MAE143 B - Linear Control - Spring 2015
Midterm, May 5th

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities
2. You have 70 minutes
3. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book"
5. Do not forget to write your name and student number

Questions:

1. **System Modeling**

   As seen in class, the first-order ordinary differential equation

   $$m\,\dot{y}(t) + b\,y(t) = p\,u(t),$$

   can model the velocity response of a car traveling in a straight line where y is the car's velocity and the control input u is the pedal excursion. When there is a delay of $\tau \geq 0$ seconds between the car response and the control input this equation becomes

   $$m\,\dot{y}(t) + b\,y(t) = p\,u(t-\tau), \qquad \tau \geq 0.$$

   (a) (2 points) Ignoring initial conditions, use the Laplace transform to show that

   $$Y(s) = G(s)\,U(s), \qquad\qquad G(s) = \frac{p}{m}\frac{e^{-s\tau}}{s + \frac{b}{m}}.$$

   Solution: Dividing by $m > 0$ and ignoring initial conditions:

   $$\mathcal{L}\left\{\dot{y}(t) + \frac{b}{m}y(t)\right\} = sY(s) + \frac{b}{m}Y(s) = \frac{p}{m}e^{-s\tau}U(s) = \mathcal{L}\left\{\frac{p}{m}u(t-\tau)\right\} \qquad \text{(+1 point)}$$

   from which

   $$Y(s) = \frac{p}{m}\frac{e^{-s\tau}}{s + \frac{b}{m}}U(s). \qquad \text{(+1 point)}$$

   (b) (2 points) Use the inverse Laplace transform to show that:

   $$y(t) = \frac{p}{b}\left(1 - e^{-\frac{b}{m}(t-\tau)}\right), \qquad t \geq \tau$$

   when $u(t) = 1$, $t \geq 0$, is a unit step.

Solution:
With $U(s) = 1/s$ we calculate

$$Y(s) = G(s)\,U(s) = \frac{p}{m}\frac{e^{-s\tau}}{s\left(s+\frac{b}{m}\right)} = e^{-s\tau}\hat{Y}(s), \qquad \hat{Y}(s) = \frac{\frac{p}{m}}{s\left(s+\frac{b}{m}\right)} \qquad \text{(+0.5 point)}$$

Expanding the rational part in partial fractions:

$$\hat{Y}(s) = \frac{\alpha}{s} + \frac{\beta}{s+\frac{b}{m}} \qquad \text{(+0.5 point)}$$

where

$$\alpha = \lim_{s\to 0} s\hat{Y}(s) = \lim_{s\to 0}\frac{\frac{p}{m}}{s+\frac{b}{m}} = \frac{p}{b}$$
$$\beta = \lim_{s\to -\frac{b}{m}}\left(s+\frac{b}{m}\right)\hat{Y}(s) = \lim_{s\to -\frac{b}{m}}\frac{\frac{p}{m}}{s} = -\frac{p}{b}. \qquad \text{(+0.5 point)}$$

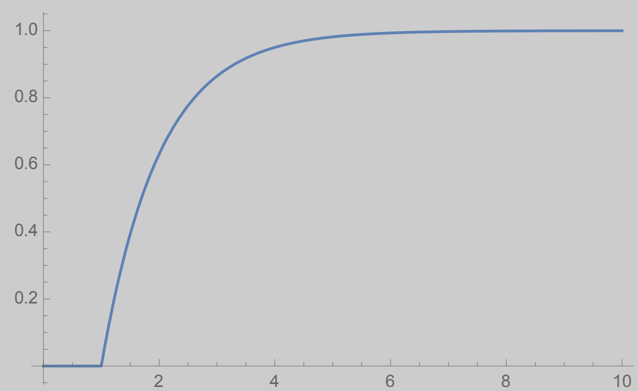
Putting it all together

$$Y(s) = \frac{p}{b}\left(\frac{e^{-s\tau}}{s} - \frac{e^{-s\tau}}{s+\frac{b}{m}}\right) \implies y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{p}{b}\left(1 - e^{-\frac{b}{m}(t-\tau)}\right), \quad t \geq \tau.$$

(+0.5 point)

(c) (2 points) Sketch $y(t)$ from (b) when $p = b = m = \tau = 1$.

Solution:
A sketch of y is:



(d) (1 point) Use the fact that $e^x \approx 1 + x$ when x is small to show that

$$G(s) \approx G_\tau(s) = \frac{p}{m}\frac{1-\tau\,s}{s+\frac{b}{m}}.$$

Solution:
Simply substitute $e^{-s\tau} \approx 1 - s\tau$ to obtain

$$G(s) = \frac{p}{m}\frac{e^{-s\tau}}{s+\frac{b}{m}} \approx \frac{p}{m}\frac{1-s\tau}{s+\frac{b}{m}} = G_\tau(s). \qquad \text{(+1 point)}$$

(e) (2 points) Let $Y(s) \approx Y_\tau(s) = G_\tau(s)\, U(s)$ and use the inverse Laplace transform to show that:

$$y_\tau(t) = \frac{p}{b}\left[1 - \left(1 + \tau\frac{b}{m}\right) e^{-\frac{b}{m}t}\right], \qquad t \geq 0,$$

when $u(t) = 1$, $t \geq 0$, is a unit step.

Solution:
With $U(s) = 1/s$ we calculate

$$Y_\tau(s) = G_\tau(s)\, U(s) = \frac{p}{m} \frac{1 - s\tau}{s\left(s + \frac{b}{m}\right)}. \qquad \text{(+0.5 point)}$$

Expanding in partial fractions:

$$Y_\tau(s) = \frac{\alpha}{s} + \frac{\beta}{s + \frac{b}{m}} \qquad \text{(+0.5 point)}$$

where

$$\begin{aligned} \alpha &= \lim_{s \to 0} sY_\tau(s) = \lim_{s \to 0} \frac{\frac{p}{m}(1 - s\tau)}{s + \frac{b}{m}} = \frac{p}{b} \\ \beta &= \lim_{s \to -\frac{b}{m}} \left(s + \frac{b}{m}\right) Y_\tau(s) = \lim_{s \to -\frac{b}{m}} \frac{\frac{p}{m}(1 - s\tau)}{s} = -\frac{p}{b}\left(1 + \tau\frac{b}{m}\right). \end{aligned} \qquad \text{(+0.5 point)}$$

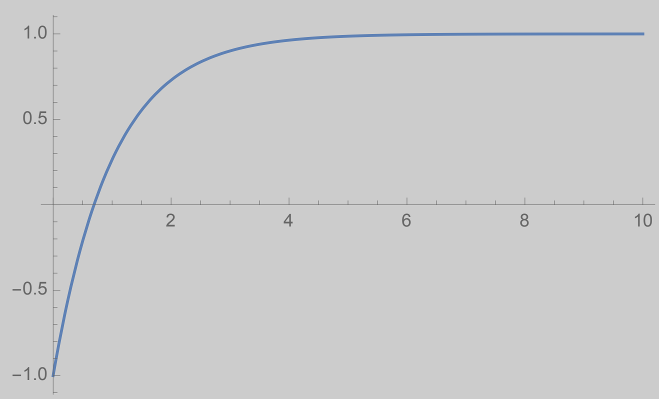
Putting it all together

$$Y_\tau(s) = \frac{p}{b}\left(\frac{1}{s} - \frac{1 + \tau\frac{b}{m}}{s + \frac{b}{m}}\right) \quad \Longrightarrow \quad y(t) = \mathcal{L}^{-1}\{Y_\tau(s)\} = \frac{p}{b}\left[1 - \left(1 + \tau\frac{b}{m}\right) e^{-\frac{b}{m}t}\right], \quad t \geq 0.$$

(+0.5 point)

(f) (2 points) Sketch $y_\tau(t)$ from (e) when $p = b = m = \tau = 1$.

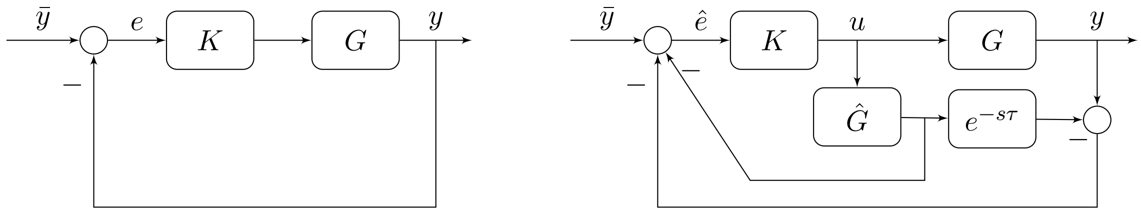
Solution:
A sketch of y_τ is:



(g) (1 point) Comment on the similarities and differences between $y(t)$ and $y_\tau(t)$.

Solution:
COMMENT TO GRADERS: Any reasonable comments earns the point.
For example:

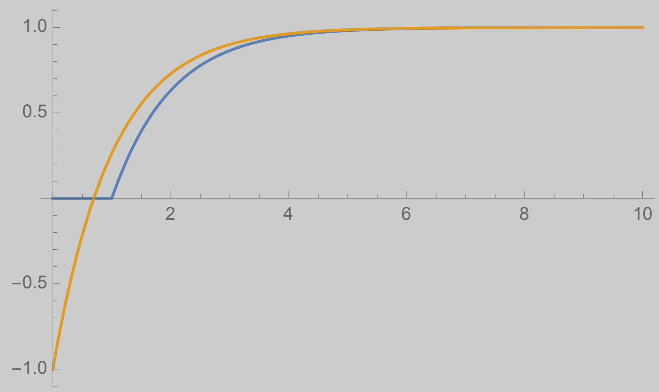
- $y(t) = 0$, $0 \leq t \leq \tau$ while $y_\tau(t)$ is not;



(a) Problem 2 (b) Problem 3

Figure 1: Figures for Problems 2 and 3

- $y_\tau(t)$ assume negative values while $y(t)$ does not;
- $y(t)$ and $y_\tau(t)$ are very similar for $t \geq \tau$.
- $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y_\tau(t) = \frac{p}{b}$.
- Compare the two sketches together:



2. **Proportional-Integral Control** $\boxed{\text{ASSUME: } p = b = m = 1.}$

Consider the standard feedback diagram shown in Fig. 1a where the controller has transfer-function:

$$K(s) = K_p + \frac{K_i}{s},$$

and let the system G be represented by the approximate transfer-function $G_\tau(s)$ from Problem 1.(d).

(a) (3 points) Calculate the transfer-function from $\bar{y}$ to e.

Solution:
Using the assumptions:

$$G(s) = G_\tau(s) = \frac{1 - \tau s}{s + 1} \qquad \text{(+0.5 point)}$$

and

$$\frac{\bar{Y}(s)}{E(s)} = S(s) = \frac{1}{1+G(s)K(s)} \qquad \text{(+0.5 point)}$$

$$= \frac{1}{1+\frac{1-\tau s}{s+1}\frac{K_p s+K_i}{s}} \qquad \text{(+1 point)}$$

$$= \frac{s(s+1)}{s^2+s+(1-\tau s)(K_p s+K_i)} \qquad \text{(+1 point)}$$

(b) (3 points) Let $K_i = K_p$ and calculate the range of positive values of K_p for which the closed-loop is asymptotically stable as a function of $\tau > 0$. What if $\tau = 0$?

Solution:
When $K_i = K_p$

$$S(s) = \frac{1}{1+\frac{1-\tau s}{s+1}\frac{K_p(s+1)}{s}} = \frac{1}{1+\frac{K_p(1-\tau s)}{s}} = \frac{s}{(1-K_p\tau)s+K_p} \qquad \text{(+1 point)}$$

which has a pole at

$$-\frac{K_p}{1-K_p\tau}$$

which is negative as long as $0 \leq K_p\tau < 1$

$$0 \leq K_p < \tau^{-1}. \qquad \text{(+1 point)}$$

If $\tau = 0$ then

$$0 \leq K_p \qquad \text{(+1 point)}$$

is enough.

(c) (3 points) Is it possible for the closed-loop to achieve asymptotic tracking of a constant reference signal $\bar{y}$? Explain why or why not.

Solution:
If $0 \leq K_p < \tau^{-1}$ (+1 point)
then yes because $S(s)$ is asymptotically stable (+1 point)
and has a zero at zero. (+1 point)

(d) (3 points) The choice $K_i = K_p$ is very special. Explain what it does and why it does not affect internal stability in this case.

Solution:
It performs a pole-zero cancellation. (+1 point)
It does not affect internal stability because $S(s)$ can be made asymptotically stable (+1 point)
and the pole-zero cancellation is that of the stable pole $s = -1$. (+1 point)

(e) (2 points (bonus)) Let G be the transfer-function $G(s)$ from Problem 1.(a) and repeat part (a). Explain why answering part (b) in this case is much more involved. What about part (c)?

Solution:
Using the assumptions:

$$G(s) = \frac{e^{-s\tau}}{s+1}$$

and

$$\frac{\bar{Y}(s)}{E(s)} = S(s) = \frac{1}{1+G(s)K(s)} = \frac{s(s+1)}{s(s+1)+e^{-s\tau}(K_p s + K_i)} \qquad \text{(+1 point)}$$

For part (b) $K_i = K_p$ and

$$\frac{\bar{Y}(s)}{E(s)} = S(s) = \frac{1}{1+G(s)K(s)} = \frac{s}{s+e^{-s\tau}K_p}$$

and evaluation of asymptotic stability requires computing the roots of the transcendental equation $s + e^{-s\tau}K_p = 0$. (+0.5 point)

The answer to part (c) is affirmative if one can assert asymptotic stability of $S(s)$ because $S(s)$ has a zero at zero. (+0.5 point)

3. **Smith Predictor**

Consider the feedback diagram shown in Fig. 1b with $\tau \geq 0$.

(a) (4 points) Show that the transfer-function from $\bar{y}$ to $\hat{e}$ is

$$\hat{S} = \frac{1}{1 + G\,K + (1 - e^{-s\tau})\,\hat{G}\,K}.$$

and that $\hat{S} = (1 + \hat{G}K)^{-1}$ when $G = e^{-s\tau}\hat{G}$.

Solution:
Follow the diagram to write:

$$\begin{aligned}\hat{e} &= \bar{y} - (y - e^{-s\tau}\hat{G}u) - \hat{G}u \\ &= \bar{y} - (G + (1 - e^{-s\tau})\hat{G})u \\ &= \bar{y} - (G + (1 - e^{-s\tau})\hat{G})K\hat{e}\end{aligned} \qquad \text{(+2 points)}$$

Solving for $\hat{e}$:

$$[1 + (G + (1 - e^{-s\tau})\hat{G})K]\hat{e} = \bar{y} \qquad \hat{e} = [1 + GK + (1 - e^{-s\tau})\hat{G}K]^{-1}\bar{y}. \qquad \text{(+1 point)}$$

If $G = e^{-s\tau}\hat{G}$ then

$$\hat{S} = \frac{1}{1 + e^{-s\tau}\hat{G}\,K + (1 - e^{-s\tau})\,\hat{G}\,K} = \frac{1}{1 + \hat{G}\,K}. \qquad \text{(+1 point)}$$

(b) (2 points) Let the system G be represented by the transfer-function $G(s)$ from Problem 1.(a). Assume that $p = b = m = 1$ and use (a) to calculate the transfer-function from $\bar{y}$ to $\hat{e}$.

Solution:

Note that

$$G(s) = \frac{p}{m}\frac{e^{-s\tau}}{s+\frac{b}{m}} = e^{-s\tau}\hat{G}(s), \qquad \hat{G}(s) = \frac{\frac{p}{m}}{s+\frac{b}{m}}. \qquad \text{(+1 point)}$$

Therefore

$$\hat{S} = \frac{1}{1+\hat{G}\,K} = \frac{1}{1+\frac{K\frac{p}{m}}{s+\frac{b}{m}}} = \frac{s+\frac{b}{m}}{s+\frac{b}{m}+K\frac{p}{m}} \qquad \text{(+1 point)}$$

(c) (2 points) Show that

$$u = \hat{K}\,(\bar{y}-y), \qquad \hat{K} = \frac{K}{1+(1-e^{-s\tau})\,\hat{G}\,K}.$$

Solution:
Follow the diagram to write:

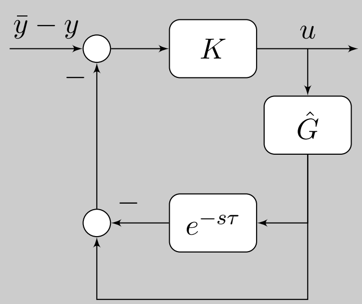
$$\begin{aligned} u &= K[\bar{y} - (y - e^{-s\tau}\hat{G}u) - \hat{G}u] \\ &= K[\bar{y} - y - (1-e^{-s\tau})\hat{G}u] \\ &= K(\bar{y}-y) - (1-e^{-s\tau})\hat{G}Ku \qquad \text{(+1 point)} \end{aligned}$$

Solving for u:

$$[1+(1-e^{-s\tau})\hat{G}K]u = K(\bar{y}-y) \qquad u = \frac{K}{1+(1-e^{-s\tau})\hat{G}K}(\bar{y}-y) \qquad \text{(+1 point)}$$

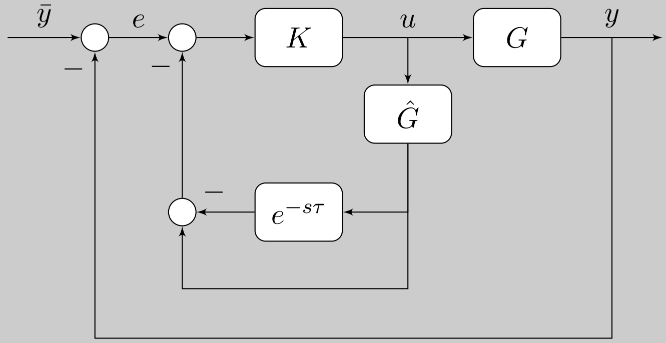
(d) (2 points) Draw a block diagram representing $\hat{K}$ in terms of the blocks $e^{-s\tau}$, K, and $\hat{G}$. Redraw the closed-loop diagram in Fig. 1b in terms of $\hat{K}$ for a bonus point. *Hint: think feedback!*

Solution:
One possible solution is:



(+2 points)

NOTE TO GRADERS: A solution with a block $1-e^{-s\tau}$ is also acceptable.
For the bonus point:



(+1 point)

(e) (2 points) When $G = e^{-s\tau}\hat{G}$, explain why the controller in Fig. 1b can only be used if $\hat{G}$ is asymptotically stable. *Hint: think pole-zero cancellation!*

Solution:
Note that

$$\hat{K} = K\frac{1}{1+\tilde{G}\,K}, \qquad \tilde{G} = (1-e^{-s\tau})\,\hat{G}.$$

Because the zeros of $(1+\tilde{G}\,K)^{-1}$ are the poles of $\tilde{G}\,K$, the zeros of $\hat{K}$ are the poles of $\tilde{G}$. (+1 point)

Because $(1-e^{-s\tau})$ has no poles, the poles of $\tilde{G}$ are the same as the poles of $\hat{G}$. When $G = e^{-s\tau}\hat{G}$, the controller $\hat{K}$ performs pole-zero cancellations of *all* poles of $\hat{G}$. Therefore, all these poles must have left real part for internal stability. (+1 point)

(f) (1 point (bonus)) A friend noticed that when $G = e^{-s\tau}\hat{G}$ the same closed-loop transfer-function from (a) can be obtained with the "way simpler" controller

$$K(s) = e^{s\tau}\,K, \quad \tau \geq 0.$$

Name one problem with this idea.

Solution:
It is true that

$$S = \frac{1}{1+G(s)K(s)} = \frac{1}{1+e^{-s\tau}\hat{G}(s)e^{-s\tau}K} = \frac{1}{1+\hat{G}(s)K}.$$

However, the controller $K(s)$ is not causal, that is it requires knowledge of error in the future. Indeed, in the case of a simple static gain K:

$$u(t) = \mathcal{L}^{-1}\{K(s)E(s)\} = \mathcal{L}^{-1}\{Ke^{s\tau}E(s)\} = Ke(t+\tau), \quad \tau > 0.$$