MAE143 B - Linear Control - Spring 2016
Final, June 7th

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your notes and the reader. You may use a calculator with no communication capabilities
2. You have 170 minutes
3. Do not get stuck! Move on and come back later if you have time
4. Write answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor
6. Initial conditions are zero unless otherwise noted
7. The exam has 6 question with for a total of 63 points and 2 bonus points

Questions:

The diagrams in Fig. 1 show pairs of open-loop pole-zero maps and their corresponding Nyquist plots. Assume that the associated transfer-function is of the form:

$$G(s) = G_0 \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}.$$

For each pair, answer the following questions: NOTE: Points shown are total for all 3 plots.

1. **Closed-loop Stability.**
   (a) (3 points) Is the open-loop system asymptotically stable?
   (b) (3 points) Can a proportional controller with unit gain stabilize the system in closed-loop? Explain.
   (c) (3 points) What are the positive values of gain for which the closed-loop system is asymptotically stable? Approximate values are fine.
   (d) (3 points) What are the positive values of gain for which the closed-loop system is not asymptotically stable? How many unstable closed-loop poles there are?

Solution:

System (a)

(a) Yes, all open-loop poles have negative real part. (+1 point)

(b) Yes, since the Nyquist plot shows no encirclements of the point $'-1'$. (+1 point)

(c) All values of gain $K > 0$ stabilize the system is closed-loop. (+1 point)

(d) There are no positive values of gain $K > 0$ for which the closed-loop system is not asymptotically stable. (+1 point)

System (b)

(a) No, one open-loop pole is on the imaginary axis. (+1 point)

(b) Yes, since the Nyquist plot shows no encirclements of the point '-1'. (+1 point)

(c) All values of gain K such that $1/K > 0.6$ (approximately), i.e. $K < 1.7$, stabilize the system is closed-loop. (+1 point)

(d) When $K > 1.7$ the Nyquist plot shows $\# = -2$ (clockwise) encirclements of the point $-1/K$ so that $N_Z = N_P - \# = 0 - (-2) = 2$ closed-loop poles are unstable.

(+1 point)

System (c)

(a) No, one open-loop pole has positive real part. (+1 point)

(b) No, since the point '-1' is on the Nyquist plot. (+1 point)

(c) All values of gain K such that $1/K < 0.18$ (approximately), i.e. $K > 5.5$, stabilize the system is closed-loop. (+1 point)

(d) When $K < 1$ the Nyquist plot shows $\# = 0$ (clockwise) encirclements of the point $-1/K$ so that $N_Z = N_P - \# = 1 - 0 = 1$ closed-loop pole is unstable. When $1 < K < 5.5$ the Nyquist plot shows $\# = -1$ (clockwise) encirclements of the point $-1/K$ so that $N_Z = N_P - \# = 1 - (-1) = 2$ closed-loop poles are unstable.

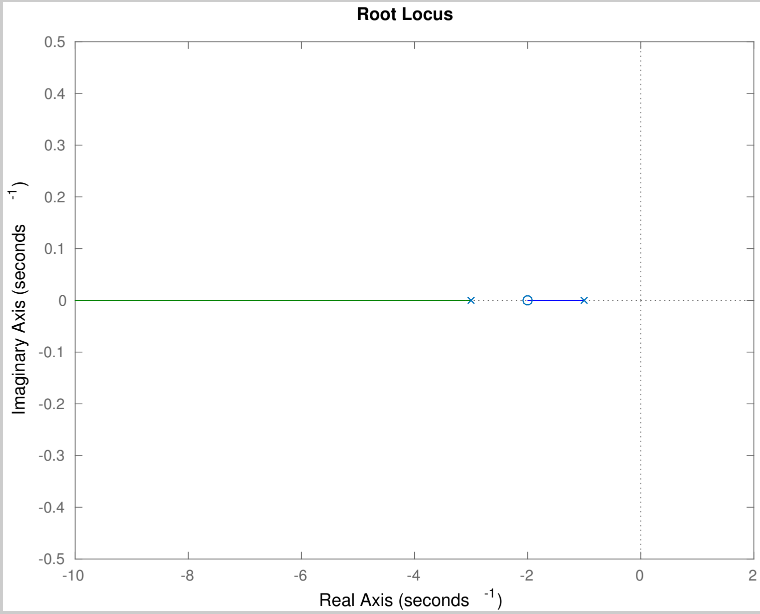
(+1 point)

2. **Root-Locus.**
   (a) (9 points) Sketch the corresponding root-locus diagram.
   (b) (3 points) Explain how your root-locus supports your conclusions in Question 1.

Solution:

System (a)

(a) The root-locus diagram should look like:



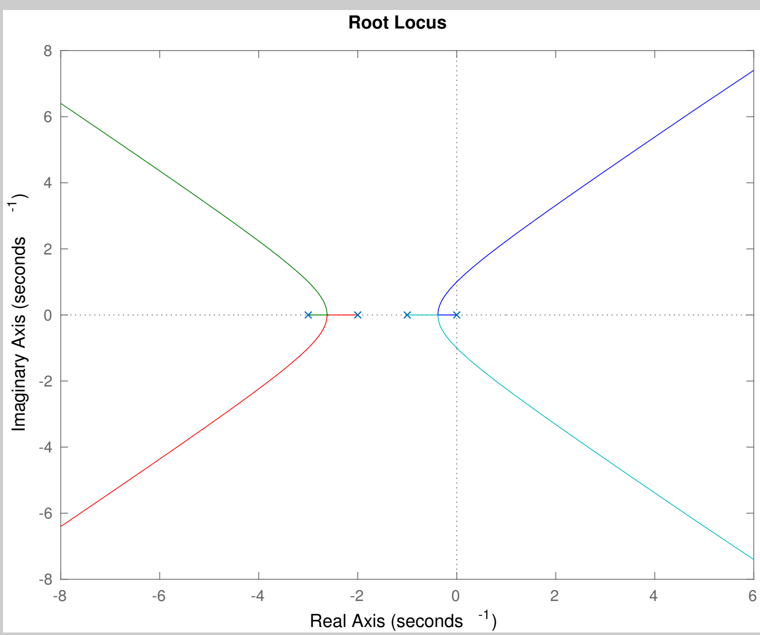
(+3 points)

(b) All closed-loop roots remain stable for any $K > 0$, as established in Q. 1.

(+1 point)

System (b)

(a) The root-locus diagram should look like:



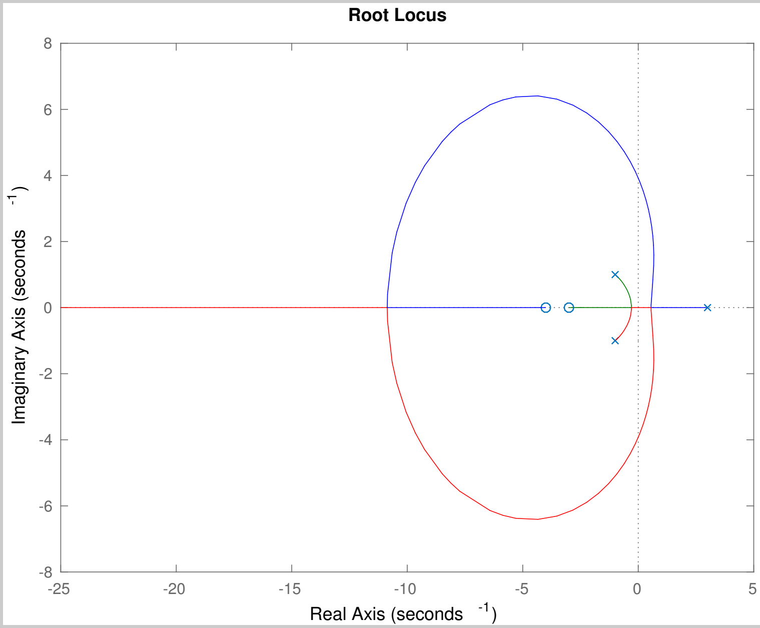
(+3 points)

(b) For some value of $K > 0$ two closed-loop poles become unstable, as established in Q. 1.

(+1 point)

System (c)

(a) The root-locus diagram should look like:



(+3 points)

(b) For small $K > 0$ there is one unstable closed-loop pole and at some point two closed-loop poles become unstable. Finally all roots become stable for large enough $K > 0$, as established in Q. 1.

(+1 point)

3. **Bode Plots.** **IMPORTANT: Use the plots provided!**

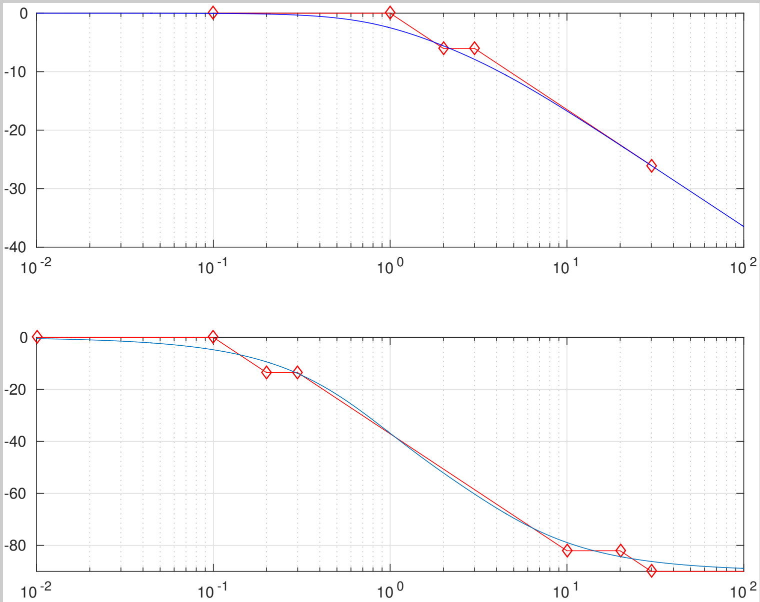
   (a) (9 points) Sketch the straight-line approximations for the corresponding Bode plots on top of the (exact) Bode plots provided in Fig. 2.

   (b) (3 points) Annotate the plots to show the points critical for the calculation of the gain-margin and phase-margin. Provide estimates for these quantities based on your reading of the plots.

Solution:

System (a)

(a) The Bode plot should look like:



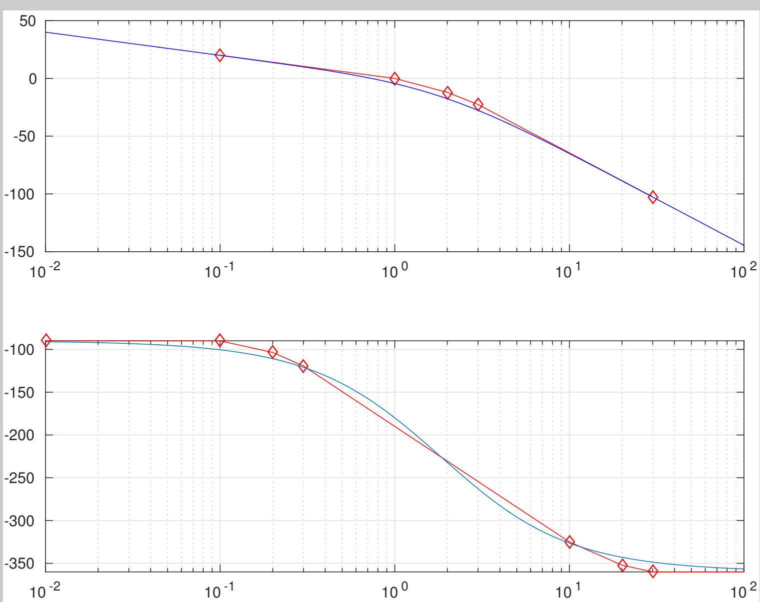
(+3 points)

(b) There is no point for which $\angle L(j\omega) = \pi$ so the gain margin in infinite; $|L(j\omega)| = 1$ at $\omega = 0$ so the phase margin is $0 - 180° = -180°$.

(+1 point)

System (b)

(a) The Bode plot should look like:



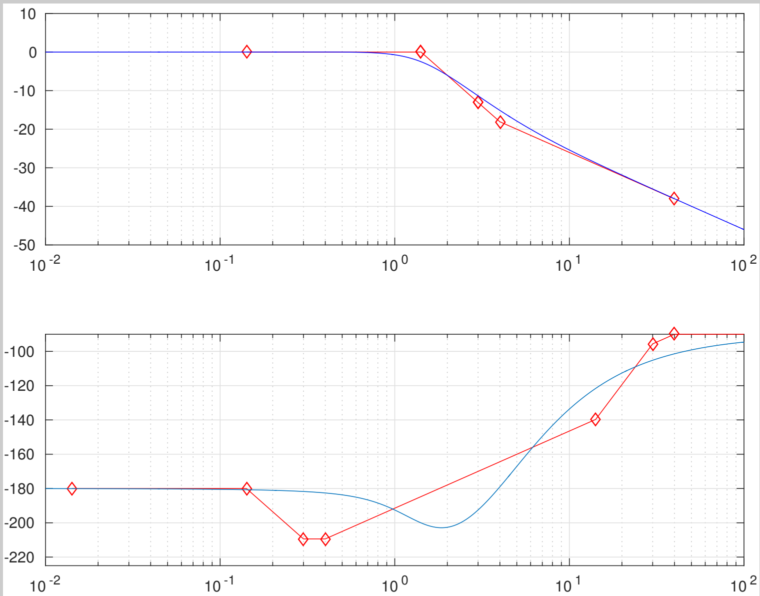
(+3 points)

(b) $\angle L(j\omega) = \pi$ at $\omega = 1$ the gain margin is approximately 3dB (single pole) or $10^{3/20} \approx 1.41$; $|L(j\omega)| = 1$ at about $\omega = 0.7$ so the phase margin is approximately $(-155 + 360) - 180 = 25°$.

(+1 point)

System (c)

(a) The Bode plot should look like:



(+3 points)

(b) The closed-loop system is not stable under unit-feedback so there are no margins!

(+1 point)

The next questions are based mostly on the paper:

[1] "Modeling β-Cell Insulin Secretion – Implications for Closed-Loop Glucose Homeostasis," by Steil, Rebrin, Janowski, Darwin and Saad, *Diabetes Technology & Therapeutics*, v. 5, n. 6, 2003.

4. **Minimal Model.**

A simplified model for the level of *glucose*, y, in humans as a function of the *insulin concentration*, γ, is the set of nonlinear ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= -a\,x_1 + b\,\gamma \\ \dot{x}_2 &= -(c + x_1)\,x_2 + d \end{aligned}$$

where $y = x_2$ and all constants are positive.

(a) (1 point) Calculate the unique equilibrium point, $(\bar{x}_1, \bar{x}_2)$ for the above nonlinear model when $\gamma = \bar{\gamma} > 0$ is constant.

Solution: At equilibrium:

$$\begin{aligned} -a\,\bar{x}_1 + b\,\bar{\gamma} &= 0 \\ -(c + \bar{x}_1)\,\bar{x}_2 + d &= 0 \end{aligned}$$

so that

$$\bar{x}_1 = \frac{b}{a}\,\bar{\gamma} > 0,$$

and

$$\bar{x}_2 = \frac{d}{c + \bar{x}_1} > 0.$$

(+1 point)

(b) (2 points (bonus)) Show that the model linearized at this equilibrium point is

$$\begin{aligned} \dot{\tilde{x}} &= \begin{bmatrix} -a & 0 \\ -\bar{x}_2 & -(c + \bar{x}_1) \end{bmatrix} \tilde{x} + \begin{bmatrix} b \\ 0 \end{bmatrix} \tilde{\gamma} \\ \tilde{y} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{x} \end{aligned}$$

with transfer-function

$$\frac{\tilde{Y}(s)}{\tilde{\Gamma}(s)} = \frac{-b\,\bar{x}_2}{(s + a)(s + c + \bar{x}_1)}.$$

Solution: System is in state-space with

$$f(x, \gamma) = \begin{pmatrix} -a\,x_1 + b\,\gamma \\ -(c + x_1)\,x_2 + d \end{pmatrix}, \qquad g(x, \gamma) = x_2.$$

Linearization leads to

$$
\begin{aligned}
A &= \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\gamma=\bar{\gamma}} = \left.\begin{bmatrix} -a & 0 \\ -x_2 & -(c+x_1) \end{bmatrix}\right|_{x=\bar{x},\gamma=\bar{\gamma}} = \begin{bmatrix} -a & 0 \\ -\bar{x}_2 & -(c+\bar{x}_1) \end{bmatrix} \\
B &= \left.\frac{\partial f}{\partial \gamma}\right|_{x=\bar{x},\gamma=\bar{\gamma}} = \left.\begin{bmatrix} b \\ 0 \end{bmatrix}\right|_{x=\bar{x},\gamma=\bar{\gamma}} = \begin{bmatrix} b \\ 0 \end{bmatrix} \\
C &= \left.\frac{\partial g}{\partial x}\right|_{x=\bar{x},\gamma=\bar{\gamma}} = \left.\begin{bmatrix} 0 & 1 \end{bmatrix}\right|_{x=\bar{x},\gamma=\bar{\gamma}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \\
D &= \left.\frac{\partial g}{\partial \gamma}\right|_{x=\bar{x},\gamma=\bar{\gamma}} = 0.
\end{aligned}
$$

(+1 bonus point)

Calculating the transfer-function:

$$
\begin{aligned}
C(sI-A)^{-1}B+D &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+a & 0 \\ \bar{x}_2 & s+c+\bar{x}_1 \end{bmatrix}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix} \\
&= \frac{1}{(s+a)(s+c+\bar{x}_1)} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+c+\bar{x}_1 & 0 \\ -\bar{x}_2 & s+a \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} \\
&= \frac{-b\,\bar{x}_2}{(s+a)(s+c+\bar{x}_1)}.
\end{aligned}
$$

(+1 bonus point)

(c) (1 point) The authors of [1] have verified that when $\bar{x}_2 = 100\text{mg/dL}$ then $a = c + \bar{x}_1 = 1/33\text{min}^{-1}$ and $b\,\bar{x}_2/a^2 = 3.3\text{mg/dL}$ per $\mu\text{U/mL}$ provide good experimental fit. Calculate the corresponding poles and zeros.

Solution:
Substitute:

$$
\frac{-b\,\bar{x}_2}{(s+a)(s+c+\bar{x}_1)} = \frac{-3.3\,a^2}{(s+a)^2} = \frac{-3.3}{(1+33\,s)^2}
$$

which has a double real pole at $s = -1/33$ and no zeros. (+1 point)

(d) (1 point) Is this equilibrium point asymptotically stable?

Solution:
Yes since the poles at $s = -1/33$ have negative real part. (+1 point)

(e) (1 point) Explain the meaning of the negative sign on the numerator.

Solution:
It means that when insulin concentration goes up glucose goes down. It is a non-minimium phase system.

(+1 point)

(f) (1 point) Can the glucose response overshoot? Explain.

Solution:
No because both poles are real. Overshoot requires complex conjugate poles.
(+1 point)

5. **Insulin Clearance.**

After insulin is released in the plasma at a rate u, its concentration, γ, does not reach steady-state values instantaneously. Instead,

$$\dot{\gamma} = -f\,\gamma + g\,u.$$

In [1], $f = g = 1/5\text{min}^{-1}$.

(a) (1 point) Show that the transfer function from $\tilde{u}$ to $\tilde{y}$ is

$$\frac{\tilde{Y}(s)}{\tilde{U}(s)} = \frac{-g\,b\,\bar{x}_2}{(s+f)(s+a)(s+c+\bar{x}_1)}$$

and substitute numerical values to calculate the corresponding poles and zeros.

Solution:
Calculate

$$\frac{\Gamma(s)}{U(s)} = \frac{g}{s+f}.$$

Because of linearity

$$\frac{\tilde{\Gamma}(s)}{\tilde{U}(s)} = \frac{g}{s+f}$$

and combine

$$\frac{\tilde{Y}(s)}{\tilde{U}(s)} = \frac{\tilde{Y}(s)}{\tilde{\Gamma}(s)}\frac{\tilde{\Gamma}(s)}{\tilde{U}(s)} = \frac{-b\,\bar{x}_2}{(s+a)(s+c+\bar{x}_1)} \times \frac{g}{s+f} = \frac{-g\,b\,\bar{x}_2}{(s+f)(s+a)(s+c+\bar{x}_1)}$$

(+1 point)

6. **Glucose Homeostasis.**

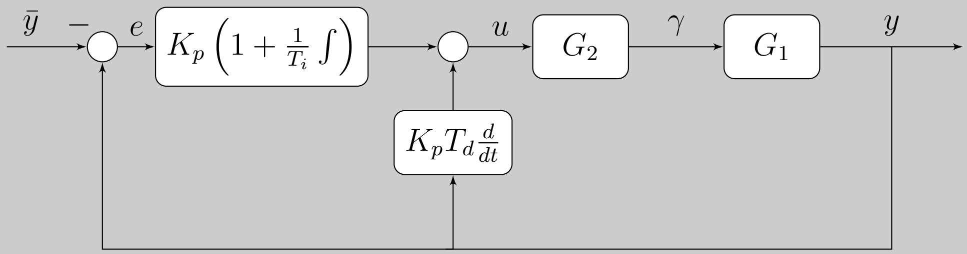
In [1], the authors propose to explain that glucose homeostasis is maintained by the following feedback mechanism:

$$u(t) = K_p(y(t) - \bar{y}) + K_p T_d \frac{dy(t)}{dt} + \frac{K_p}{T_i}\int_0^t (y(\tau) - \bar{y})\; d\tau \tag{1}$$

(a) (2 points) Draw a block diagram representing the complete closed-loop insulin homeostasis system, including the signals $\bar{y}$, y, γ, and u.

Solution:

Block diagram should look like:



IMPORTANT: Note the "positive" feedback loop!

(+2 points)

(b) (1 point) Show that when $\bar{y}$ is constant then (1) is equivalent to:

$$u(t) = K_p(y(t) - \bar{y}) + K_pT_d\frac{d(y(t) - \bar{y})}{dt} + \frac{K_p}{T_i}\int_0^t (y(\tau) - \bar{y})\, d\tau. \tag{2}$$

What kind of "controller" is (2)?

Solution:

When $\bar{y}$ is constant

$$\frac{dy(t)}{dt} = \frac{d(y(t) - \bar{y})}{dt}.$$ (+1/2 point)

Except for the positive feedback this is a Proportional-Integral-Derivative (PID) controller.

(+1/2 point)

(c) (2 points) Explain why the feedback controllers (1) and (2) can be defined in terms of the actual glucose level, y, rather then its variation from equilibrium, $\tilde{y}$.

Solution:

As discussed in §5.8 of the textbook, implementation of a controller based on a linearized system requires offsetting the input and outputs as

$$u = \bar{u} + K(\bar{y} - g(\bar{x}, \bar{u}) - y)$$ (+1 point)

Because $\bar{y}$ is constant and K is linear

$$u = K(\bar{y} - y) + \bar{w}, \qquad \bar{w} = \bar{u} - g(\bar{x}, \bar{u})$$

which can be seen as a constant disturbance to the input, $\bar{w}$. Because of the integrator in the controller, this disturbance will be rejected in closed-loop.

(+1 point)

(d) (2 points) In [1] the authors have determined experimentally the values $T_i = 100$, $T_d = 38$, and $K_p = 0.17$. Calculate the transfer-function corresponding to the controller (2).

Solution:
Applying the Laplace transform

$$\begin{aligned}
U(s) &= \left(K_p + K_p T_d s + \frac{K_p}{T_i s}\right)(Y(s) - \bar{Y}(s)) \\
&= K_p \frac{1 + T_i s + T_i T_d s^2}{T_i s}(Y(s) - \bar{Y}(s)) \\
&= K_p\, K(s)(Y(s) - \bar{Y}(s)), \qquad K(s) = \frac{1 + T_i s + T_i T_d s^2}{T_i s}. \qquad \text{(+1 point)}
\end{aligned}$$

Substituting numeric values

$$K(s) = \frac{1 + 100s + 3800s^2}{100s}. \qquad \text{(+1 point)}$$

(e) (2 points) Use the values of T_i and T_d from [1] and calculate the loop transfer-function, $L(s)$, that can be used for feedback analysis of the closed-loop glucose homeostasis system with respect to the proportional gain $K_p > 0$.

Solution:
The loop transfer-function is obtained from

$$\begin{aligned}
\frac{\tilde{Y}(s)}{E(s)} &= \frac{\tilde{Y}(s)}{\tilde{Y}(s) - Y(s)} \\
&= \frac{\tilde{Y}(s)}{\tilde{U}(s)} \frac{\tilde{U}(s)}{\tilde{Y}(s) - Y(s)} \\
&= -K_p\, K(s) \frac{-g\, b\, \bar{x}_2}{(s+f)(s+a)(s+c+\bar{x}_1)} \\
&= K_p\, L(s), \qquad L(s) = \frac{g\, b\, \bar{x}_2 (1 + T_i s + T_i T_d s^2)}{T_i s (s+f)(s+a)(s+c+\bar{x}_1)} \qquad \text{(+1 point)}
\end{aligned}$$

Substituting numeric values

$$L(s) = 3.3(1/5)(1/33)^2 \frac{1 + 100s + 3800s^2}{s(s+1/5)(s+1/33)^2} = 3.3 \frac{1 + 100s + 3800s^2}{s(1+5\,s)(1+33\,s)^2}$$

(+1 point)

(f) (3 points) Sketch the Bode plot for the loop transfer-function, $L(s)$.

Solution:
From part (e):

$$L(s) = 3.3(1/5)(1/33)^2 \frac{1 + 100s + 3800s^2}{s(s+1/5)(s+1/33)^2} = 3.3 \frac{1 + 100s + 3800s^2}{s(1+5\,s)(1+33\,s)^2}$$

Critical points are complex zeros at

$$\omega_n = 1/\sqrt{3800} \approx 0.016, \qquad \zeta = 100\omega_n/2 \approx 0.8$$

then two real poles at

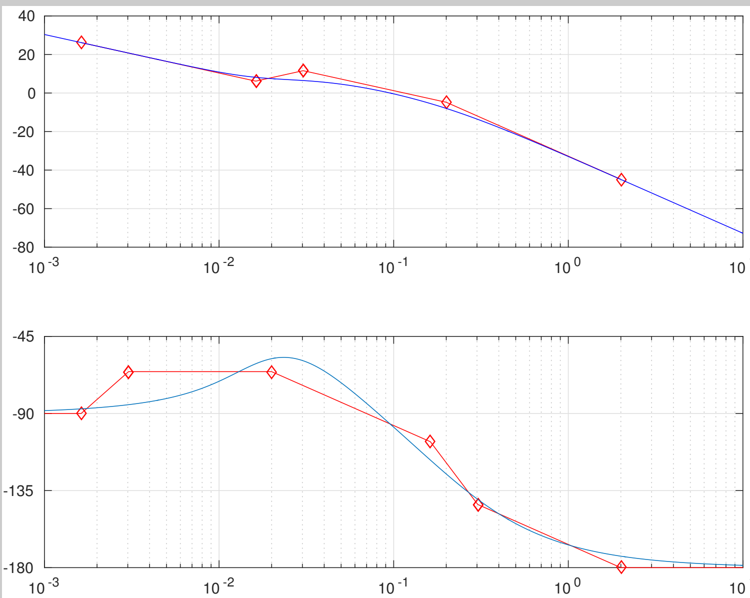
$$\omega = 1/33 \approx 0.03$$

and a real pole at

$$\omega = 1/5 = 0.2$$

(+1 point)

A sketch should look like:



(+2 points)

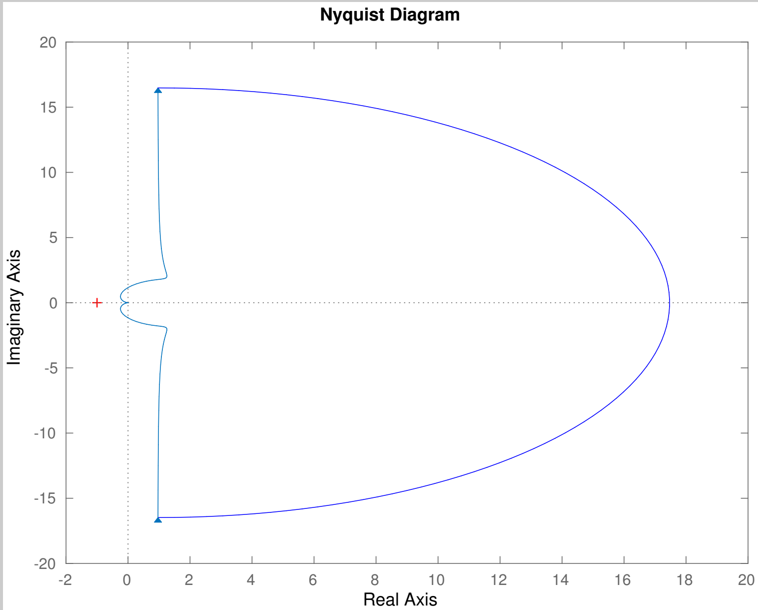
(g) (3 points) Sketch the Nyquist plot for the loop transfer-function, $L(s)$.

Solution:

From part (f) we expect the Nyquist plot to cross the negative imaginary once then converge to zero without crossing the negative real axis. A large clockwise arc closes the plot due to the pole at zero.

(+1 point)

A sketch should look like:



(+2 points)

(h) (3 points) Use any technique you like to prove that the closed-loop glucose homeostasis system is internally stable for those values of parameters. Calculate the steady-state closed-loop response to a constant $\bar{y} \neq 0$.

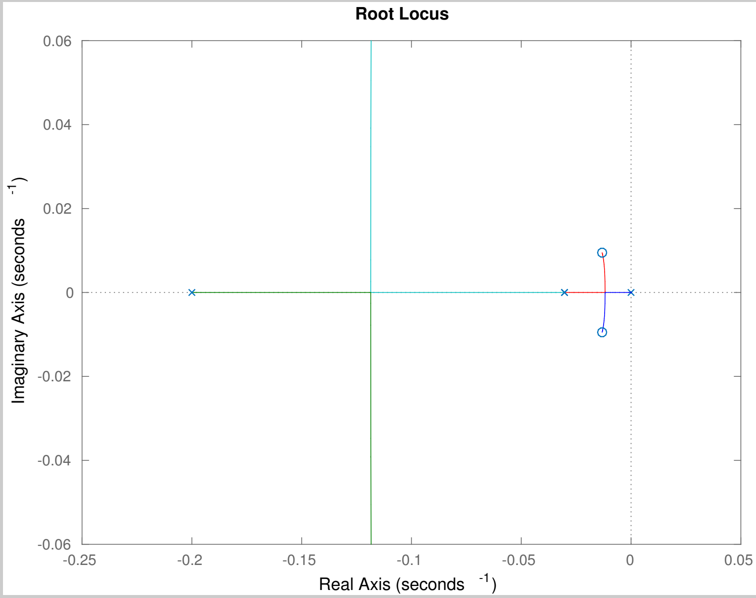
Solution:

Simplest solution is using the Nyquist plot from part (g).

Since $L(s)$ has no poles on the right-hand side of the complex plane and never crosses the negative real axis it should be asymptotically stable for all values of gain $K_p > 0$. It is internally stable because there are no pole-zero cancellations. Because the controller has a pole at $s = 0$ the closed-loop system can asymptotically track constant inputs, therefore $y_{ss}(t) = \bar{y}$.

(+3 points)

Alternatively, one could draw the same conclusions from a root-locus plot:



(i) (3 points) Can you determine $K_p > 0$ so that the closed-loop is internally stable when $T_d = 0$? Name one advantage and one disadvantage of having $T_d = 0$.

Solution:

With one less zero, the Nyquist plot would now cross the negative real axis. However, by picking K_p such that $-1/K_p$ is to the left of the crossing guarantees closed-loop stability.

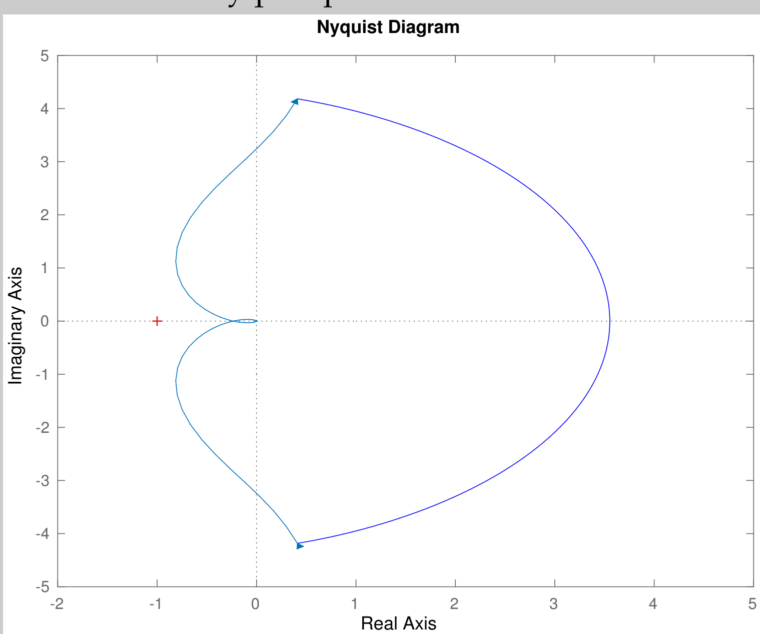
(+1 point)

With $T_d = 0$ and $T_i = 100$

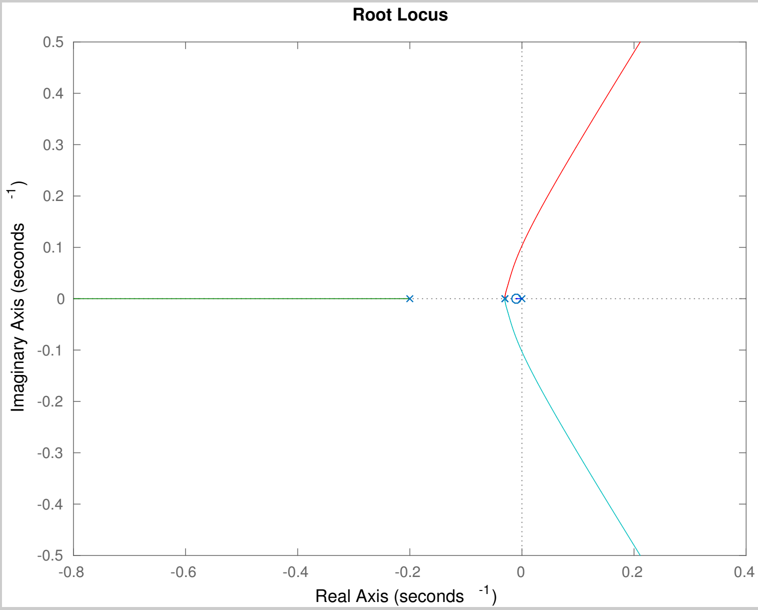
$$K(s) = \frac{1 + T_i s}{T_i s} = \frac{1 + 100s}{100s}$$

which places a single zero at $s = -0.01$.

The modified Nyquist plot would look like:



or the root locus:



(+1 point)

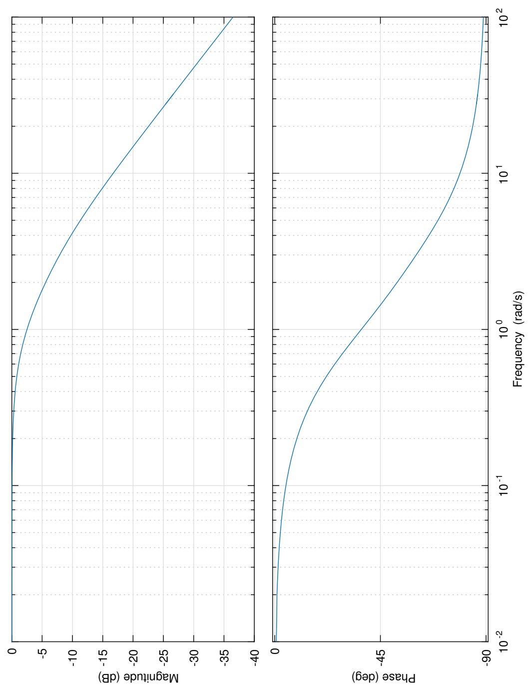
One disadvantage of T_d is the limited gain margin, which is infinite when $T_d \neq 0$. One advantage is a simpler feedback system.

(+1 point)

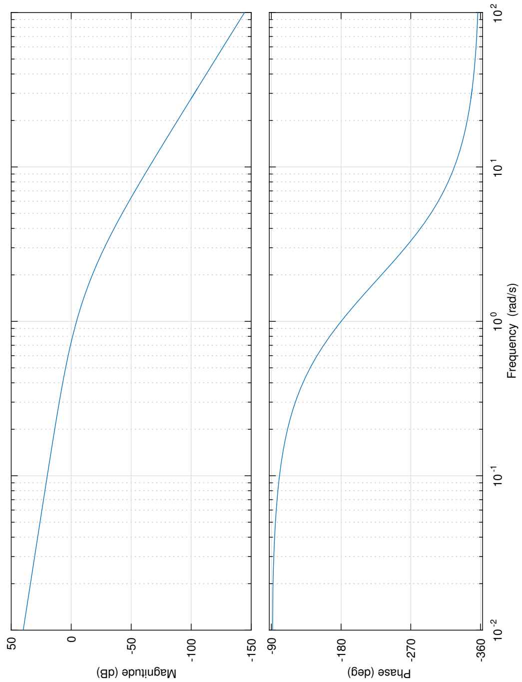
Have a great summer!

NAME: **STUDENT ID:**

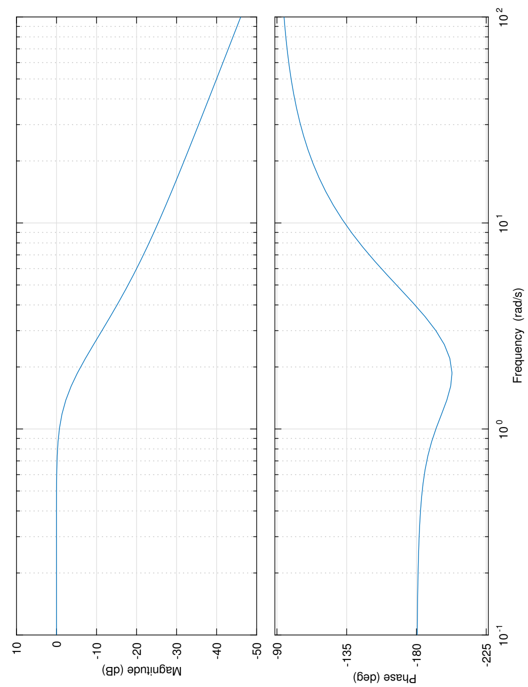
WRITE YOUR NAME AND ID AND
DO NOT FORGET TO RETURN THIS PAGE WITH YOUR EXAM!



(a)



(b)



(c)

Figure 2: Diagrams for Question 3

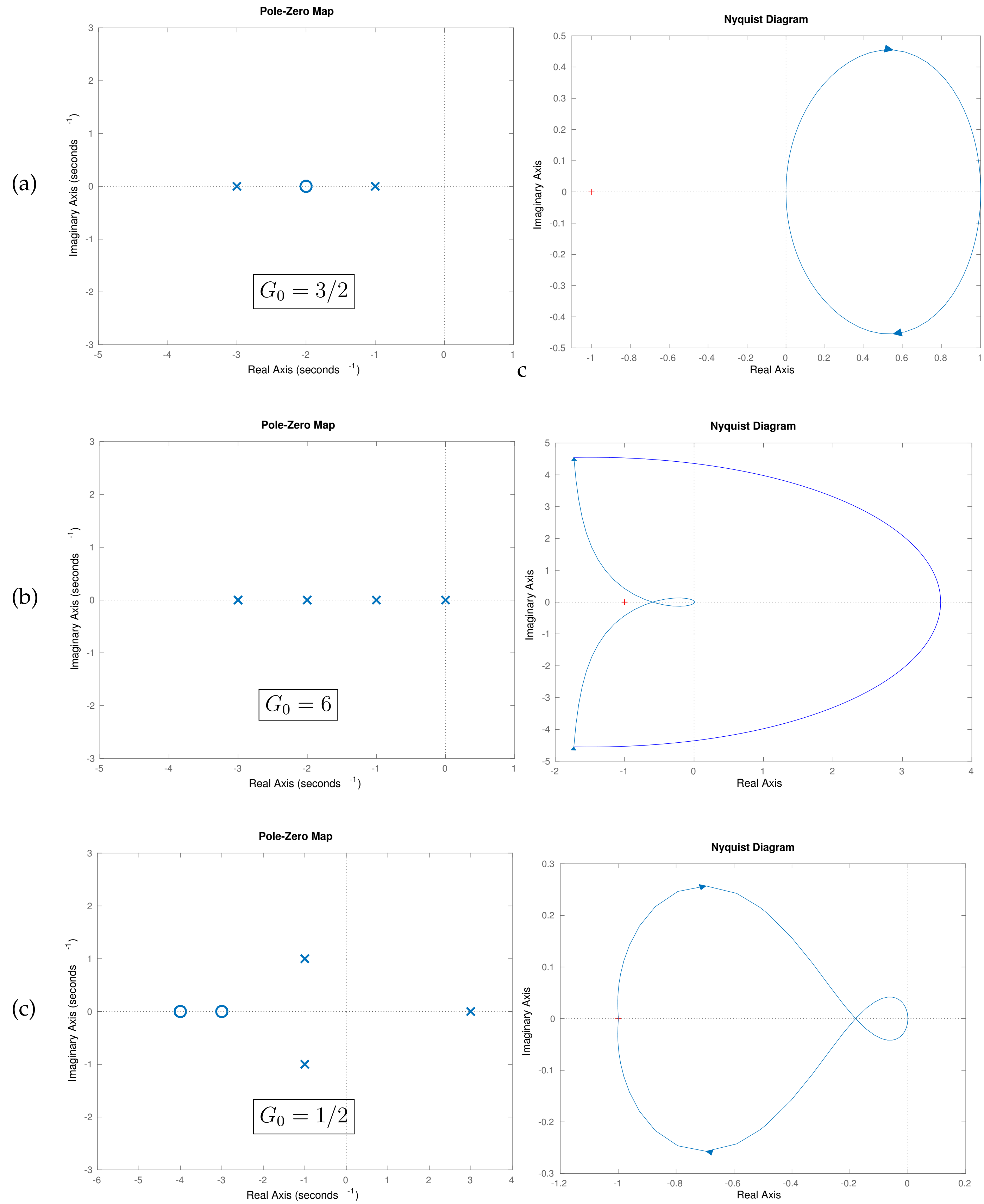


Figure 1: Diagrams for Questions 1 and 2