MAE143 B - Linear Control - Spring 2016
Final, June 7th

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your notes and the reader. You may use a calculator with no communication capabilities
2. You have 170 minutes
3. Do not get stuck! Move on and come back later if you have time
4. Write answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor
6. Initial conditions are zero unless otherwise noted
7. The exam has 6 question with for a total of 63 points and 2 bonus points

Questions:

The diagrams in Fig. 1 show pairs of open-loop pole-zero maps and their corresponding Nyquist plots. Assume that the associated transfer-function is of the form:

$$G(s) = G_0 \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}.$$

For each pair, answer the following questions: NOTE: Points shown are total for all 3 plots.

1. **Closed-loop Stability.**
   (a) (3 points) Is the open-loop system asymptotically stable?
   (b) (3 points) Can a proportional controller with unit gain stabilize the system in closed-loop? Explain.
   (c) (3 points) What are the positive values of gain for which the closed-loop system is asymptotically stable? Approximate values are fine.
   (d) (3 points) What are the positive values of gain for which the closed-loop system is not asymptotically stable? How many unstable closed-loop poles there are?

2. **Root-Locus.**
   (a) (9 points) Sketch the corresponding root-locus diagram.
   (b) (3 points) Explain how your root-locus supports your conclusions in Question 1.

3. **Bode Plots.** **IMPORTANT: Use the plots provided!**
   (a) (9 points) Sketch the straight-line approximations for the corresponding Bode plots on top of the (exact) Bode plots provided in Fig. 2.
   (b) (3 points) Annotate the plots to show the points critical for the calculation of the gain-margin and phase-margin. Provide estimates for these quantities based on your reading of the plots.

The next questions are based mostly on the paper:

[1] "Modeling β-Cell Insulin Secretion – Implications for Closed-Loop Glucose Homeostasis," by Steil, Rebrin, Janowski, Darwin and Saad, *Diabetes Technology & Therapeutics*, v. 5, n. 6, 2003.

4. **Minimal Model.**

   A simplified model for the level of *glucose*, y, in humans as a function of the *insulin concentration*, γ, is the set of nonlinear ordinary differential equations:

$$\begin{aligned}\dot{x}_1 &= -a\,x_1 + b\,\gamma \\ \dot{x}_2 &= -(c + x_1)\,x_2 + d\end{aligned}$$

   where $y = x_2$ and all constants are positive.

   (a) (1 point) Calculate the unique equilibrium point, $(\bar{x}_1, \bar{x}_2)$ for the above nonlinear model when $\gamma = \bar{\gamma} > 0$ is constant.

   (b) (2 points (bonus)) Show that the model linearized at this equilibrium point is

$$\begin{aligned}\dot{\tilde{x}} &= \begin{bmatrix} -a & 0 \\ -\bar{x}_2 & -(c + \bar{x}_1) \end{bmatrix} \tilde{x} + \begin{bmatrix} b \\ 0 \end{bmatrix} \tilde{\gamma} \\ \tilde{y} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{x}\end{aligned}$$

   with transfer-function

$$\frac{\tilde{Y}(s)}{\tilde{\Gamma}(s)} = \frac{-b\,\bar{x}_2}{(s+a)(s+c+\bar{x}_1)}.$$

   (c) (1 point) The authors of [1] have verified that when $\bar{x}_2 = 100\text{mg/dL}$ then $a = c + \bar{x}_1 = 1/33\text{min}^{-1}$ and $b\,\bar{x}_2/a^2 = 3.3\text{mg/dL per }\mu\text{U/mL}$ provide good experimental fit. Calculate the corresponding poles and zeros.

   (d) (1 point) Is this equilibrium point asymptotically stable?

   (e) (1 point) Explain the meaning of the negative sign on the numerator.

   (f) (1 point) Can the glucose response overshoot? Explain.

5. **Insulin Clearance.**

   After insulin is released in the plasma at a rate u, its concentration, γ, does not reach steady-state values instantaneously. Instead,

$$\dot{\gamma} = -f\,\gamma + g\,u.$$

   In [1], $f = g = 1/5\text{min}^{-1}$.

   (a) (1 point) Show that the transfer function from $\tilde{u}$ to $\tilde{y}$ is

$$\frac{\tilde{Y}(s)}{\tilde{U}(s)} = \frac{-g\,b\,\bar{x}_2}{(s+f)(s+a)(s+c+\bar{x}_1)}$$

   and substitute numerical values to calculate the corresponding poles and zeros.

6. **Glucose Homeostasis.**

In [1], the authors propose to explain that glucose homeostasis is maintained by the following feedback mechanism:

$$u(t) = K_p(y(t) - \bar{y}) + K_p T_d \frac{dy(t)}{dt} + \frac{K_p}{T_i} \int_0^t (y(\tau) - \bar{y}) \, d\tau \tag{1}$$

(a) (2 points) Draw a block diagram representing the complete closed-loop insulin homeostasis system, including the signals $\bar{y}$, y, γ, and u.

(b) (1 point) Show that when $\bar{y}$ is constant then (1) is equivalent to:

$$u(t) = K_p(y(t) - \bar{y}) + K_p T_d \frac{d(y(t) - \bar{y})}{dt} + \frac{K_p}{T_i} \int_0^t (y(\tau) - \bar{y}) \, d\tau. \tag{2}$$

What kind of "controller" is (2)?

(c) (2 points) Explain why the feedback controllers (1) and (2) can be defined in terms of the actual glucose level, y, rather then its variation from equilibrium, $\tilde{y}$.

(d) (2 points) In [1] the authors have determined experimentally the values $T_i = 100$, $T_d = 38$, and $K_p = 0.17$. Calculate the transfer-function corresponding to the controller (2).

(e) (2 points) Use the values of T_i and T_d from [1] and calculate the loop transfer-function, $L(s)$, that can be used for feedback analysis of the closed-loop glucose homeostasis system with respect to the proportional gain $K_p > 0$.

(f) (3 points) Sketch the Bode plot for the loop transfer-function, $L(s)$.

(g) (3 points) Sketch the Nyquist plot for the loop transfer-function, $L(s)$.

(h) (3 points) Use any technique you like to prove that the closed-loop glucose homeostasis system is internally stable for those values of parameters. Calculate the steady-state closed-loop response to a constant $\bar{y} \neq 0$.

(i) (3 points) Can you determine $K_p > 0$ so that the closed-loop is internally stable when $T_d = 0$? Name one advantage and one disadvantage of having $T_d = 0$.

Have a great summer!

NAME: **STUDENT ID:**

WRITE YOUR NAME AND ID AND
DO NOT FORGET TO RETURN THIS PAGE WITH YOUR EXAM!

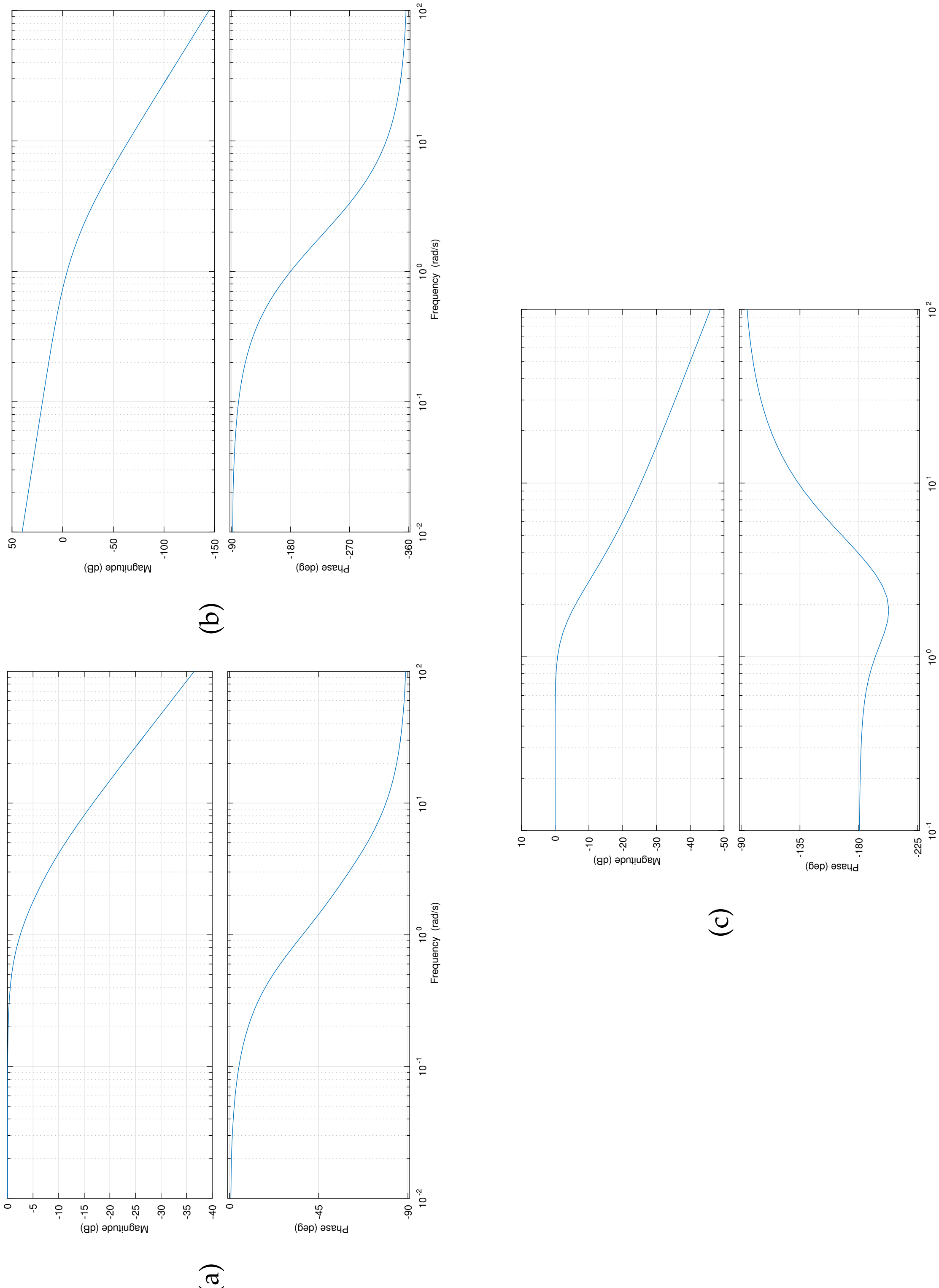

Figure 2: Diagrams for Question 3

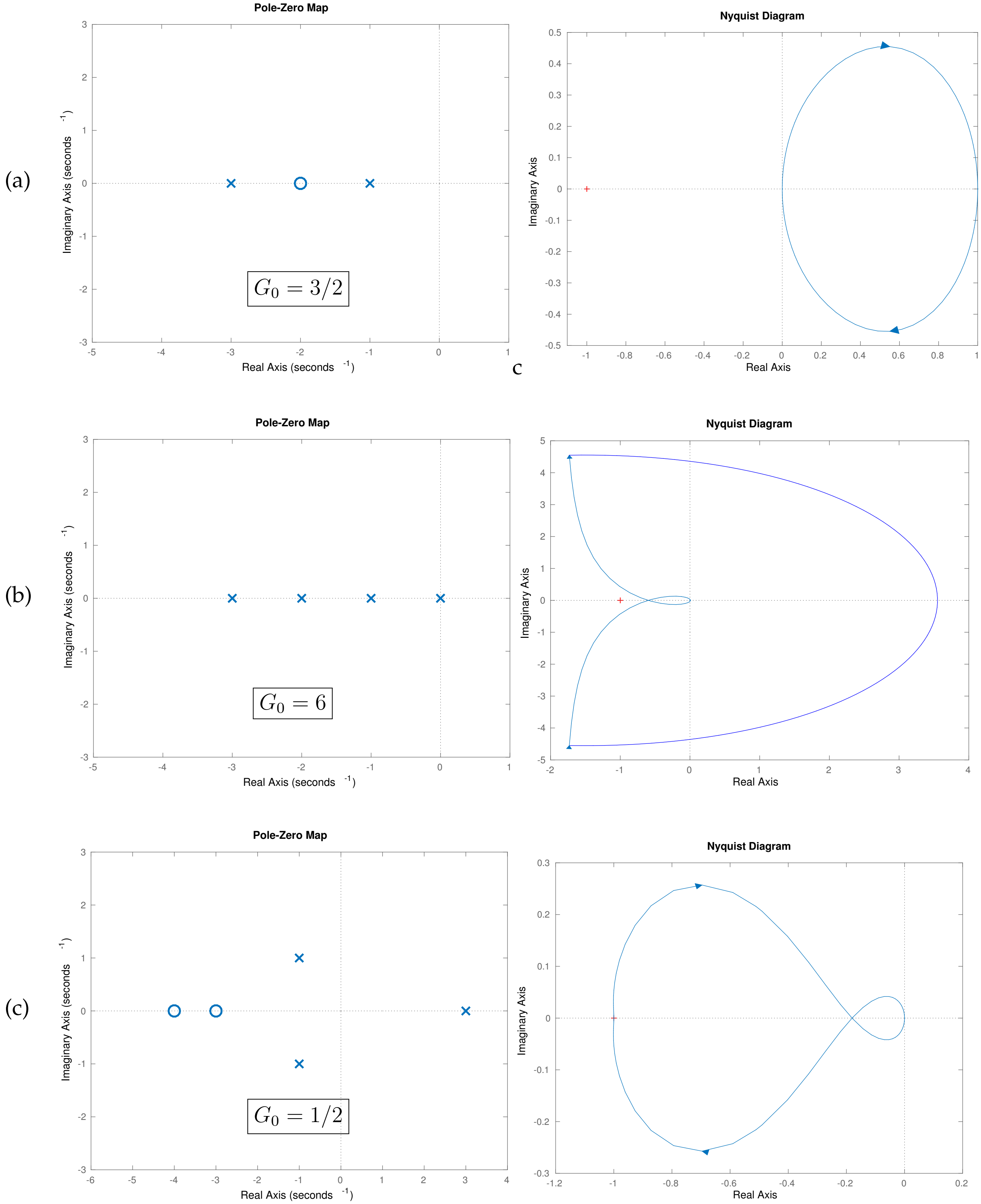

Figure 1: Diagrams for Questions 1 and 2