MAE 143B - Homework 9

7.1

a)

We have stable first-order poles at $p_1 = -1$ and $p_2 = -10$. For small values of ω, we recover the DC gain

$$\tilde{K} = \lim_{\omega \downarrow 0} |G(j\omega)| = \frac{1}{10} = -20\text{dB}.$$

Having this finite limit, our straight-line approximation for the magnitude plot must start with a horizontal line at -20dB. The resulting straight-line approximation and Bode plot are shown in Fig. 1, where the exact plot can be reconstructed in MATLAB using the commands `zpk` (you could also use `tf` or `ss`, whichever you prefer) and `bode`:

```
>> sys = zpk([],[-1 -10],1)

sys =

       1
  ------------
  (s+1) (s+10)

Continuous-time zero/pole/gain model.

>> bode(sys);
```

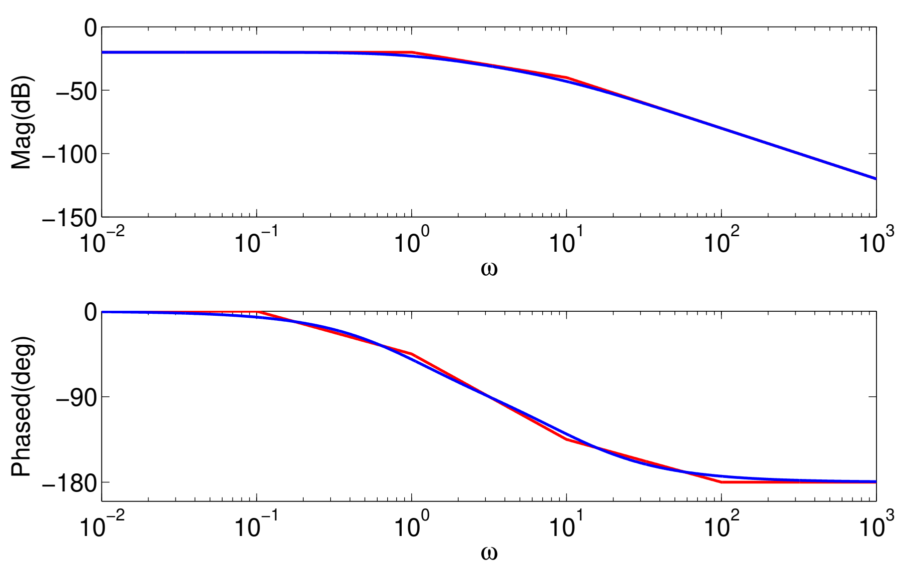

Figure 1: Bode plot for part a), straight-line approximation in red.

c)

We have a simple zero at $z_1 = -10$, stable second-order poles with $\omega_n = 1$ rad/s and $\zeta = 0.05$ and DC gain

$$\tilde{K} = \lim_{\omega \downarrow 0} |G(j\omega)| = 10 = 20\text{dB}.$$

Resulting straight-line approximation and Bode plot are shown in Fig. 2.

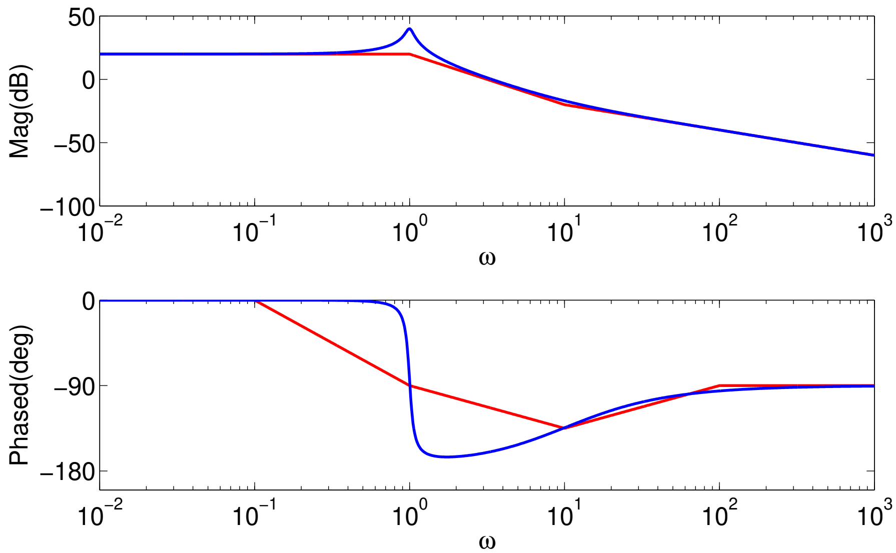

Figure 2: Bode plot for part c), straight-line approximation in red.

g)

In this case, we have a simple zero at $z_1 = -10$ and a pole of multiplicity 2 at $p_1 = -1$. The DC gain is

$$\tilde{K} = \lim_{\omega \downarrow 0} |G(j\omega)| = 10 = 20\text{dB}.$$

Resulting straight-line approximation and Bode plot are shown in Fig. 3 below.

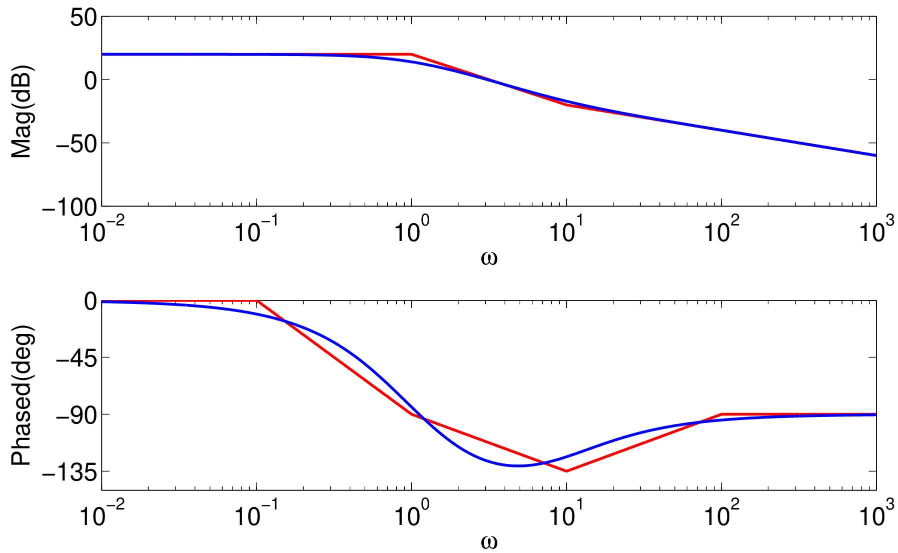

Figure 3: Bode plot for part g), straight-line approximation in red.

7.2

The three plots are shown in Figures 4-6. You can reconstruct these using the MATLAB command `nyquist`. None of the plots have encirclements of the point -1, implying stability under negative unit feedback by the Nyquist stability criterion.

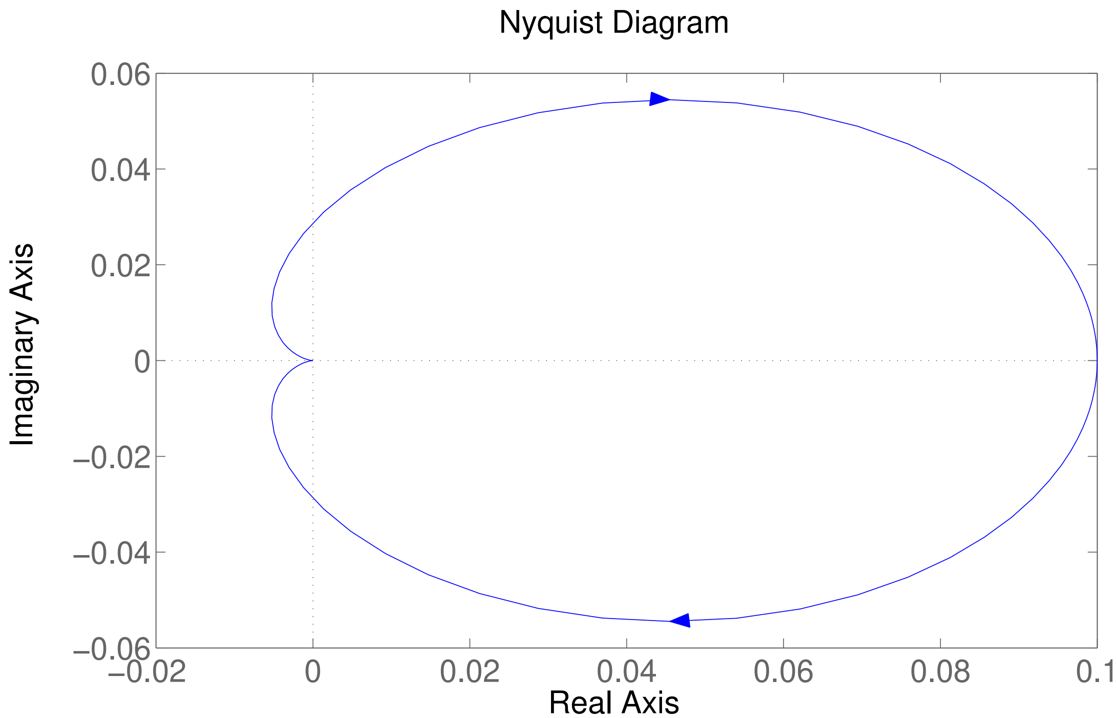

Figure 4: Polar plot for part a).

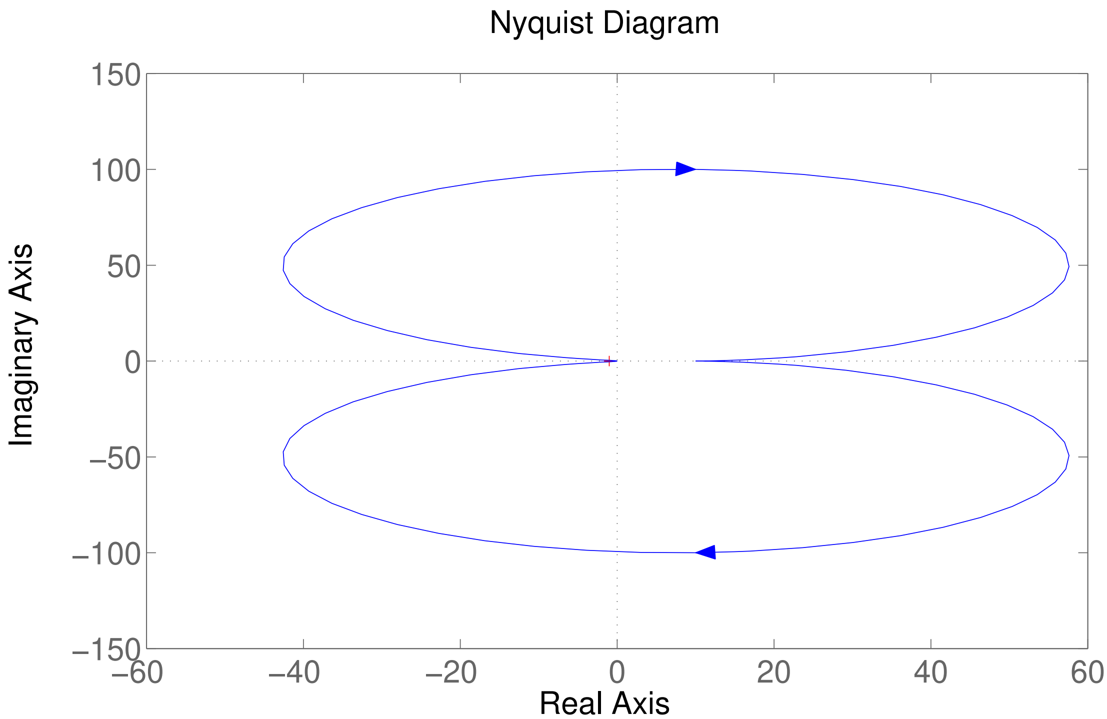

Figure 5: Polar plot for part c).

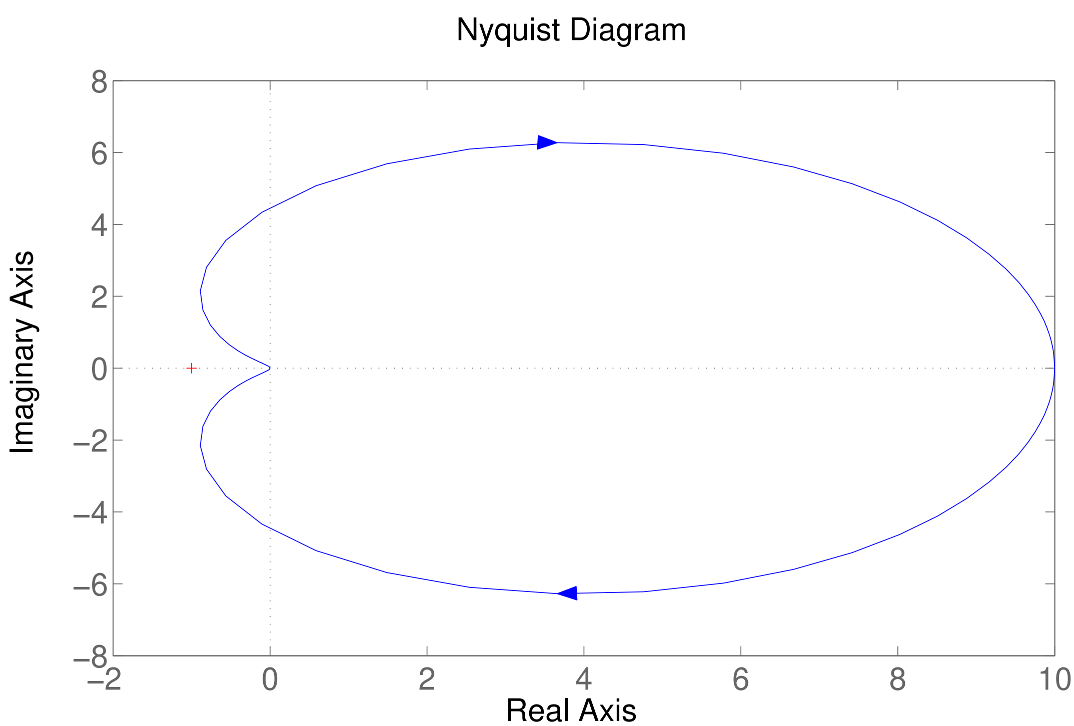

Figure 6: Polar plot for part g).

7.3

For the solutions, we assume all frequency axes are in rad/s instead Hz. Otherwise, factors of 2π would have to be added for all poles and zeros.

a)

We have a DC gain $\tilde{K} = -20\text{dB} = 0.1$, straight-line slope of 20 dB per decade on $\omega \in [1, 10)$ rad/s and zero slope elsewhere. For the transfer function to be minimum-phase, we thus need a zero at $z_1 = -1$ and a pole at $p_1 = -10$, resulting in the transfer function

$$G(s) = 0.1 \cdot \frac{s+1}{0.1s+1} = \frac{s+1}{s+10}$$

e)

We have a DC gain $\tilde{K} = 0\text{dB} = 1$. The straight-line slope is -20 dB per decade on $\omega \in [1, 10)$ rad/s, -40 dB on $\omega \geq 10$ rad/s and zero elsewhere. This behavior is achieved by the minimum-phase transfer function

$$G(s) = \frac{1}{(s+1)(0.1s+1)} = 10 \cdot \frac{1}{(s+1)(s+10)}.$$

j)

Again, the DC gain is $\tilde{K} = 0\text{dB} = 1$. The straight-line approximation has slope -40 dB per decade on $\omega \in [1, 10)$ rad/s, -60 dB on $\omega \geq 10$ rad/s and zero elsewhere. Thus, we need a simple pole at $p_1 = -10$ and second-order poles with $\omega_n = 1$ and $\zeta \approx 0.005$ (ζ needs to be very small for the peak of approximately 40dB). This yields the minimum-phase transfer function

$$G(s) = \frac{1}{(0.1s+1)(s^2+0.01s+1)} = 10 \cdot \frac{1}{(s+10)(s^2+0.01s+1)}$$

7.4

Our transfer function from 7.3 already match the given phase diagrams.

7.8

From P6.8, we have the transfer function

```
sys =

            -s^4 - 597.3 s^3 - 1.737e04 s^2
  ---------------------------------------------------
  s^4 + 597.3 s^3 + 1.737e04 s^2 + 1.867e05 s + 3.867e06
```

The Bode plot for this solution is displayed in Figure 7. As we can see, we have pole-zero cancellations with the non-dominant poles. Moreover, the phase is shifted upwards by π twice. The first shift is because of the two zeros at the origin of our transfer function, whereas the second shift is due to the negative sign in our transfer function (see setup in P6.8). You could equivalently have the phase start from 0.

In P6.10, we found that the worst-case velocity to drive over a sinusoidal road profile with wavelength λ was $v \approx 2.5\lambda$/s, which coincides with the peak in our Bode Magnitude plot. In P6.9, we computed the oscillation

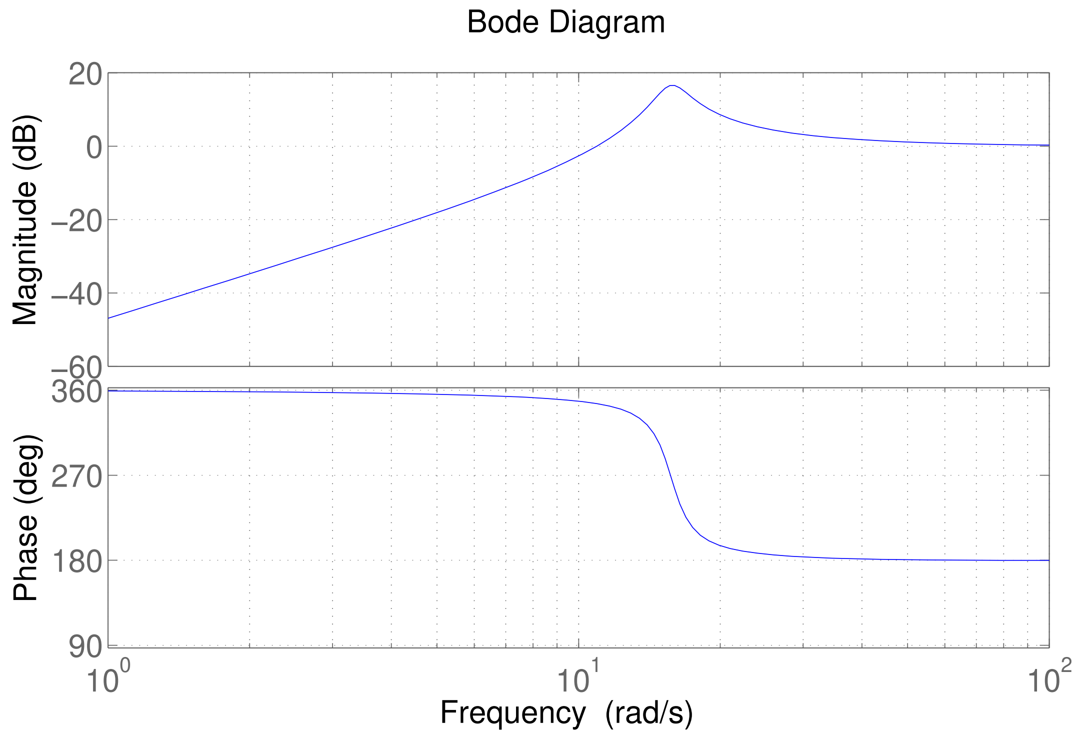

Figure 7: Bode plot for P7.8.

response to a rectangular pothole traversed at different velocities. The Fourier transform of a rectangle of height d and width T centered at the origin is a scaled *sinc* function, i.e.

$$\mathcal{F}\{d \cdot \mathrm{rect}(t/T)\} = dT\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right).$$

Figure 8 shows the Fourier transform of the pothole traversed at the two velocities. Notice that a time shift does not change the magnitude of our Fourier transform, only the phase. As expected, it does not matter when we hit the pothole, provided our velocity does not change. Zooming into the Fourier transform plots reveals approximately the same frequency content at the resonance frequency of approximately 15.7 rad/s. Given that the system is a high-pass filter, the large peak of the low-velocity Fourier transform at low frequencies does not affect the output significantly. At high frequencies, the two signals have similar total frequency content. Based on these observations, it seems reasonable that we did not observe a fundamental difference in the responses computed in assignment 7 for these two velocities.

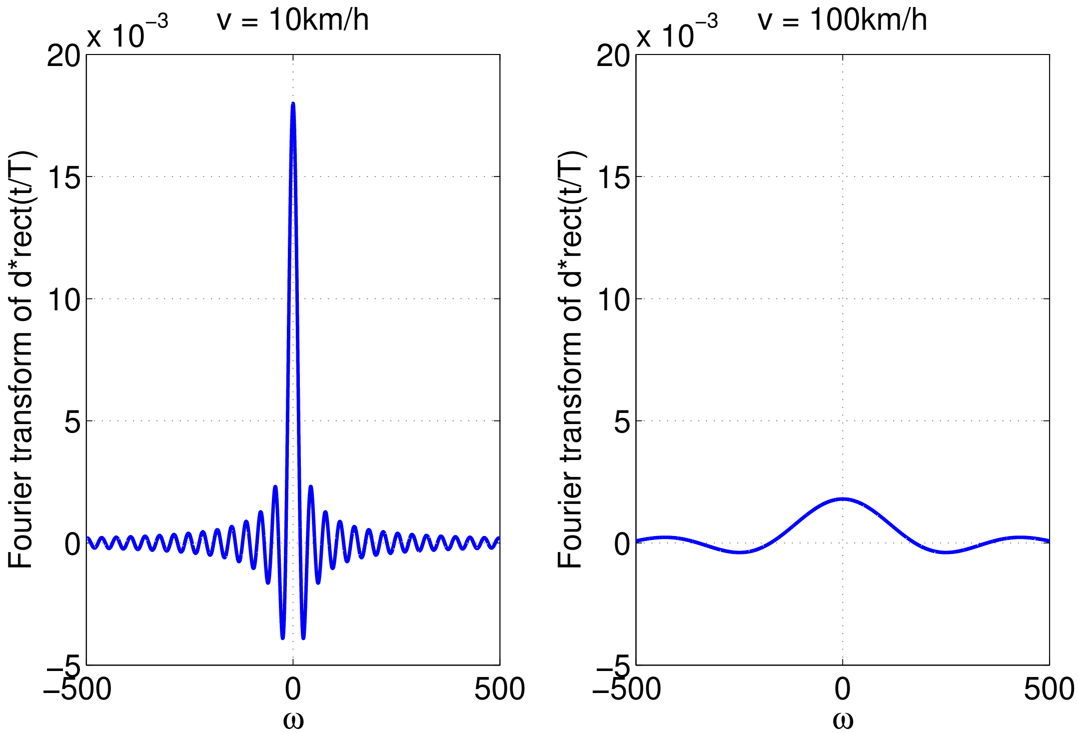

Figure 8: Fourier transform of centered pothole with $d = 1m$ traversed at given velocities.

7.9

As we know from previous assignments, the transfer function of this plant model is

$$G(s) = \frac{\Omega_2(s)}{Y(s)} = \frac{r_1\Omega_1(s)}{r_2T(s)} = \frac{r_1r_2/J_r}{s+b_r/J_r},$$

with I-controller $K(s) = K_i/s$ for some $K_i \neq 0$. Independent of our design, the closed-loop system will be able to track a constant reference due to the controller pole at the origin. Also, using Bode or Nyquist plots and positivity of all parameters, it is easily verified that the closed-loop system is asymptotically stable for any $K_i > 0$, which is not surprising after P6.16 on assignment 7. The plots are shown in Figure 9.

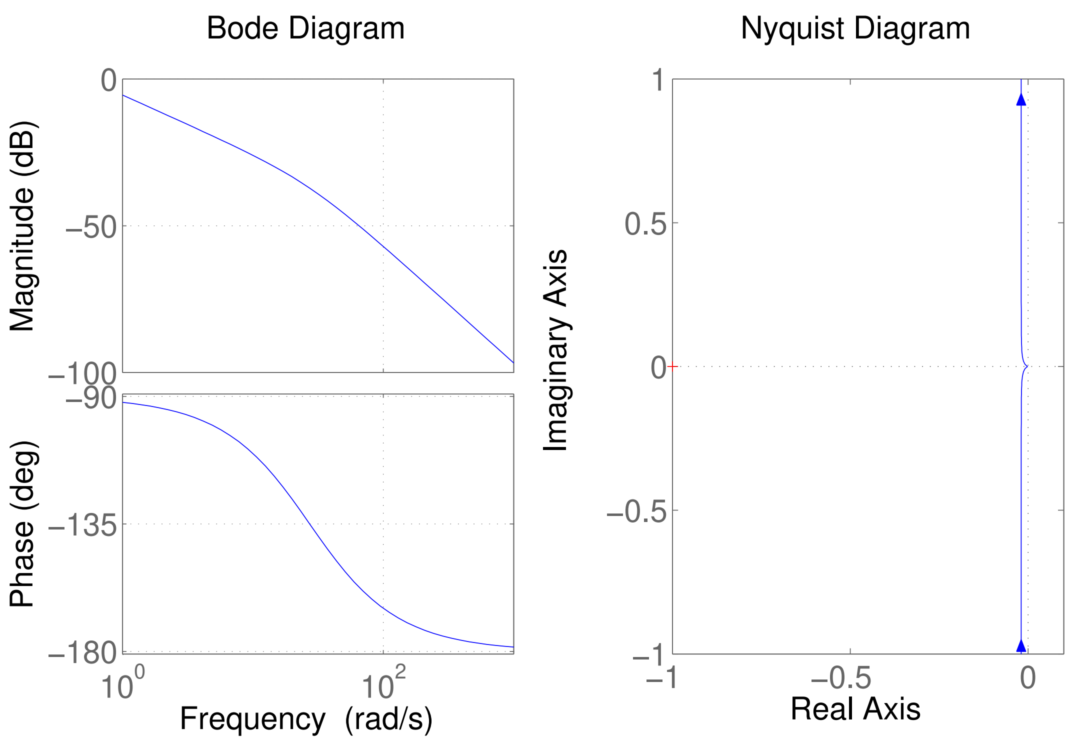

Figure 9: Plots for P7.9.

7.13

Given our parameters, we can use MATLAB to compute the transfer function of our system. One possible controller achieving closed-loop stability can be constructed by placing complex conjugate zeros on the imaginary axis at approximately half the imaginary parts of the complex system poles on the axis. The two poles required to make the controller proper can be placed on the negative part of the real axis, for instance at $p_{c,1/2} = -10$.

Bode and Nyquist plots corresponding to this setup are displayed in Figure 10, where the Bode plot includes gain and phase margins and the Nyquist plot has information on several points for positive frequencies to help track the behavior. We clearly have positive margins and no encirclements of the point at $s = -1$, implying asymptotic stability of the closed-loop transfer function with the controller described above.

The plots can be reconstructed using the following MATLAB code, where the Nyquist plot requires some zooming due to the very large and very small gains of our transfer function at different frequencies.

```
% P 7.13
m = 1600;
T = 11*3600;
W = 2*pi/T;

A = [    0      1    0;
     3*W^2      0  2*W;
         0  -2*W    0];
B = [ 0 ; 0 ; 1/m ];
C = [ 1   0     0 ];
D = 0;

sys   = ss(A,B,C,D);
sys   = zpk(sys);
sys.p = {zeros(3,1) + j*imag(sys.p{:})};
% To correct numerical errors in computing the transfer function

C   = zpk([-j*0.15e-3/2,j*0.15e-3/2],[-10,-10],1);

figure;
subplot(1,2,1)
margin(sys*C);
grid on;
subplot(1,2,2)
nyquist(sys*C,{1e-5,100});
```

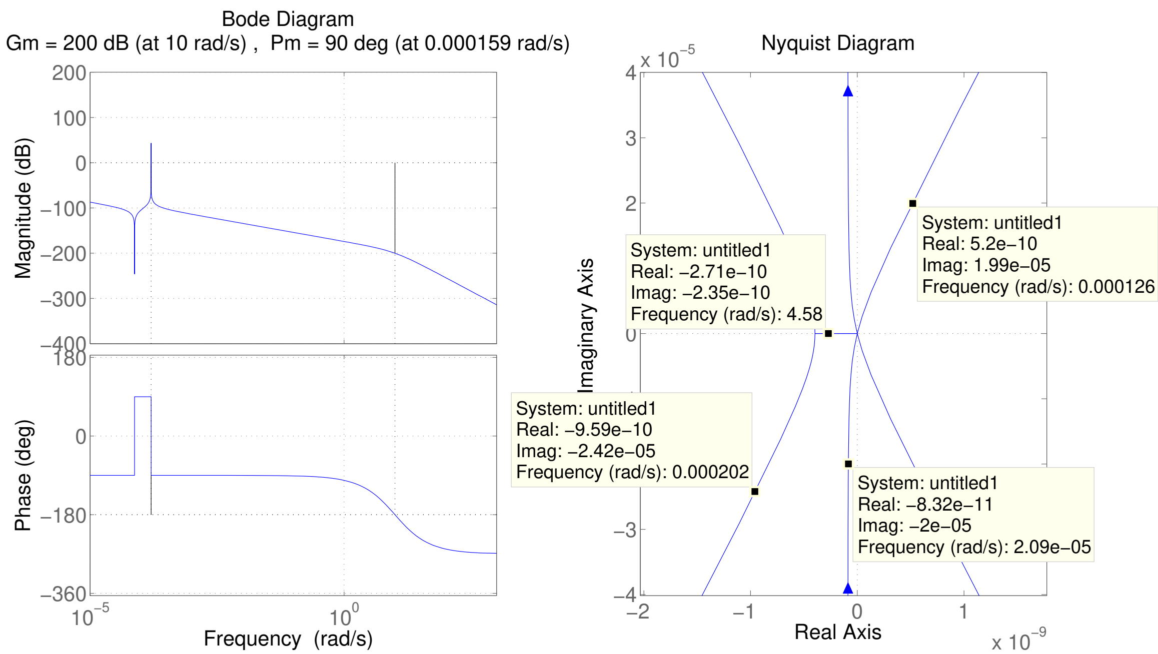

Figure 10: Plots for P7.13.