MAE 143B - Homework 6 Solutions

P5.1

b)

Let x_1 and x_2 be the outputs of the first and second integrators, respectively. Next, write the ODE at the input of each integrator. We have the following.

$$\begin{aligned}\dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= x_1\end{aligned}$$

Then write the equation for the output y.

$$y = 2\dot{x}_1 + x_2 = 2(x_2 + u) + x_2 = 3x_2 + 2u$$

Finally put the above equations in matrix form to obtain the state-space representation of the system.

$$\begin{aligned}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 \\ 0\end{bmatrix} u = Ax + Bu \\ y &= \begin{bmatrix}0 & 3\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + 2u = Cx + Du\end{aligned}$$

The transfer function can be calculated as

$$\begin{aligned}G(s) = C(sI - A)^{-1}B + D &= \begin{bmatrix}0 & 3\end{bmatrix}\begin{bmatrix}s & -1 \\ -1 & s\end{bmatrix}^{-1}\begin{bmatrix}1 \\ 0\end{bmatrix} + 2 \\ &= \frac{1}{s^2 - 1}\begin{bmatrix}0 & 3\end{bmatrix}\begin{bmatrix}s & 1 \\ 1 & s\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} + 2 \\ &= \frac{2s^2 + 1}{s^2 - 1}\end{aligned}$$

d)

Redraw the block diagram and name the state variables as in Figure 1.

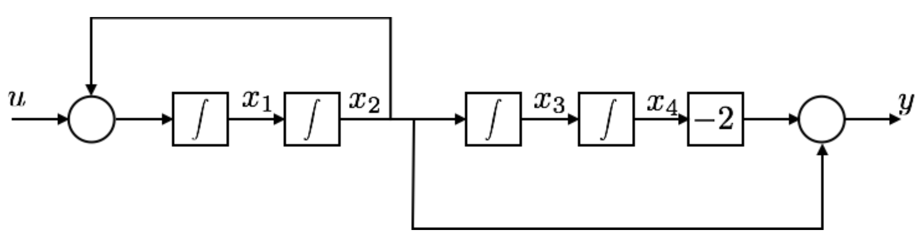

Figure 1: Block diagram P5.1d redrawn.

We can see the cascade system: let the transfer function from u to x_2 be $G_1(s)$ and from x_2 to y be $G_2(s)$. Then the transfer function of the complete system is $G(s) = G_1(s)G_2(s)$.

Writing the ODE at each of the inputs of the integrators, we have the following.

$$\begin{aligned}\dot{x}_1 &= x_2 + u\\ \dot{x}_2 &= x_1\\ \dot{x}_3 &= x_2\\ \dot{x}_4 &= x_3\end{aligned}$$

and the output is $y = x_2 - 2x_4$. Put the equations in state-space form and we have

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} &= \begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}u\\ y &= \begin{bmatrix}0 & 1 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}\end{aligned}$$

To find $G_1(s)$, notice that $\ddot{x}_2 = \dot{x}_1 = x_2 + u$. Taking Laplace transform of this ODE with zero initial conditions we have

$$G_1(s) = \frac{1}{s^2 - 1}.$$

On the other hand, take double derivative of the output $y = x_2 - 2x_4$ and we have

$$\ddot{y} = \ddot{x}_2 - 2\ddot{x}_4 = \ddot{x}_2 - 2x_2.$$

Taking Laplace transform of the above with initial conditions yields

$$s^2 Y = (s^2 - 2)X_2,$$

so then

$$G_2(s) = \frac{s^2 - 2}{s^2}.$$

The transfer function of the whole system is

$$G(s) = G_1(s)G_2(s) = \frac{s^2 - 2}{s^2(s^2 - 1)}.$$

P5.14

Let the state be $x = \omega$, the input $u = v_a$ and output $y = \omega = x$. The ODE can be rewritten as follows in terms of the new variables.

$$J\dot{x} + \left(b + \frac{K_t K_e}{R_a}\right)x = \frac{K_t}{R_a}u$$

And so the state-space form of the system is

$$\begin{aligned}\dot{x} &= -\frac{b + \frac{K_t K_e}{R_a}}{J}x + \frac{K_t}{JR_a}u\\ y &= x + 0u\end{aligned}$$

The block diagram of the system is Figure 2.

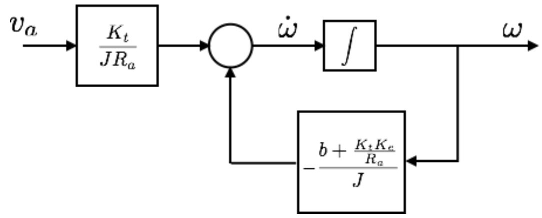

Figure 2: Block diagram for P5.14.

P5.15

Let the states $x_1 = \dot{\theta} = \omega$ and $x_2 = \theta$, so that $\dot{x}_2 = x_1$. Let the input $u = v_a$ and output $y = \theta = x_2$. For shorthand notation, let

$$a = \frac{b + \frac{K_tK_e}{R_a}}{J}$$

and

$$c = \frac{K_t}{JR_a}.$$

The ODE can be rewritten as

$$\dot{x}_1 = -ax_1 + cu.$$

The state-space representation of the system can then be expressed as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-a & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}c\\ 0\end{bmatrix}u = Ax + Bu$$
$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + 0u = Cx + Du$$

Block diagram for the system is in Figure 3

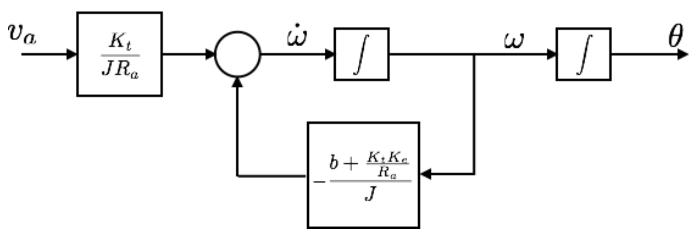

Figure 3: Block diagram for P5.15.

P5.16

From $\tau = K_ti_a$ we have $i_a = \tau/K_t$. Substitute this into $v_a = R_ai_a + K_e\omega$ we have

$$\tau = \frac{K_t}{R_a}(v_a - K_e\omega).$$

Let the state $x = \omega$, the input $u = v_a$ and output $y = \tau$. The above equation becomes

$$y = -\frac{K_tK_e}{R_a}x + \frac{K_t}{R_a}u$$

The state-space form of the system is

$$\dot{x} = -\frac{b + \frac{K_t K_e}{R_a}}{J} x + \frac{K_t}{J R_a} u$$
$$y = -\frac{K_t K_e}{R_a} x + \frac{K_t}{R_a} u$$

Block diagram for the system is in Figure 4

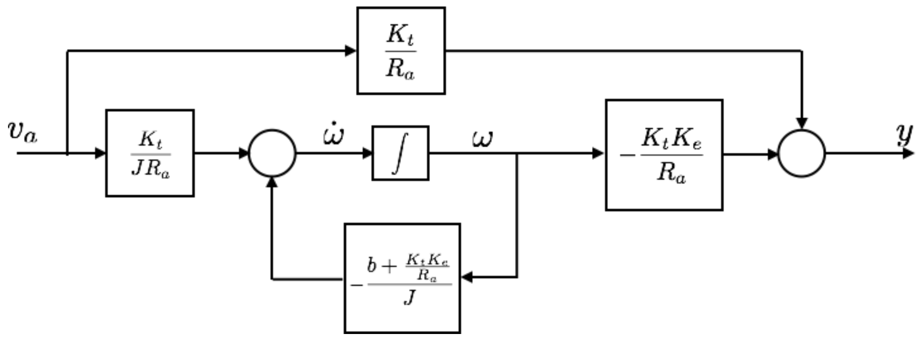

Figure 4: Block diagram for P5.16.

With a constant voltage, we have

$$v_a = R_a i_a + K_e \omega = R_a \frac{\tau}{K_t} + K_e \omega,$$

or

$$\tau = \frac{K_t}{R_a}(v_a - K_e \omega).$$

And so the torque is at its maximum when $\omega = 0$.

P5.24

Let

$$\dot{x} = f(x) = r_0 x - \frac{r_0}{k} x^2.$$

To find the equilibrium points, let $f(x) = 0$. Then we have two equilibria: $x = 0$ and $x = k$. Linearization about $x = 0$:

$$A_1 = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} = r_0.$$

With the assumption that all constant parameters are greater than zero, A_1 is not Hurwitz because its only eigenvalue is $r_0 > 0$. Therefore, $x = 0$ is an unstable equilibrium point.

Linearization about $x = k$:

$$A_2 = \left.\frac{\partial f(x)}{\partial x}\right|_{x=k} = -r_0.$$

This time A_2 is Hurwitz. So then $x = k$ is an asymptotically stable equilibrium point.

The block diagram is in Figure 5.

Simulink model for this problem is in Figure 6.

The simulation results with different initial conditions are in Figure 7.

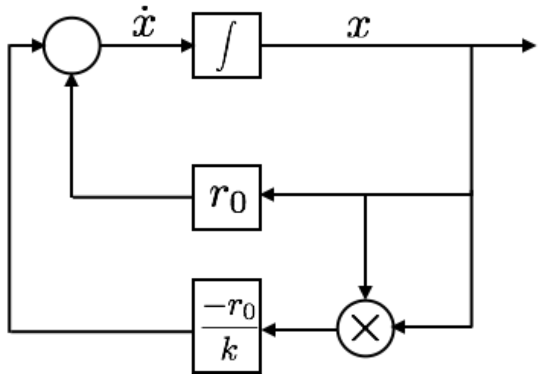

Figure 5: Block diagram for P5.24.

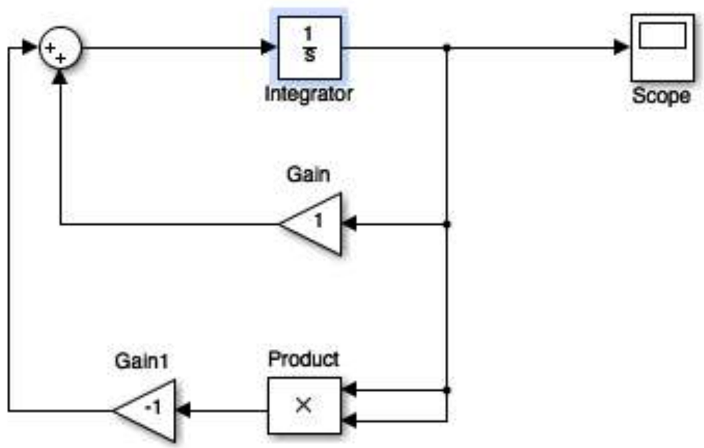

Figure 6: Simulink model for P5.24. The initial condition for the integrator block can be changed in its setting by double-clicking on it.

P5.25

Let the states be

$$\mathcal{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The state equation is then

$$f(\mathcal{X}) = \dot{\mathcal{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} rx - axy \\ eaxy - my \end{bmatrix}$$

To find the equilibrium points, let $f(\mathcal{X}) = 0$. There are two of them

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{m}{ea} \\ \frac{r}{a} \end{bmatrix}$$

Linearization about $(0; 0)$

$$A_1 = \left.\frac{\partial f(\mathcal{X})}{\partial \mathcal{X}}\right|_{\mathcal{X}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} r & 0 \\ 0 & -m \end{bmatrix}$$

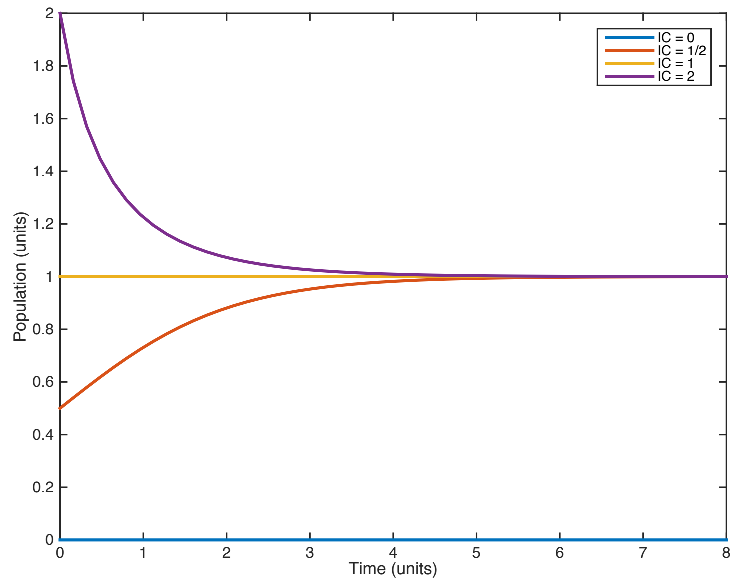

Figure 7: Simulation results for P5.24 with various initial conditions.

A_1 is not Hurwitz because its eigenvalues are $r > 0$ and $-m$. So the equilibrium point $(0; 0)$ is unstable.

Linearization about $(\frac{m}{ea}; \frac{r}{a})$

$$A_2 = \frac{\partial f(\mathcal{X})}{\partial \mathcal{X}}\bigg|_{\mathcal{X}=\begin{bmatrix}\frac{m}{ea}\\ \frac{r}{a}\end{bmatrix}} = \begin{bmatrix} r & -m/e \\ er & 0 \end{bmatrix}$$

The eigenvalues of A_2 can be by solving its characteristic equation, $|sI - A_2| = 0$. There are two eigenvalues at $\pm jmr$. Lemma 5.1 is inconclusive when the eigenvalues of A_2 are on the $j\omega$ axis. This equilibrium point is called a *stable focus.*

The block diagram is in Figure 8.

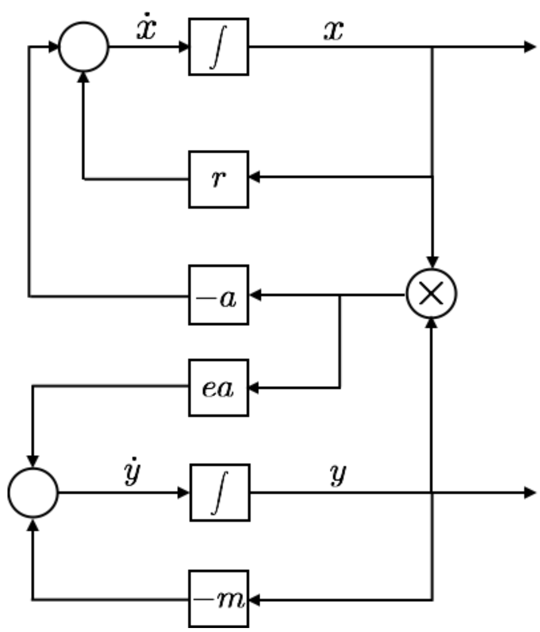

Figure 8: Block diagram for P5.25.

Simulink model for this problem is in Figure 9. The values of the gain blocks are found by using the constants given in the problem.

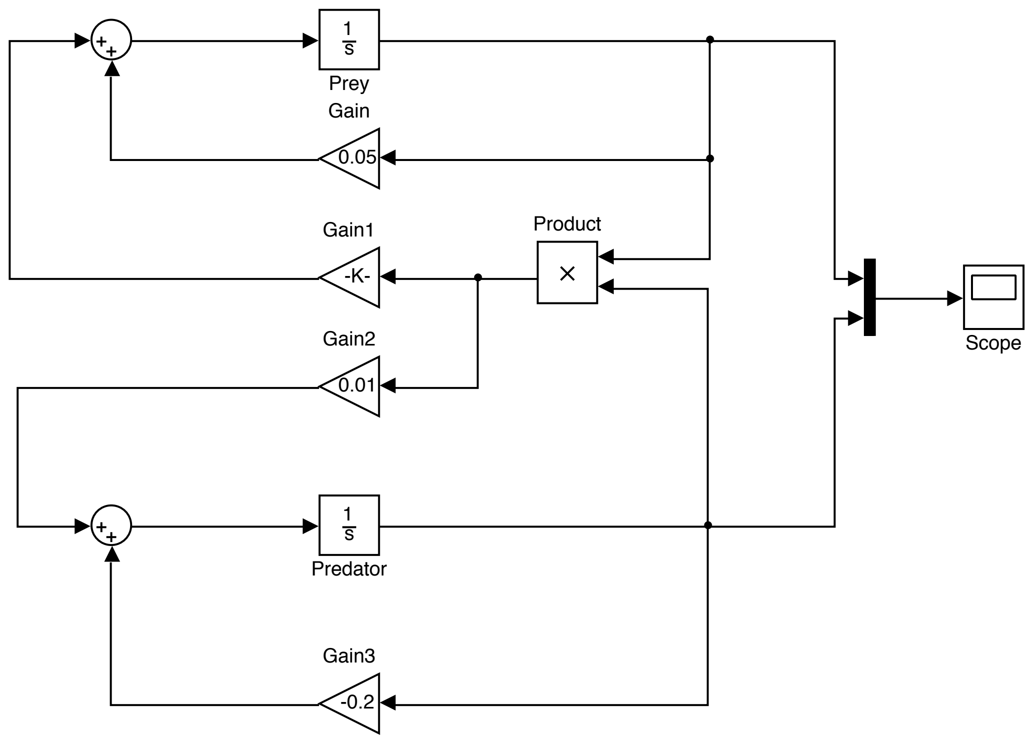

Figure 9: Simulink model for P5.25. The initial condition for the integrator block can be changed in its setting by double-clicking on it.

The simulation results are in Figures 10, 11, 12, 13.

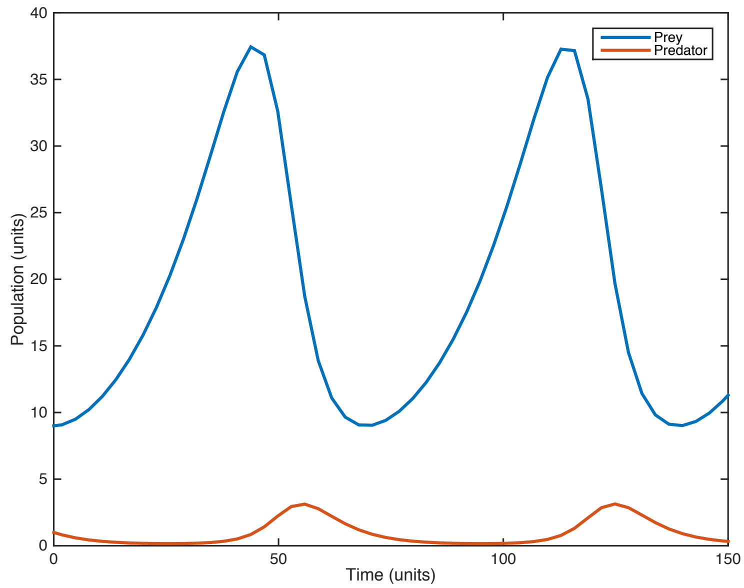

Figure 10: Simulation results for P5.25. Initial conditions are $(9, 1)$.

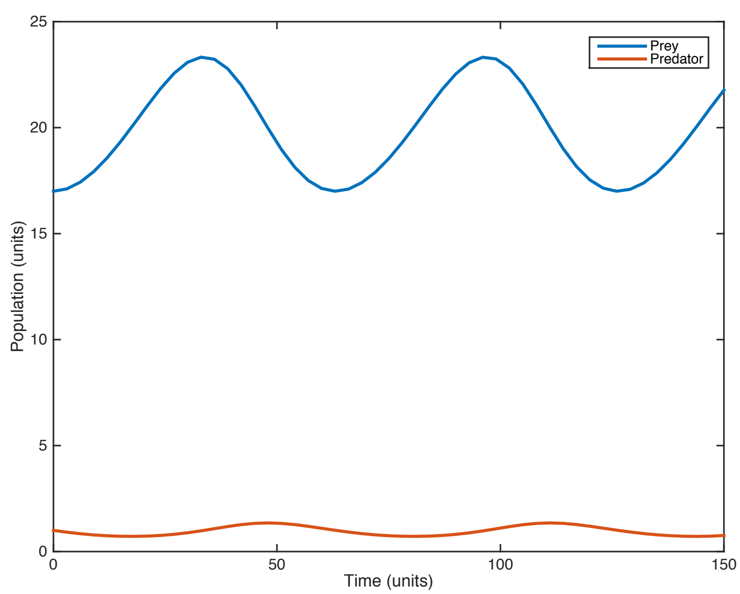

Figure 11: Simulation results for P5.25. Initial conditions are $(17, 1)$.

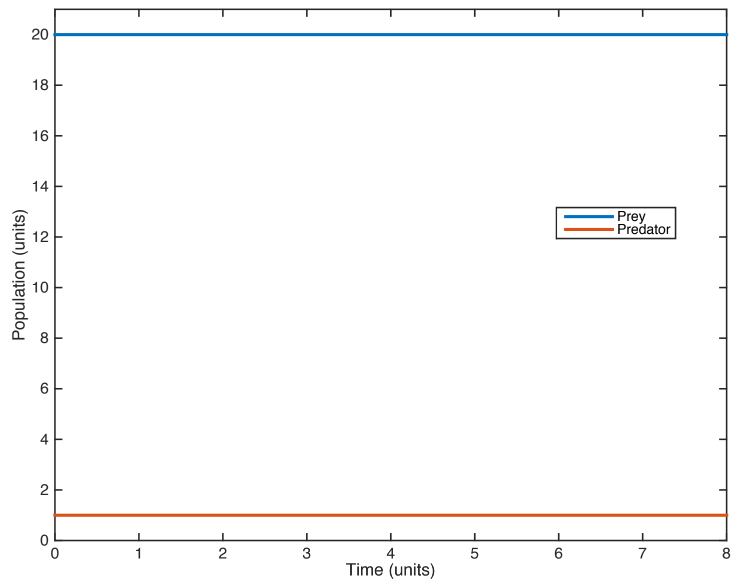

Figure 12: Simulation results for P5.25. Initial conditions are $(20, 1)$, which is one of the equilibrium point.

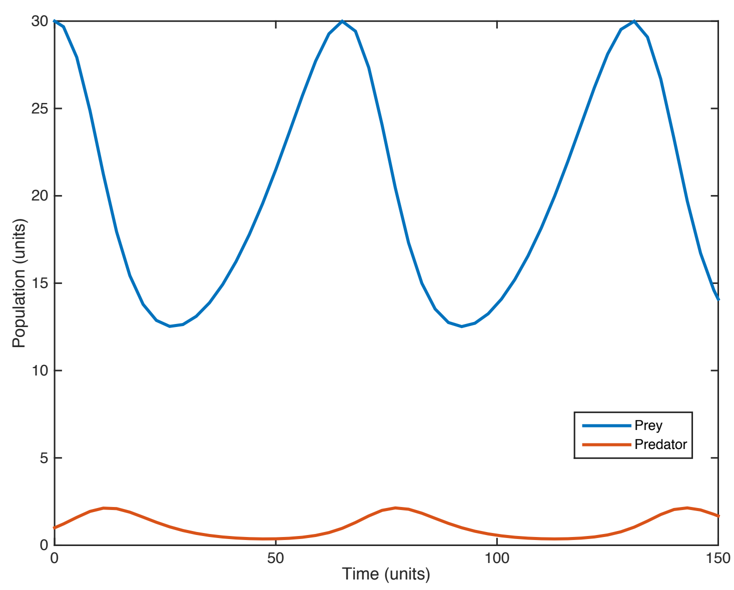

Figure 13: Simulation results for P5.25. Initial conditions are $(30, 1)$.