MAE 143B - Homework 2

Note about Steady-State Components

Notice that in this class, we define the steady-state component of a signal as in Chapter 3 of your textbook. That is, every term of a signal corresponding to poles of its Laplace transform on the imaginary axis – even if these are repeated – is considered part of the signal's steady-state component.

Graders: For this homework, it's alright to only consider contributions of simple imaginary axis poles in the steady-state components.

2.29

From the mechanical diagram, we immediately get

$$m\ddot{x} + b\dot{x} + kx = mg\cos(\pi/2 - \theta) = mg\sin(\theta).$$

2.30

Using the transformation $y = x - k^{-1}mg\sin(\theta)$, we have

$$m\ddot{y} + b\dot{y} + ky = 0.$$

This allows us to analyze a horizontal version of the system instead.

3.3

b)

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s+1}\right\} = \mathcal{L}^{-1}\left\{1\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} = \delta(t) - 2e^{-t}1(t)$$

e)

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-1)}\right\} = \sum_{i=1}^{2}\operatorname*{Res}_{s=p_i}\left\{\frac{1}{(s+1)(s-1)}e^{st}\right\} = \frac{1}{2}\left(e^{t} - e^{-t}\right)1(t)$$

h)

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} = 1(t) - 1(t-1)$$

3.9

b)

The transfer function is

$$H(s) = \mathcal{L}\left\{e^{t} - e^{-t}\right\} = \frac{1}{s-1} - \frac{1}{s+1} = \frac{2}{(s-1)(s+1)}.$$

Given that one of the two poles has positive real part, this system is not asymptotically stable. A constant input $u(t) = \tilde{u} \cdot 1(t)$ has Laplace transform $U(s) = \tilde{u}/s$, so that

$$Y(s) = H(s)U(s) = \frac{2}{s(s-1)(s+1)}\tilde{u},$$

with poles at $p_1 = -1$, $p_2 = 0$, $p_3 = 1$. The output $y(t)$ follows as

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \sum_{i=1}^{3} \operatorname*{Res}_{s=p_i}\left\{Y(s)e^{st}\right\} = \tilde{u}\left(e^{t} + e^{-t} - 2\right)1(t),$$

with respective transient and steady-state components

$$y_{tr}(t) = \tilde{u}e^{-t} \cdot 1(t), \qquad y_{ss}(t) = -2\tilde{u} \cdot 1(t).$$

d)

The transfer function is

$$H(s) = \mathcal{L}\left\{e^{-t}\cos(t)\right\} = \frac{1/2}{s+1-j} + \frac{1/2}{s+1+j} = \frac{s+1}{s^2+2s+2}.$$

Given that both poles have strictly negative real parts, this system is asymptotically stable. The response to a constant input as above is

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \sum_{i=1}^{3} \operatorname*{Res}_{s=p_i}\left\{Y(s)e^{st}\right\} = \left(\frac{1}{2} + e^{-t}\left(\frac{-1-j}{4}e^{jt} + \frac{-1-j}{4}e^{-jt}\right)\right)\tilde{u} =$$
$$\frac{\tilde{u}}{2}\left(1 + e^{-t}\left(\sin(t) - \cos(t)\right)\right)1(t) = \frac{\tilde{u}}{2}\left(1 + \sqrt{2}e^{-t}\cos(t - 3\pi/4)\right)1(t),$$

with respective transient and steady-state components

$$y_{tr}(t) = \frac{\sqrt{2}}{2}\tilde{u}e^{-t}\cos(t - 3\pi/4)1(t), \qquad y_{ss}(t) = \frac{\tilde{u}}{2}1(t).$$

3.54

From the ordinary differential equation, we have

$$(J_1 r_1^2 + J_2 r_2^2)s\Omega_1(s) = r_2^2 T(s),$$

so that

$$H_1(s) = \frac{\Omega_2(s)}{T(s)} = \frac{r_1}{r_2}\frac{\Omega_1(s)}{T(s)} = \frac{r_1 r_2}{(J_1 r_1^2 + J_2 r_2^2)s}$$

and

$$H_2(s) = \frac{\Theta_2(s)}{T(s)} = \frac{\Omega_2(s)}{sT(s)} = \frac{H_1(s)}{s} = \frac{r_1 r_2}{(J_1 r_1^2 + J_2 r_2^2)s^2}.$$

Given their poles at the origin, neither of these transfer functions is asymptotically stable.

3.55

The Laplace transform of a constant input $\tau(t) = \tilde{\tau} \cdot 1(t)$ is $T(s) = \tilde{\tau}/s$, resulting in

$$\theta_2(t) = \mathcal{L}^{-1}\left\{\frac{r_1 r_2}{(J_1 r_1^2 + J_2 r_2^2)s^3}\tilde{\tau}\right\} = \frac{r_1 r_2 \tilde{\tau}}{2(J_1 r_1^2 + J_2 r_2^2)} t^2 \cdot 1(t)$$

and

$$\omega_2(t) = \mathcal{L}^{-1}\left\{\frac{r_1 r_2}{(J_1 r_1^2 + J_2 r_2^2)s^2}\tilde{\tau}\right\} = \frac{r_1 r_2 \tilde{\tau}}{J_1 r_1^2 + J_2 r_2^2} t \cdot 1(t).$$

Both of these signals have zero transient components and equal their steady-state components, i.e.

$$\theta_2(t) = \theta_{2,ss}(t), \qquad \omega_2(t) = \omega_{2,ss}(t).$$

For the sinusoidal input $\tau(t) = \tilde{\tau}\cos(\omega t)1(t)$, we get

$$T(s) = \frac{s}{s^2+1}\tilde{\tau} = \frac{s}{(s+j)(s-j)}\tilde{\tau}.$$

and

$$\theta_2(t) = \frac{r_1 r_2 \tilde{\tau}}{J_1 r_1^2 + J_2 r_2^2} \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2(s+j)(s-j)}\right\} = \frac{r_1 r_2 \tilde{\tau}}{J_1 r_1^2 + J_2 r_2^2}\left(1 - \frac{1}{2}\left(e^{jt} + e^{-jt}\right)\right) = \frac{r_1 r_2 \tilde{\tau}}{J_1 r_1^2 + J_2 r_2^2}\left(1 - \cos(t)\right)1(t),$$

with transient component $\theta_{2,tr}(t) = 0$ and steady-state component $\theta_{2,ss}(t) = \theta_2(t)$. The response for $\omega_2(t)$ is

$$\omega_2(t) = \frac{r_1 r_2 \tilde{\tau}}{J_1 r_1^2 + J_2 r_2^2} \cdot \mathcal{L}^{-1}\left\{\frac{s}{s(s+j)(s-j)}\right\} = \frac{r_1 r_2 \tilde{\tau}}{J_1 r_1^2 + J_2 r_2^2}\frac{1}{2j}\left(e^{jt} - e^{-jt}\right) = \frac{r_1 r_2 \tilde{\tau}}{J_1 r_1^2 + J_2 r_2^2}\sin(t)1(t),$$

with transient component $\omega_{2,tr}(t) = 0$ and steady-state component $\omega_{2,ss}(t) = \omega_2(t)$.