MAE 143B - Homework 4 Solutions

P4.2

b)

$$K = \frac{K_p s + K_i}{s} \Rightarrow GK = \frac{aK_p s + aK_i}{s(s+a)} \text{ and } 1 + GK = \frac{s^2 + (a + aK_p)s + aK_i}{s(s+a)}.$$

The order of $\frac{1}{1+GK}$, which is 2nd order, is the same as the order of H, S, D and Q.

$$\begin{aligned}
S &= \frac{1}{1+GK} = \frac{s(s+a)}{s^2 + (a + aK_p)s + aK_i} \\
D &= \frac{G}{1+GK} = \frac{as}{s^2 + (a + aK_p)s + aK_i} \\
H &= \frac{GK}{1+GK} = \frac{aK_p s + aK_i}{s^2 + (a + aK_p)s + aK_i} \\
Q &= \frac{K}{1+GK} = \frac{(K_p s + K_i)(s+a)}{s^2 + (a + aK_p)s + aK_i}
\end{aligned}$$

d)

$$GK = \frac{K_p a_2}{s^2 + a_1 s + a_2} \Rightarrow 1 + GK = \frac{s^2 + a_1 s + a_2 + K_p a_2}{s^2 + a_1 s + a_2}$$

Similarly to Part (b), H, S, D and Q are 2nd order transfer functions.

$$\begin{aligned}
S &= \frac{1}{1+GK} = \frac{s^2 + a_1 s + a_2}{s^2 + a_1 s + a_2 + K_p a_2} \\
D &= \frac{G}{1+GK} = \frac{a_2}{s^2 + a_1 s + a_2 + K_p a_2} \\
H &= \frac{GK}{1+GK} = \frac{K_p a_2}{s^2 + a_1 s + a_2 + K_p a_2} \\
Q &= \frac{K}{1+GK} = \frac{K_p s^2 + K_p a_1 s + K_p a_2}{s^2 + a_1 s + a_2 + K_p a_2}
\end{aligned}$$

P4.4

For the closed-loop system to be asymptotically stable (a.s.), the closed-loop transfer function's poles must be all in the open left half of the complex plane. Note that because $H = \frac{GK}{1+GK}$, the poles of H is the zeros of $1 + GK$.

Lemma 4.1 allows one to see if the system is able to achieve asymptotic tracking (a.t.). First $S = (1+GK)^-1$ needs to be a.s., and then if GK has a pole at $s = 0$ then the a.t. is achieved.

a)

We have $S = \frac{1}{1+GK} = \frac{s(s+1)}{s(s+1)+1}$ which has poles at $s = -0.5 \pm j\frac{\sqrt{3}}{2}$, meaning S is a.s.. Therefore the closed-loop transfer function is also a.s., that makes the closed-loop system a.s.

Also GK has a pole at the origin, which makes a.t. achievable.

b)

$1 + GK = \frac{s^2+1}{s^2}$ that has zeros at $s = \pm j$. That means the closed-loop system is not a.s. and a.t. cannot be achieved.

d)

We have $S = \frac{1}{1+GK} = \frac{s(s+1)}{s(s+1)+1}$ which has poles at $s = -0.5 \pm j\frac{\sqrt{3}}{2}$, meaning S is a.s.. Therefore the closed-loop transfer function is also a.s., that makes the closed-loop system a.s.

Also GK has a pole at the origin, which makes a.t. achievable.

P4.5

Use Lemma 4.2 to solve this problem.

a)

From P4.4(a), we have the closed-loop system and the sensitivity S being a.s., but the controller K has no pole at the origin. So the system cannot achieve a.t. and asymptotic disturbance rejection (a.d.r.).

Alternatively we can look at

$$D = \frac{G}{1+GK} = \frac{s+1}{s(s+1)+1},$$

that has only one zero at $s = -1$, or $j\omega = 1$, while the frequency of the constant $\bar{w}$ is $\omega = 0$. So it seems that we cannot make both $S(0)$ and $D(0)$ zero at the same time. That means the system cannot achieve a.t. and a.d.r.

b)

From P4.4(b), we have that S is not a.s., and it follows that a.t. and a.d.r. are not achievable.

d)

From P4.4(d), we have S being a.s. The controller K has a pole at $s = 0$, and so a.t. and a.d.r are achievable.

Alternatively we can look at

$$D = \frac{G}{1+GK} = \frac{s}{s(s+1)+1}.$$

This time D has a zero at the origin, which means D and S can both be zero at $s = 0$, which is the frequency of the constant disturbance $\bar{w}$. This indicates that a.t. and a.d.r. are achievable.

P4.7

From the block diagram, the closed-loop system takes the form

$$Y(s) = \frac{GK}{1+GK}(\bar{Y}(s) - V(s)),$$

where the transfer function is

$$H(s) = \frac{GK}{1+GK} = \frac{1}{s(s+1)+1}$$

We have $Y(s) = \frac{\bar{y}}{s}$ and $V(s) = \frac{\bar{v}}{s} + \frac{s}{s^2+\omega^2}$. Substitute those equation into the above to yield

$$Y(s) = H(s)\left(\frac{\bar{y}}{s} - \frac{\bar{v}}{s}\right) + H(s)\left(\frac{s}{s^2+\omega^2}\right),$$

where we can see the first that the first term's steady-state response exists because it only has one pole at the origin and thus can be found by the final value theorem, while the second term will result in a sinusoidal steady-state response, which can be found by frequency response.

Let $Y_1(s) = H(s)\left(\frac{\bar{y}}{s} - \frac{\bar{v}}{s}\right)$. Apply the final value theorem to find the steady-state value of this response

$$y_{ss1} = \lim_{s\to 0} sY_1(s) = \bar{y} - \bar{v}.$$

Let $Y_2(s) = H(s)\left(\frac{s}{s^2+\omega^2}\right)$. The sinusoidal steady-state value of this response can be found using the frequency response to the input $\cos(\omega t)$ of the system with the transfer function $H(s)$

$$y_{sss2} = |H(j\omega)|\cos(\omega t + \angle H(j\omega))$$

$H(s) = \frac{1}{s(s+1)+1}$ yields $|H(j\omega)| = \frac{1}{\sqrt{\omega^4-\omega^2+1}}$ and $\angle H(j\omega) = -\text{atan2}(\omega, 1-\omega^2)$. Therefore

$$y_{sss2} = \frac{1}{\sqrt{\omega^4-\omega^2+1}}\cos(\omega t - \text{atan2}(\omega, 1-\omega^2)).$$

The total steady-state response is

$$y_{ss}(t) = y_{ss1} - y_{sss2} = \bar{y} - \bar{v} - \frac{1}{\sqrt{\omega^4-\omega^2+1}}\cos(\omega t - \text{atan2}(\omega, 1-\omega^2)).$$

The steady state tracking error is

$$e_{ss}(t) = \bar{v} + \frac{1}{\sqrt{\omega^4-\omega^2+1}}\cos(\omega t - \text{atan2}(\omega, 1-\omega^2)),$$

which is not zero. That means the system does not achieve asymptotic tracking and asymptotic disturbance rejection.

If $\bar{v} = 0$ and $\omega \to \infty$, then $e_{ss}(t) \to 0$ because $\omega^4 - \omega^2 + 1 \to \infty$, which indicates asymptotic tracking can be achieved.

P4.8

Use Lemma 4.3 to solve this problem.

a)

$$\begin{aligned}
GK &= \frac{s+1}{(s+1)(s+2)} \\
S &= \frac{1}{1+GK} = \frac{s+2}{s+3} \\
D &= \frac{G}{1+GK} = \frac{(s+2)}{(s+1)(s+3)} \\
H &= \frac{GK}{1+GK} = \frac{1}{s+3} \\
Q &= \frac{K}{1+GK} = \frac{s+1}{s+3}
\end{aligned}$$

S is a.s. and GK has a pole-zero cancelation at $s = -1$, which means the system is internally stable.

d)

$$\begin{aligned}
GK &= \frac{s-1}{(s+1)(s-1)} \\
S &= \frac{1}{1+GK} = \frac{s+1}{s+2} \\
D &= \frac{G}{1+GK} = \frac{s-1}{s+2} \\
H &= \frac{GK}{1+GK} = \frac{1}{s+2} \\
Q &= \frac{K}{1+GK} = \frac{s+1}{(s-1)(s+2)}
\end{aligned}$$

Because GK has a pole-zero cancelation at $s = 1$, the system is not internally stable.

Additional Problem

From P2.16 we have the following ODE:

$$\dot{\omega} + \frac{b + \frac{K_t K_e}{R_a}}{J}\omega = \frac{K_t}{JR_a}v_a$$

Let $a = \frac{K_t}{JR_a}$ and $c = \frac{b+\frac{K_t K_e}{R_a}}{J}$ so that the ODE can be written compactly as

$$\dot{\omega} + c\omega = av_a.$$

From P2.16, using $\tau = 0.1s$ we have $c = 1/\tau = 10$. Additionally, $\tilde{\omega} = 5000rpm = 83.3rps$ is achieved when $v_a = 12V$ (note that we are using RPS, seconds and volts here). By this we can calculate a via

$$\tilde{\omega} = \frac{av_a}{c} = \frac{a}{c} \times 12,$$

such that $a = 69$.

Taking Laplace transform of the above ODE with zero initial condition yields

$$(s + c)\Omega(s) = aV_a(s),$$

such that the open-loop transfer function is

$$G(s) = \frac{\Omega(s)}{V_a(s)} = \frac{a}{s+c} = \frac{69}{s+10}.$$

Let the PI controller be

$$K(s) = K_p + \frac{K_i}{s},$$

so that

$$V_a(s) = K(s)E(s) = K(s)[\bar{\Omega}(s) - \Omega(s)]$$

Substitute the above into the system's equation and we find

$$(1 + GK)\Omega(s) = GK\bar{\Omega}(s), \tag{1}$$

and so the closed-loop transfer function is

$$H(s) = \frac{GK}{1 + GK} = \frac{aK_p s + aK_i}{s^2 + (aK_p + c)s + aK_i}$$

To achieve asymptotic tracking, first we need $S(s) = (1 + GK)^{-1}$ to be asymptotically stable. That means $S(s)$ has to have roots on open left half of the complex plane. The criterion for this can be found as

$$4aK_i > 0 \ \& \ (aK_p + c) > 0. \tag{2}$$

Once the above criterion has been met, GK must have a pole at $s = jw$ where w is the frequency of the reference signal, which is zero in this case since we are tracking a constant reference. This is already satisfied since

$$GK = \frac{aK_p s + aK_i}{s(s + c)}$$

does indeed have a pole at $s = 0$.

Now go back to criterion (2). Using the numeric values we have found for a and c, (2) becomes

$$276K_i > 0 \ \& \ (69K_p + 10) > 0.$$

The pair $K_p = 1$, $K_i = 1$ satisfies this criterion. Our PI controller now takes the form

$$K = \frac{s + 1}{s}.$$

The closed-loop transfer function can be expressed as

$$H(s) = \frac{GK}{1 + GK} = \frac{69(s + 1)}{s^2 + 79s + 69}.$$

Now we will find the tracking error when using this controller to track the reference signal

$$\bar{\omega} = 2500 + 100\cos(2\pi t)RPM = 41.7 + 1.7\cos(2\pi t)RPS$$

Taking Laplace transform of the above yields

$$\bar{\Omega}(s) = \frac{41.7}{s} + 1.7\frac{s}{s + 4\pi^2}$$

Substitute the above into (1) to find the response

$$\Omega(s) = H(s)(\frac{41.7}{s}) + 1.7H(s)\frac{s}{s + 4\pi^2}. \tag{3}$$

The steady-state value of the response ω_{ss} can now be seen as the sum

$$\omega_{ss} = \omega_{ss1} + \omega_{sss2},$$

where ω_{ss1} is the steady-state response due to the constant part of $\bar{\omega}$, which can be found by the final value theorem, and ω_{sss2} is the sinusoidal steady-state response due to the sinusoidal part of $\bar{\omega}$, which can be found by frequency response.

Apply the final value theorem to the first term in (3) to find that

$$\omega_{ss1} = \lim_{s\to 0} sH(s)\frac{41.7}{s} = 41.7.$$

The sinusoidal steady-state response can be found by frequency response as

$$\omega_{sss2} = 1.7|H(jw)|\cos(2\pi t + \angle H(jw)),$$

where $w = 2\pi$ is the frequency of the sinusoidal wave in $\bar{\omega}$.

Using the numeric values we have so far,

$$H(jw) = H(j2\pi) = \frac{10 - j}{12},$$

and so

$$|H(jw)| = 0.83,$$

and

$$\angle H(jw) = -\text{atan}0.1.$$

Then

$$\omega_{sss2} = 1.4\cos(2\pi t - \text{atan}0.1).$$

The steady-state tracking error can be found as

$$e_{ss} = \bar{\omega} - \omega_{ss} = 1.7\cos(2\pi t) - 1.4\cos(2\pi t - \text{atan}0.1).$$