MAE 143B - Homework 5

3.21

Taking Laplace transforms with zero initial conditions, we get

$$H_1(s) = \frac{\Omega_2(s)}{T(s)} = \frac{r_1\Omega_1(s)}{r_2 T(s)} = \frac{r_1 r_2}{(J_1 r_1^2 + J_2 r_2^2)s}$$

and

$$H_2(s) = \frac{\Theta_2(s)}{T(s)} = \frac{1}{s} H_1(s) = \frac{r_1 r_2}{(J_1 r_1^2 + J_2 r_2^2)s^2}.$$

Both transfer functions have poles on the imaginary axis, implying that they are not asymptotically stable.

3.23

Notice that both transfer functions $H_1(s)$ and $H_2(s)$ have poles on the imaginary axis, meaning that we can not use the frequency response as in Section 3.8 to compute the steady-state responses based on these transfer functions. However, we can compute the outputs by simply taking inverse Laplace transforms and finding the steady-state components of the solutions. For the remainder of the problem, we write $A := r_1 r_2/(J_1 r_1^2 + J_2 r_2^2)$.

For $\tau = \tilde{\tau}$ at $t \geq 0$, we have $T(s) = \tilde{\tau}/s$ and

$$\begin{aligned} \omega_2(t) &= \mathcal{L}^{-1}\{H_1(s)T(s)\} = \mathcal{L}^{-1}\left\{\frac{A\tilde{\tau}}{s^2}\right\} = A\tilde{\tau}t = \omega_{2,ss}(t), \\ \theta_2(t) &= \mathcal{L}^{-1}\{H_2(s)T(s)\} = \mathcal{L}^{-1}\left\{\frac{A\tilde{\tau}}{s^3}\right\} = \frac{A\tilde{\tau}}{2}t^2 = \theta_{2,ss}(t). \end{aligned}$$

For $\tau = \tilde{\tau}\cos(\omega t)$ at $t \geq 0$, we have $T(s) = \tilde{\tau}s/(s^2+\omega^2)$ and

$$\begin{aligned} \omega_2(t) &= \mathcal{L}^{-1}\{H_1(s)T(s)\} = \mathcal{L}^{-1}\left\{\frac{A\tilde{\tau}}{s^2+\omega^2}\right\} = \frac{A\tilde{\tau}}{\omega}\cos(\omega t) = \omega_{2,ss}(t), \\ \theta_2(t) &= \mathcal{L}^{-1}\{H_2(s)T(s)\} = \mathcal{L}^{-1}\left\{\frac{A\tilde{\tau}}{s(s^2+\omega^2)}\right\} = \frac{A\tilde{\tau}}{\omega^2} - \frac{A\tilde{\tau}}{2\omega^2}e^{j\omega t} + \frac{A\tilde{\tau}}{2\omega^2}e^{-j\omega t} = \frac{A\tilde{\tau}}{\omega^2}(1-\sin(\omega t)) = \theta_{2,ss}(t). \end{aligned}$$

4.12

For the closed-loop system to be internally stable, we need to choose a controller K such that the sensitivity transfer function S is asymptotically stable and GK does not have pole-zero cancellations in the closed right-half plane. We have

$$G(s) = \frac{\Omega_2(s)}{T(s)} = \frac{r_1}{r_2} \cdot \frac{\Omega_1(s)}{T(s)} = \frac{r_1 r_2/J_r}{s + b_r/J_r},$$

which given positive parameters has one stable pole and no zeros, such that internal stability reduces to asymptotic stability of the sensitivity transfer function

$$S(s) = \frac{1}{1+GK} = \frac{s + b_r/J_r}{s + (b_r + r_1 r_2 K)/J_r},$$

which has a single pole at

$$s_0 = -\frac{b_r + r_1 r_2 K}{J_r},$$

such that the closed-loop system is internally stable for any $K > -b_r/(r_1 r_2)$. However, this controller is not going to be able to asymptotically track a constant reference $\omega_2(t) = \bar{\omega}$, $t \geq 0$ because GK does not have a pole at the origin (see Lemma 4.1 in the notes).

4.13

As in the previous problem, we only need to worry about asymptotic stability of the sensitivity transfer function to guarantee internal stability of the closed-loop system. We have

$$G(s) = \frac{\Theta_2(s)}{T(s)} = \frac{r_1 r_2/J_r}{s(s + b_r/J_r)}, \qquad S(s) = \frac{1}{1+GK} = \frac{s(s + b_r/J_r)}{s(s + b_r/J_r) + (r_1 r_2 K)/J_r},$$

resulting in poles of the sensitivity transfer function S at

$$s_{1/2} = -\frac{b_r}{2J_r} \pm \sqrt{\frac{b_r^2}{4J_r^2} - \frac{r_1 r_2}{J_r} K} = \frac{-b_r \pm \sqrt{b_r^2 - 4 r_1 r_2 J_r K}}{2J_r},$$

which are both in the open left-half plane for any $K > 0$. Given that GK has a pole at the origin through G, any such controller achieves asymptotic tracking of a constant reference $\omega_2(t) = \bar{\omega}$, $t \geq 0$.

Additional problem

From the previous homework assignment, we have

$$\frac{\Omega(s)}{V_a(s)} = \frac{69}{s+10},$$

such that the transfer function from the voltage input $v_a(t)$ to the angular position $\theta(t)$ is

$$G(s) = \frac{\Theta(s)}{V_a(s)} = \frac{\Omega(s)}{sV_a(s)} = \frac{69}{s(s+10)}.$$

For asymptotic tracking of constant position references, we require GK to have a pole at the origin and the sensitivity transfer function S to be asymptotically stable. Given that $G(s)$ has a pole at the origin, we only need to worry about asymptotic stability of the sensitivity transfer function provided our controller $K(s)$ has no zeros at the origin. Using a proportional controller $K(s) = K$, we get the sensitivity transfer function

$$S(s) = \frac{s(s+10)}{s(s+10) + 69K},$$

with poles at

$$s_{1/2} = -5 \pm \sqrt{25 - 69K},$$

which are in the open left-half plane provided any $K > 0$. Hence, we can use any such controller to achieve asymptotic tracking of constant position references. If $\theta(t) = 2\pi$ rad, $t \geq 0$, we get the steady-state error

$$e_{ss}(t) = |S(j0)| \cos(\angle S(j0)) 2\pi \text{ rad} = \frac{1}{69K} \cos(0) 2\pi \text{ rad} = \frac{2\pi}{69K} \text{ rad}.$$