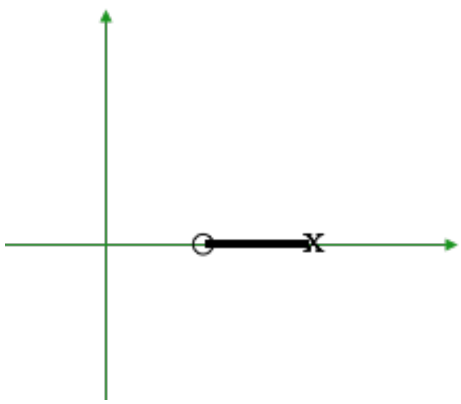
MAE 143B - Homework 8 Solutions

P6.14

b)

With this system, the root locus simply starts at the pole and ends at the zero. Sketches by hand and `matlab` are in Figure 1. In `matlab`, use `zpk` to build the system with desired poles and zeros and gain, then use `rlocus` to plot its root locus. For example,

```
sys = zpk([1],[2],1);
rlocus(sys)
```



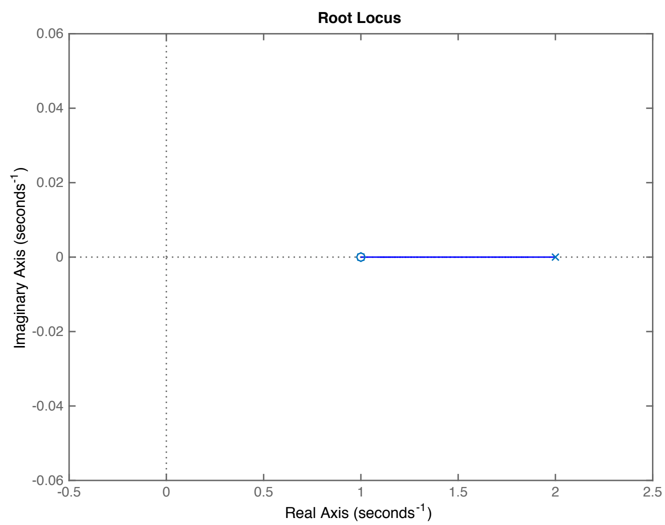


Figure 1: Root locus for P6.14(b): by hand (left) and `matlab` (right).

d)

The portion from -2 to 0 of the real axis belongs to the root locus. Since we have 3 poles and 1 zeros, there are 2 asymptotes. The intersection of the asymptotes is

$$c = \frac{(j - j - 2) - 0}{3 - 1} = -1,$$

and the angles that they make with the real axis are

$$\phi_1 = \frac{\pi}{2} \text{ and } \phi_2 = \frac{\pi + 2\pi}{2} = \frac{3\pi}{2}.$$

Root locus sketches by hand and `matlab` are in Figure 2.

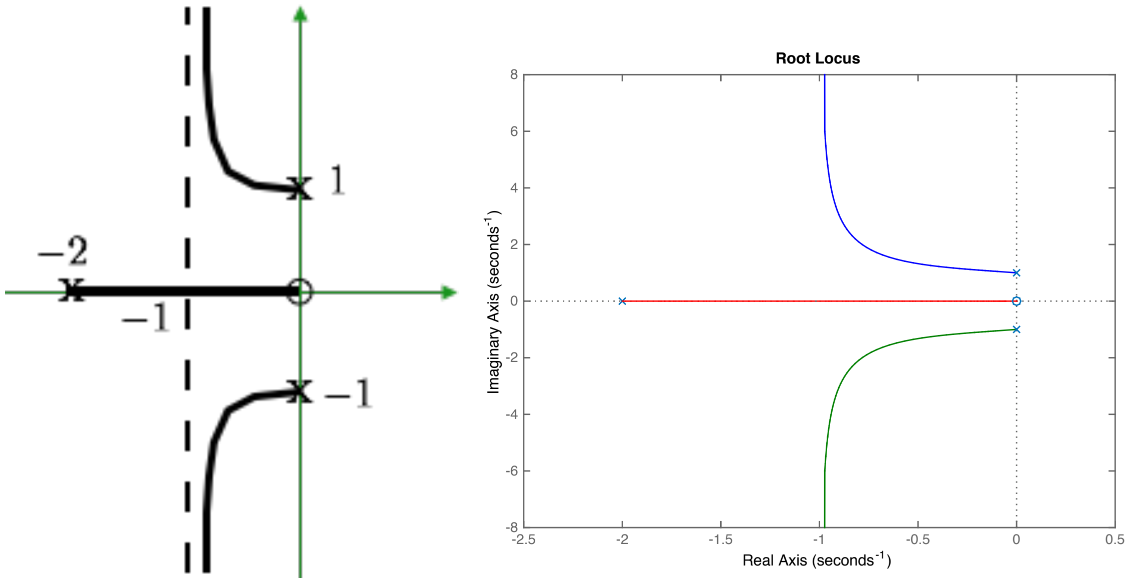


Figure 2: Root locus for P6.14(d): by hand (left) and `matlab` (right).

e)

The portion from -1 to 1 of the real axis belongs to the root locus. There are no zeros and 4 poles, which means there are 4 asymptotes. Their intersection is

$$c = \frac{(1 - 1 + j - j) - 0}{4 - 0} = 0,$$

and the angles they make with the real axis are

$$\phi_1 = \frac{\pi}{4}, \phi_2 = \frac{3\pi}{4}, \phi_3 = \frac{5\pi}{4}, \phi_4 = \frac{7\pi}{4}.$$

Root locus sketches by hand and `matlab` are in Figure 3.

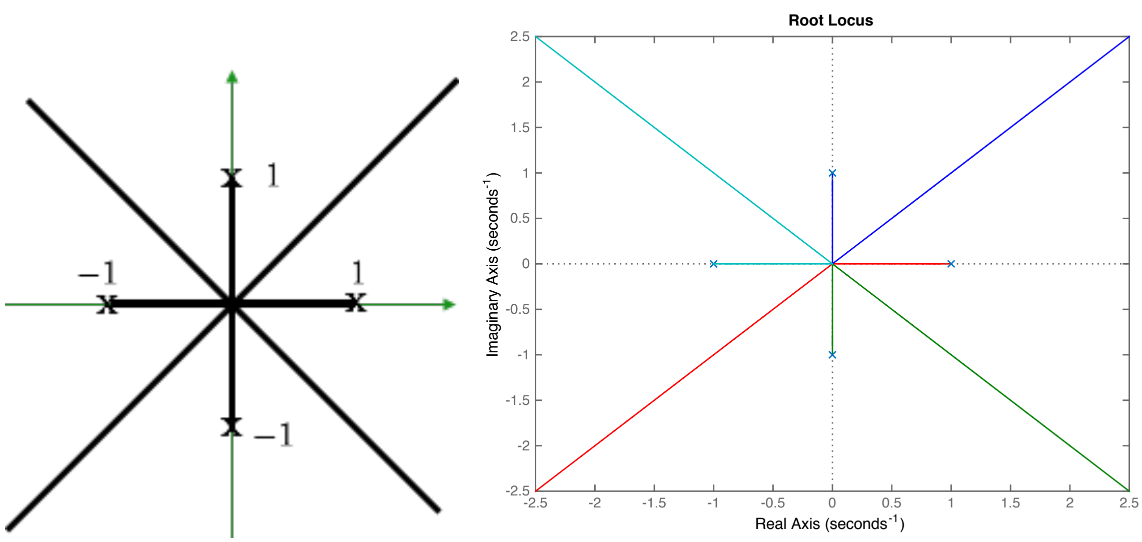


Figure 3: Root locus for P6.14(e): by hand (left) and `matlab` (right).

P6.15

b)

Using `sisotool` in `matlab`, one possible controller is

$$C(s) = 1.8 \times \frac{s+6}{s-5}$$

which produces the root locus in Figure 4. Note that the gain can be chosen so that the closed-loop system is stable. The controller is not asymptotically stable because it has a pole at 5.

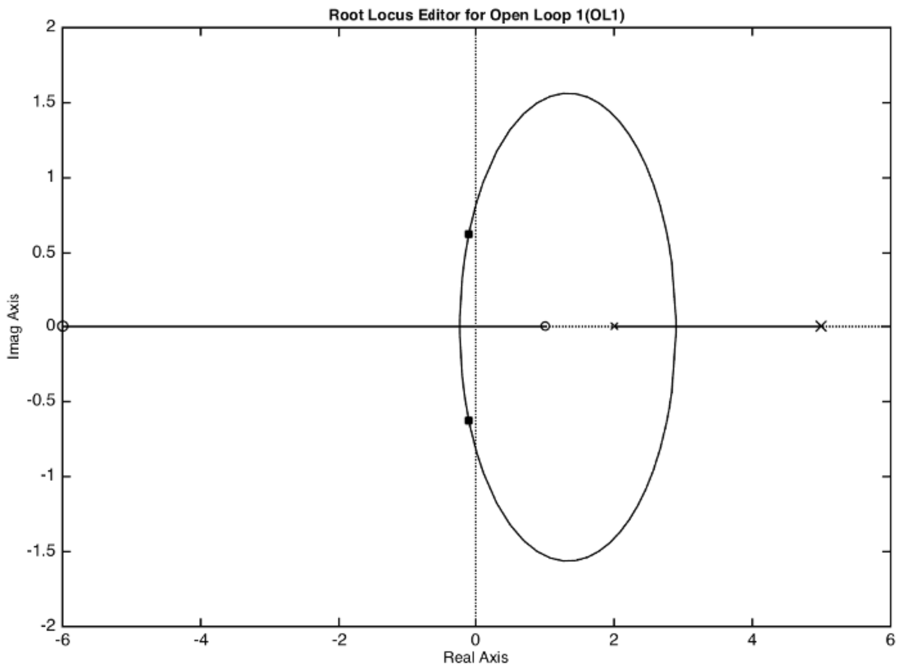


Figure 4: Root locus for P6.15(b).

d)

Referring to Part (d) of P6.14, we can see that the system is already stable. Therefore any proportional control of gain $K > 0$ will suffice.

e)

A possible controller has a gain of 0.8 and complex zeros at $0.6 \pm j1.3$ and a real zero at -1 so that a pole/zero cancelation happens, and complex poles at $-6.5 \pm j2$ and a real pole at -19. The resulting root locus is in Figure 5. Again, the gain can be select differently, making sure that the closed-loop poles are in the left half plane.

P6.21

Let the states

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} r \\ \dot{r} \\ \omega \end{bmatrix},$$

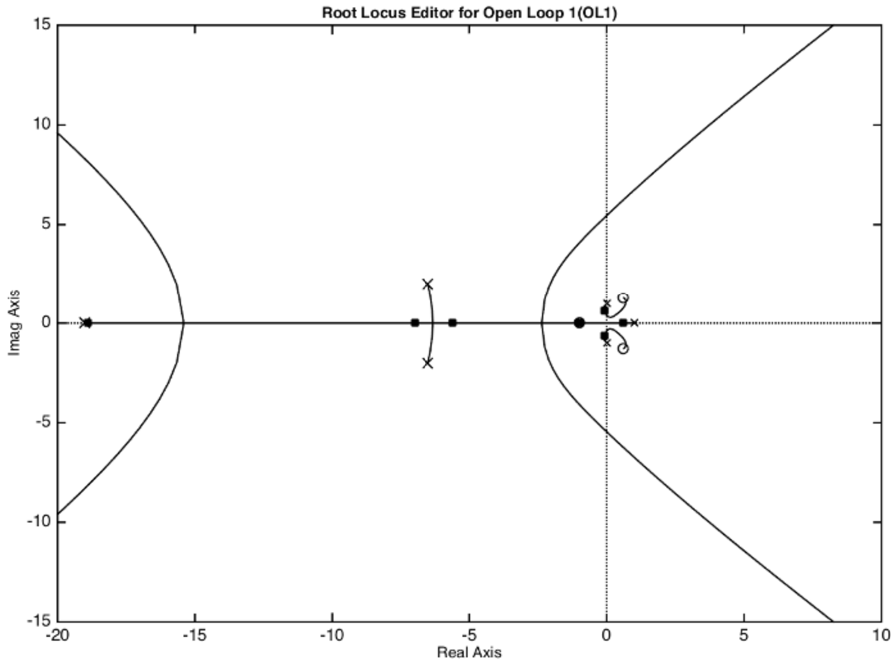


Figure 5: Root locus for P6.15(e).

and the output $y = r = w_1$. Rearrange the differential equations to get the following state-space form.

$$\begin{aligned}
\dot{w}_1 &= w_2 = f_1(w, u) \\
\dot{w}_2 &= w_1\omega^2 - \frac{GM}{w_1^2} = f_2(w, u) \\
\dot{w}_3 &= \frac{\frac{u}{m} - 2w_2\omega}{w_1} = f_3(w, u) \\
y &= w_1 = h(w, u)
\end{aligned}$$

The block diagram is in Figure 6.

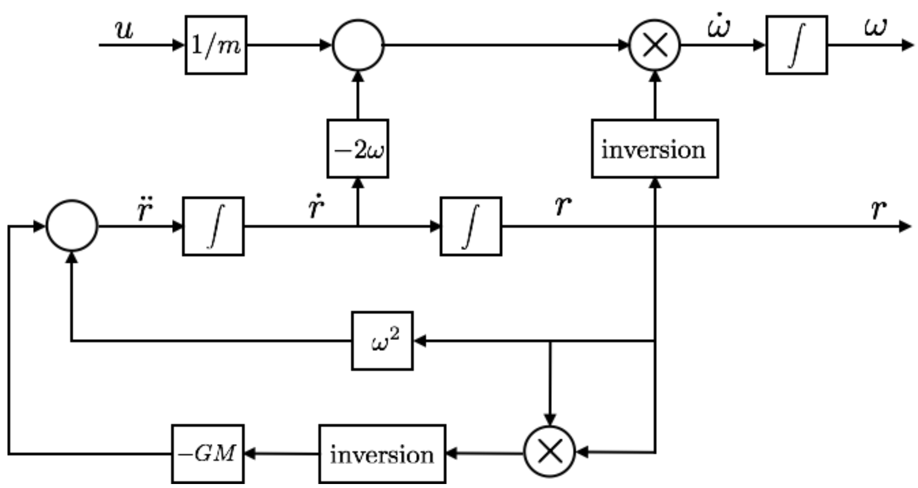


Figure 6: Block diagram for P6.21.

P6.22

To find the equilibrium points, from the state-space found in P6.21, set $\dot{w} = 0$.

$$\begin{aligned}
&\dot{w}_1 = 0 \Rightarrow \dot{r} = 0 \\
&\dot{w}_2 = 0 \Rightarrow w_1\omega^2 - \frac{GM}{w_1^2} = 0 : \text{ true if } w_1 = r = R, \omega = \Omega \text{ and } \Omega^2 R^3 = GM \\
&\dot{w}_3 = 0 \Rightarrow u = 0
\end{aligned}$$

And so, if $w_1 = r = R, \omega = \Omega$ and $\Omega^2 R^3 = GM$, then $u = \dot{r} = 0, r = R, \omega = \Omega$ is the equilibrium point of the non-linear system.

Define the new states as follows.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r - R \\ \dot{r} \\ R\omega - R\Omega \end{bmatrix},$$

and the new output $\tilde{y} = r - R$.

Then the equilibrium above becomes

$$\begin{bmatrix} u \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Linearization of the system around the above equilibrium point is

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 3\Omega^2 & 0 & 2\Omega \\ 0 & -2\Omega & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} u = Ax + Bu \\
\tilde{y} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x = Cx + Du.
\end{aligned}$$

Using symbolic `matlab`, the transfer function can be calculated as

$$G(s) = C(sI - A)^{-1}B + D = \frac{2\Omega}{ms(s^2 + \Omega^2)}.$$

For the period $T = 11h = 39600s$, the angular velocity is

$$\Omega = \frac{2\pi}{T} = 1.59 \times 10^{-4} rad/s.$$

With $m = 1600kg$, the transfer function $G(s)$ becomes

$$G(s) = \frac{2 \times 10^{-7}}{s(s^2 + 2.5 \times 10^{-8})},$$

that has no zeros and 2 poles at 0 and $\pm j1.59 \times 10^{-4}$.

Using `sisotool` in `matlab`, a possible controller can be

$$C(s) = 0.00012 \times \frac{(s + 0.0002)(s + 0.0004)}{(s + 0.004)^2},$$

which produces the root locus in Figure 7. Note that with the above poles and zeros of the controller, a range of gain K could be selected such that the closed-loop poles are stable.

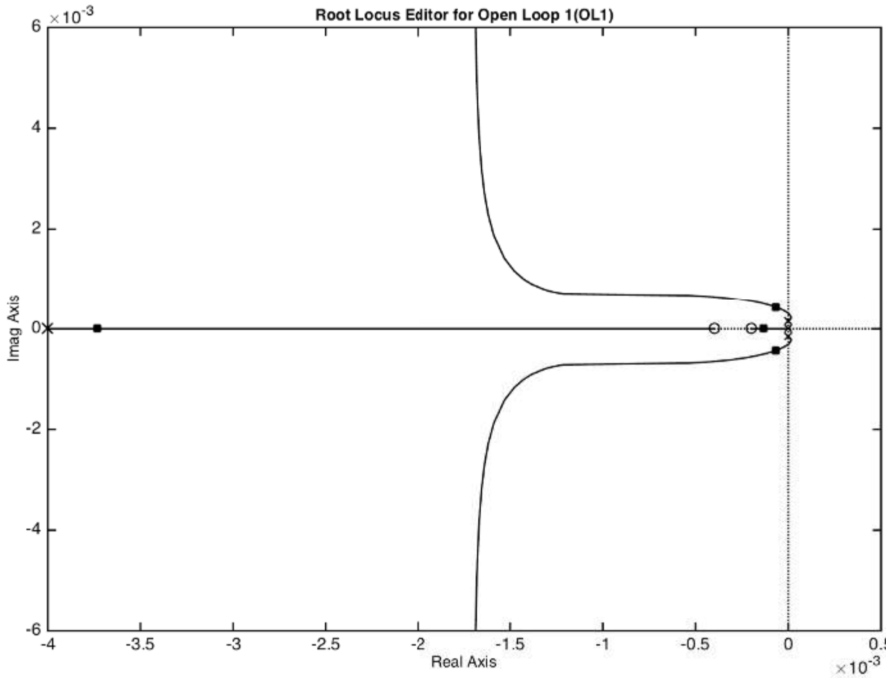


Figure 7: Root locus for P6.22.