_e},$$

confirming our answer from 3.28. For the second input, we have $\phi = 0$, $A = \tilde{v}_a$ and

$$\omega_{ss}(t) = \tilde{v}_a|H(jw)|\cos(wt + \angle H(jw)).$$

From 3.27, we have

$$H(jw) = \frac{K_t}{bR_a + K_t K_e + jJR_a w} = \frac{bR_a + K_t K_e}{(bR_a + K_t K_e)^2 + (JR_a w)^2}K_t - j\frac{JR_a w}{(bR_a + K_t K_e)^2 + (JR_a w)^2}K_t,$$
$$|H(jw)| = \frac{K_t}{(bR_a + K_t K_e)^2 + (JR_a w)^2}\sqrt{(bR_a + K_t K_e)^2 + (JR_a w)^2},$$
$$\angle H(jw) = 2\pi - \tan^{-1}\left(\frac{bR_a + K_t K_e}{JR_a w}\right),$$

such that

$$\begin{aligned}
\omega_{ss}(t) &= \frac{K_t\tilde{v}_a}{(bR_a+K_tK_e)^2+(JR_aw)^2}\sqrt{(bR_a+K_tK_e)^2+(JR_aw)^2}\cos\left(wt+2\pi-\tan^{-1}\left(\frac{bR_a+K_tK_e}{JR_aw}\right)\right)\\
&= \frac{K_t\tilde{v}_a\sqrt{(bR_a+K_tK_e)^2+(JR_aw)^2}}{(bR_a+K_tK_e)^2+(JR_aw)^2}\cdot\\
&\quad\left(\cos(wt)\cos\left(2\pi-\tan^{-1}\left(\frac{bR_a+K_tK_e}{JR_aw}\right)\right)-\sin(wt)\sin\left(2\pi-\tan^{-1}\left(\frac{bR_a+K_tK_e}{JR_aw}\right)\right)\right).
\end{aligned}$$

Now notice that

$$\begin{aligned}
\cos\left(2\pi-\tan^{-1}\left(\frac{bR_a+K_tK_e}{JR_aw}\right)\right) &= \frac{\Re\{H(jw)\}}{|H(jw)|},\\
\sin\left(2\pi-\tan^{-1}\left(\frac{bR_a+K_tK_e}{JR_aw}\right)\right) &= \frac{\Im\{H(jw)\}}{|H(jw)|},
\end{aligned}$$

which together with the above gives

$$\begin{aligned}
&\omega_{ss}(t) = \tilde{v}_a\left(\Re\{H(jw)\}\cos(wt)+\Im\{H(jw)\}\sin(wt)\right) =\\
&\qquad\qquad \frac{K_t\tilde{v}_a}{(bR_a+K_tK_e)^2+(JR_aw)^2}\left((bR_a+K_tK_e)\cos(wt)+JR_aw\sin(wt)\right),
\end{aligned}$$

confirming our answer from 3.28.