MAE143B HW #3 SOLUTIONS

3.60/

$$J\dot{\omega} + \left(b + \frac{K_t K_e}{R_a}\right)\omega = \frac{K_t}{R_a} V_a \qquad (1)$$

Assuming zero initial conditions, and taking the Laplace transform of both sides of (1) gives us.

$$J s W(s) + \left(b + \frac{K_t K_e}{R_a}\right) W(s) = \frac{K_t}{R_a} V_a(s)$$

$$\Rightarrow \quad W(s)\left[sJ + \left(b + \frac{K_t K_e}{R_a}\right)\right] = \frac{K_t}{R_a} V_a(s) \qquad (2).$$

Let $H(s) = \frac{\omega(s)}{V_a(s)}$ denote the transfer function with output $\omega(s)$ and input $V_a(s)$. Rearranging (2) above, we get :

$$H(s) = \frac{W(s)}{V_a(s)} = \frac{K_t / R_a}{sJ + b + \frac{K_t K_e}{R_a}} = \underline{\frac{K_t}{R_a \cdot J} \cdot \frac{1}{s + \frac{b}{J} + \frac{K_t K_e}{R_a J}}}$$

If all constants are assumed to be positive, then

$$\underline{s = -\frac{b}{J} - \frac{K_t K_e}{R_a J}}$$

is the only pole of the system. As this pole is strictly negative, we have asymptotic stability.

3.61

Calculate the motor's angular velocity $\omega(t)$ obtained in response to a constant armature voltage input $v_a(t) = \hat{v}_a$, $t \geq 0$.

So, if $v_a(t) = \hat{v}_a$ for $t \geq 0$, then by taking the Laplace transform we have that.

$$v_a(s) = \frac{\hat{v}_a}{s}.$$

From (3.60), we have that.

$$H(s) = \frac{\omega(s)}{v_a(s)} = \frac{K_t}{R_a J} \cdot \frac{1}{s + \frac{b}{J} + \frac{K_t K_e}{R_a J}}$$

$$\Rightarrow \omega(s) = H(s) \cdot v_a(s)$$

$$= \frac{K_t}{R_a J} \cdot \frac{1}{s + \frac{b}{J} + \frac{K_t K_e}{R_a J}} \cdot \frac{\hat{v}_a}{s}$$

$$\omega(s) = \frac{\frac{K_t \hat{v}_a}{R_a J}}{s\left(s + \frac{b}{J} + \frac{K_t K_e}{R_a J}\right)}$$

To take the inverse Laplace transform, we can first resolve $\omega(s)$ with a partial fraction expansion. Namely:

$$\omega(s) = \frac{A}{s} + \frac{B}{s + \frac{b}{J} + \frac{K_t K_e}{R_a J}}$$

$\Rightarrow \cancel{\omega(t)} =$

We can equate coefficients to determine the value of A and B.

$$\frac{A}{s} + \frac{B}{s + \frac{b}{J} + \frac{K_t K_e}{R_a J}} = \frac{\frac{K_t}{R_a J}\hat{V}_a}{s\left(s + \frac{b}{J} + \frac{K_t K_e}{R_a J}\right)}$$

$$\Rightarrow \quad \frac{A\left(s + \frac{b}{J} + \frac{K_t K_e}{R_a J}\right) + Bs}{s\left(s + \frac{b}{J} + \frac{K_t K_e}{R_a J}\right)} = \frac{\frac{K_t \hat{V}_a}{R_a J}}{s\left(s + \frac{b}{J} + \frac{K_t K_e}{R_a J}\right)}$$

$$\Rightarrow \quad A + B = 0 \qquad \text{(equating 's' coefficient)}$$

$$A\left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right) = \frac{K_t \hat{V}_a}{R_a J}$$

$$\Rightarrow \quad \underline{A = \frac{K_t \hat{V}_a}{bR_a + K_t K_e}} \qquad B = -\frac{K_t \hat{V}_a}{bR_a + K_t K_e}$$

Now, we can use the Laplace transform table to take the inverse Laplace transform.

$$\omega(t) = A1(t) + Be^{-t\left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)}$$

$$= \frac{K_t \hat{V}_a}{bR_a + K_t K_e}\left(1(t) - e^{-t\left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)}\right)$$

We note that as $t \to \infty$, $\dfrac{K_t \hat{V}_a}{bR_a + K_tK_e}\, e^{-t\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)} \to 0$, so this is the ~~steady~~ transient state component. The other component, $\dfrac{K_t\hat{V}_a}{bR_a + K_tK_e}\, 1(t)$ persists in time, so it is the steady state response.

Now, we have to repeat the question for the input $v_a(t) = \hat{V}_a \cos(\omega t)$, $t \geq 0$.
First, let us take the Laplace transform of the input,

$$V_a(s) = \frac{\hat{V}_a s}{s^2 + \omega^2}$$

Now, following the same steps as before, we have, $\omega(s) = H(s) \cdot \hat{V}_a(s)$

i.e.

$$\omega(s) = \frac{K_t}{R_aJ} \cdot \frac{1}{\left(s + \frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)} \cdot \frac{\hat{V}_a s}{(s^2 + \omega^2)}$$

$$= \frac{K_t\hat{V}_a}{R_aJ} \cdot \frac{s}{\left(s + \frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)\cdot\left(s^2 + \omega^2\right)}$$

Again, we can take a partial fractions approach and get the following:

$$\omega(s) = \frac{A}{s + \frac{b}{J} + \frac{K_tK_e}{R_aJ}} + \frac{B}{s + j\omega} + \frac{B^*}{s - j\omega}$$

Here, we have complex conjugate roots. Let us use a different method to solve our partial fraction expression

$$A = \lim_{s \to -\left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)} \left(s + \frac{b}{J} + \frac{K_t K_e}{R_a J}\right) \cdot \frac{K_t \tilde{V}_a}{R_a J} \cdot \frac{s}{\left(s + \frac{b}{J} + \frac{K_t K_e}{R_a J}\right) \cdot (s^2 + \omega^2)}$$

$$= \frac{-\frac{K_t}{R_a J} \hat{V}_a \left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)}{\left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)^2 + \omega^2}$$

$$B = \lim_{s \to -j\omega} \frac{(s + j\omega)\left(\frac{K_t \tilde{V}_a}{R_a J}\right) s}{\left(s + \frac{b}{J} + \frac{K_t K_e}{R_a J}\right)(s + j\omega)(s - j\omega)}$$

$$B = \frac{\frac{K_t}{R_a J} \tilde{V}_a}{2\left(\frac{b}{J} + \frac{K_t K_e}{R_a J} - j\omega\right)} = \frac{K_t \tilde{V}_a}{2 R_a J \left(\frac{b}{J} + \frac{K_t K_e}{R_a J} - j\omega\right)}$$

$$B^* = \frac{K_t \hat{V}_a}{2 R_a J \left(\frac{b}{J} + \frac{K_t K_e}{R_a J} + j\omega\right)}$$

We have that

$$W(s) = \frac{A}{\left(s + \frac{b}{J} + \frac{K_t K_e}{R_a J}\right)} + \frac{B}{s + j\omega} + \frac{B^*}{s - j\omega}$$

Taking the inverse Laplace transform (by looking at the table),

$$\omega(t) = \left[A e^{-t\left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)} + B e^{-j\omega t} + B^* e^{j\omega t} \right] 1(t)$$

We have,

$$B^* = \frac{K_t \hat{V}_a}{2 R_a J} \cdot \frac{1}{\left(\frac{b}{J} + \frac{K_t K_e}{R_a J} + j\omega\right)}$$

so,

$$\frac{b}{J} + \frac{K_t K_e}{R_a J} + j\omega = \sqrt{\left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)^2 + \omega^2}\; e^{j\,\mathrm{atan2}\left(\omega,\, \frac{b}{J} + \frac{K_t K_e}{R_a J}\right)}$$

so.

$$B^* = \frac{K_t \hat{V}_a}{2 R_a J} \cdot \left(\left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)^2 + \omega^2\right)^{-\frac{1}{2}} e^{-j\,\mathrm{atan2}\left(\omega,\, \frac{b}{J} + \frac{K_t K_e}{R_a J}\right)}$$

$$B = \underbrace{\frac{K_t \hat{V}_a}{2 R_a J} \cdot \left(\left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)^2 + \omega^2\right)^{-\frac{1}{2}}}_{|B|} e^{+j\,\underbrace{\mathrm{atan2}\left(\omega,\, \frac{b}{J} + \frac{K_t K_e}{R_a J}\right)}_{\theta}}$$

so,

$$B^* = |B| e^{-j\theta}$$

$$B = |B| e^{j\theta}$$

so,

$$w(t) = Ae^{-t\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)} + |B|e^{j\theta}e^{-j\omega t} + |B|e^{-j\theta}e^{j\omega t}$$

$$= Ae^{-t\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)} + |B|\left(e^{j\theta}e^{-j\omega t} + e^{-j\theta}e^{j\omega t}\right)$$

$$= Ae^{-t\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)} + |B|\left(e^{j\theta - j\omega t} + e^{j\omega t - j\theta}\right)$$

$$= Ae^{-t\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)} + |B|\left(e^{j(\theta - \omega t)} + e^{-j(\theta - \omega t)}\right)$$

$$= Ae^{-t\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)} + |B|\cos(\theta - \omega t)$$

$$= Ae^{-t\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)} + |B|\cos(\omega t - \theta)$$ (by the properties of cosine)

$$= Ae^{-t\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)} + |B|\cos\left(\omega t - \operatorname{atan2}\left(\omega, \frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)\right)$$

$$= \underbrace{Ae^{-t\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)}}_{\text{transient}} + \underbrace{\frac{K_t\tilde{V}_a}{2R_aJ}\left(\left(\frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)^2 + \omega^2\right)^{-\frac{1}{2}} \cos\left(\omega t + \operatorname{atan2}\left(-\omega, \frac{b}{J} + \frac{K_tK_e}{R_aJ}\right)\right)}_{\text{steady state}}$$

(3.62)

$$J\dot{\omega} + \left(b + \frac{K_t K_e}{R_a}\right)\omega = \frac{K_t}{R_a} V_a$$

Taking the Laplace transform of both sides gives us

$$sJ\omega(s) + \left(b + \frac{K_t K_e}{R_a}\right)\omega(s) = \frac{K_t}{R_a} V_a(s)$$

$$\Rightarrow \quad \frac{\omega(s)}{V_a(s)} = \frac{K_t / R_a}{sJ + b + \frac{K_t K_e}{R_a}}$$

$$= \frac{K_t / R_a J}{s + \frac{b}{J} + \frac{K_t K_e}{R_a J}}$$

Now we substitute $s = j\omega$ into the transfer function above and determine the phase and magnitude.

$$\frac{\omega(j\omega)}{V_a(j\omega)} = \frac{K_t / R_a J}{j\omega + \frac{b}{J} + \frac{K_t K_e}{R_a J}}$$

Let us first compute the magnitude.

$$\left|\frac{\omega(j\omega)}{V_a(j\omega)}\right| = \frac{K_t}{R_a J} \cdot \frac{1}{\sqrt{\omega^2 + \left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)^2}}$$

Now, let us compute the phase.

$$\frac{\omega(j\omega)}{V_a(j\omega)} = \frac{K_t}{R_a J} \cdot \frac{1}{j\omega + \frac{b}{J} + \frac{K_t K_e}{R_a J}}$$

$$= \frac{K_t}{R_a J} \cdot \frac{\frac{b}{J} + \frac{K_t K_e}{R_a J} - j\omega}{\omega^2 + \left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)^2}$$

$$= \frac{\frac{K_t}{R_a J}\left(\frac{K_t K_e + R_a b}{R_a J} - j\omega\right)}{\left(\frac{K_t K_e + R_a b}{R_a J}\right)^2 + \omega^2}$$

so, $$\angle\left(\frac{\omega(j\omega)}{V_a(j\omega)}\right) = \operatorname{atan}\left(-\omega, \frac{K_t K_e + R_a b}{R_a J}\right)$$

So, we can write:

$$y_{ss}(t) = A|G(j\omega)|\cos(\omega t + \phi + \angle G(j\omega))$$

For the first signal from (3.61), we have $A = \hat{V}_a$, $\omega = 0$, $\phi = 0$.

$$y_{ss}(t) = \hat{V}_a \cdot \frac{K_t}{R_a J} \cdot \frac{1}{\frac{b}{J} + \frac{K_t K_e}{R_a J}} \cos(0) \quad 1(t),$$

which is the same expression as we got before.

Similarly, if $v_a(t) = \tilde{V}_a \cos(\omega t)$,
we have $A = \tilde{V}_a$,

$\omega = \omega$,

$\phi = 0$,

which gives us.

$$y_{ss}(t) = \tilde{V}_a \cdot \frac{K_t}{R_a J} \cdot \frac{1}{\sqrt{\omega^2 + \left(\frac{b}{J} + \frac{K_t K_e}{R_a J}\right)^2}} \cos\left(\omega t + \operatorname{atan}\left(-\omega, \frac{K_t K_e + R_a b}{R_a J}\right)\right)$$

4.i)

$$G(s) = \frac{a}{s+a} \qquad K(s) = K,$$

$$S = \frac{1}{1+GK} = \frac{1}{\frac{Ka}{s+a}+1} = \frac{s+a}{Ka+s+a}$$

$$S = \underline{\underline{\frac{s+a}{s+a(K+1)}}} \quad \text{(1st order)}$$

$$D = \frac{G}{1+GK} = \frac{\frac{a}{s+a}}{1+\frac{Ka}{s+a}} = \underline{\underline{\frac{a}{s+a(K+1)}}} \quad \text{(1st order)}$$

$$H = \frac{GK}{1+GK} = \frac{\frac{a}{s+a}\cdot K}{1+\frac{Ka}{s+a}} = \frac{\frac{Ka}{s+a}}{\frac{Ka+a+s}{s+a}} = \underline{\underline{\frac{Ka}{s+a(1+K)}}}$$

(1st order)

$$Q = \frac{K}{1+GK} = \frac{K}{1+\frac{Ka}{s+a}} = \frac{K}{\frac{s+a+Ka}{s+a}}$$

$$= \underline{\frac{K/(s+a)}{s+a(1+K)}} \quad \text{(1st order)}$$

(4.2)

(c) $G = \dfrac{a}{s+a}$ $\qquad K = K_e \dfrac{s+c}{s+b}$

$$S = \frac{1}{1+GK} = \frac{1}{1 + \frac{a}{s+a} \cdot K_e \frac{s+c}{s+b}}$$

$$= \frac{1}{\frac{(s+a)(s+b) + K_e a(s+c)}{(s+a)(s+b)}} = \underline{\underline{\frac{(s+a)(s+b)}{(s+a)(s+b) + K_e a(s+c)}}}$$

which is a second order transfer function.

$$D = \frac{\frac{a}{s+a}}{1 + \frac{a}{s+a} \cdot K_e \frac{s+c}{s+b}} = \underline{\underline{\frac{a(s+b)}{(s+a)(s+b) + aK_e(s+c)}}}$$

which is a second order transfer function.

$$H = \frac{\frac{a}{s+a} \cdot K_\ell \frac{s+c}{s+b}}{1 + \frac{a}{s+a} \cdot K_\ell \frac{s+c}{s+b}} = \frac{\frac{a}{s+a} \cdot K_\ell \cdot \frac{s+c}{s+b}}{\frac{(s+a)(s+b) + aK_\ell(s+c)}{(s+a)(s+b)}}$$

$$= \frac{aK_\ell(s+c)}{(s+a)(s+b) + aK_\ell(s+c)}$$

which is a second order transfer function.

$$Q = \frac{\frac{K_\ell(s+c)}{(s+b)}}{1 + \frac{a}{s+a} \cdot K_\ell \frac{s+c}{s+b}} = \frac{\frac{K_\ell(s+c)}{(s+b)}}{\frac{(s+a)(s+b) + aK_\ell(s+c)}{(s+a)(s+b)}}$$

$$= \frac{K_\ell(s+c)(s+a)}{(s+a)(s+b) + aK_\ell(s+c)}$$

Also second order

(4.2d)

$$G = \frac{a_2}{s^2 + a_1 s + a_2}, \qquad K = K_p \, *$$

$$S = \frac{1}{1+GK} = \frac{1}{1 + \dfrac{K_p a_2}{s^2 + a_1 s + a_2}}$$

$$= \frac{s^2 + a_1 s + a_2}{s^2 + a_1 s + a_2 + K_p a_2}$$ 2^d order transfer function

$$D = \frac{\dfrac{a_2}{s^2 + a_1 s + a_2}}{1 + \dfrac{K_p a_2}{s^2 + a_1 s + a_2}} = \frac{a_2}{s^2 + a_1 s + a_2 + K_p a_2}$$ 2^nd order transfer function

$$H = D \cdot K_p = \frac{K_p a_2}{s^2 + a_1 s + a_2 + K_p a_2}$$

$$Q = K_p(S) = \frac{K_p(s^2 + a_1 s + a_2)}{s^2 + a_1 s + a_2 + K_p a_2}$$ 2^nd order T.F.

(4.A)

Here, we have a system which is connected in a Positive feedback arrangement where the output is added back into the system. This differs from the typical negative feedback configurations that we employ.

Positive feedback typically results in a unstable system, because the output continues to grow in an unbounded fashion.

There are situations where we can use positive feedback, e.g. Schmitt Triggers.

NOTE: FOR THIS HW, WE CONSIDERED ASYMPTOTIC STABILITY FOR 4.16, NOT INTERNAL STABILITY.

4.16)

$$J_r \dot{\omega}_1 + b_r \omega_1 = r_2^2 \tau \quad , \qquad \omega_2 = \left(\frac{r_1}{r_2}\right)\omega_1 \qquad (1)$$

$$J_r = J_1 r_2^2 + J_2 r_1^2 , \qquad b_r = b_1 r_2^2 + b_2 r_1^2$$

let us first take a Laplace transform of (1). we note that τ is the input here and ω_2 is the output. So,

$$J_r \left(\frac{\dot{\omega}_2 r_2}{r_1}\right) + \frac{b_r \omega_2 r_2}{r_1} = r_2^2 \tau$$

$$\frac{s J_r r_2}{r_1} \omega_2(s) + \omega_2(s) \frac{b_r r_2}{r_1} = r_2^2 \tau(s)$$

$$\omega_2(s) \left[\frac{s J_r r_2}{r_1} + \frac{b_r r_2}{r_1}\right] = r_2^2 \tau(s) \qquad (2)$$

~~$\omega_2(s)$~~ we also have that $\tau = K(\bar{\omega}_2 - \omega_2)$. so, let us take the Laplace transform of our expression for τ.

$$\tau(s) = K(\bar{\omega}_2(s) - \omega_2(s)) \qquad (3)$$

(3) $\Rightarrow$ (2)

$$\omega_2(s)\left[\frac{s J_r r_2}{r_1} + \frac{b_r r_2}{r_1}\right] = r_2^2 K\left[\bar{\omega}_2(s) - \omega_2(s)\right]$$

$$\omega_2(s)\left[\frac{s J_r r_2}{r_1} + \frac{b_r r_2}{r_1} + r_2^2 K\right] = r_2^2 K \bar{\omega}_2(s)$$

Divide out by r_2^2

$$\omega_2(s)\left[\frac{sJ_r}{r_1 r_2} + \frac{b_r}{r_1 r_2} + K\right] = K\bar{\omega}_2(s)$$

$$\frac{\omega_2(s)}{\bar{\omega}_2(s)} = \frac{K}{\frac{sJ_r}{r_1 r_2} + \frac{b_r}{r_1 r_2} + K}$$

$$= \frac{K/(r_1 r_2 J_r)}{s + \frac{b_r}{J_r} + \frac{K r_1 r_2}{J_r}}$$

For system to be asymptotically stable, we require,

$$\cancel{s + \frac{b}{J_r} + \frac{K r_1 r_2}{J_r} < 0}$$

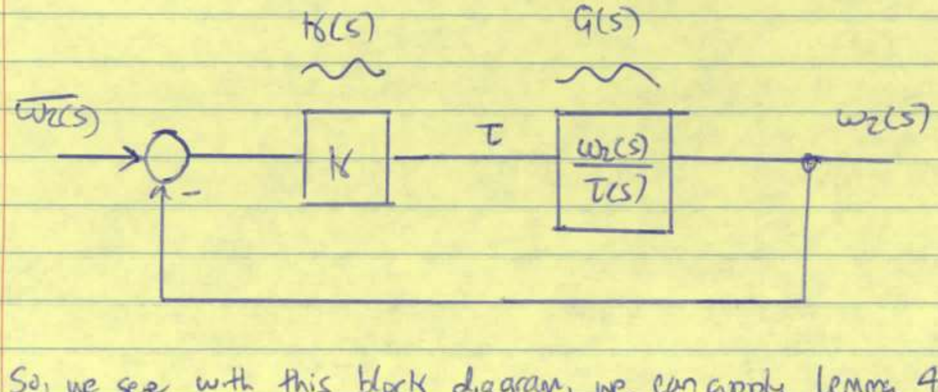
$$\frac{b_r}{J_r} + \frac{K r_1 r_2}{J_r} > 0$$

$$\Rightarrow b_r + K r_1 r_2 > 0$$

$$\Rightarrow K > -\frac{b_r}{r_1 r_2}$$

$$K > -1.875$$

We will use lemma 4.1 to solve this question, although other methods can be used. Let us first draw a block diagram of our system.



So, we see with this block diagram, we can apply lemma 4.1 directly. We first need to check that the corresponding sensitivity TF is asymptotically stable, but we know for $K > -1.875$, this is the case.

Thus, we need to check that $K(s)G(s)$ has a pole at $s=0$ (because we want to track a constant reference).

We have that $$G(s) = \frac{\omega_2(s)}{\tau(s)} = \frac{r_2^2}{\frac{sJr r_2}{r_1} + \frac{br r_2}{r_1}}$$

$$= \frac{r_1 r_2 / Jr}{s + \frac{br}{Jr}}$$

So; we can see easily that $G(s)K(s)$ does not have a pole at $s=0$, so we do not satisfy our asymptotic tracking criteria.

NOTE:

In the first part, we derived the closed loop transfer function directly. However, we could have looked at the sensativity TF of the block diagram above to show if we achieve A.S or not.

We will show this here.

$$S = \frac{1}{1+GK} = \frac{1}{\frac{1}{1} + \frac{Kr_1r_2/J_r}{s+\frac{b_r}{J_r}}}$$

$$= \frac{s+\frac{b_r}{J_r}}{s+\frac{b_r}{J_r}+\frac{Kr_1r_2}{J_r}}$$

so, we require $\frac{b_r}{J_r} + \frac{Kr_1r_2}{J_r} > 0$

$$\Rightarrow \quad b_r + Kr_1r_2 > 0$$

$$K > -\frac{b_r}{r_1r_2}$$

which is the same condition as we derived before.