MAE 143B - Homework 5

5.18

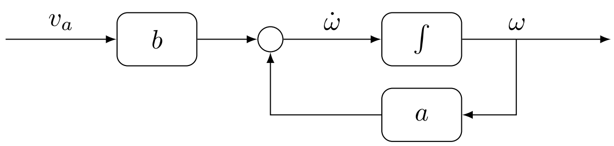


Figure 1: Block Diagram for 5.18

Have a look at Figure 1 for a block diagram representing the given equations, where

$$a = -\left(\frac{b}{J} + \frac{K_t K_e}{J R_a}\right), \qquad b = \frac{K_t}{J R_a}.$$

With state-space variables

$$x = \omega, \qquad u = v_a, \qquad y = \omega$$

we have the state-space equations

$$\dot{x} = a\,x + b\,u,$$
$$y = x.$$

5.19

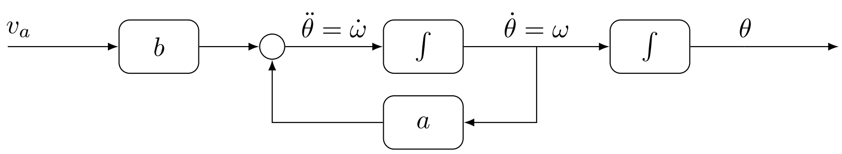


Figure 2: Block Diagram for 5.19

Have a look at Figure 2 for a block diagram representing the given equations, where a, b are as in 5.18. With state-space variables

$$x_1 = \theta, \qquad x_2 = \dot{\theta}, \qquad u = v_a, \qquad y = \theta$$

we have the state-space equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

5.26

For $b > m$, the model describes exponential growth. Similarly, we have exponential decay for $b < m$ and constant population size when $b = m$.

5.27

This nonlinear model has equilibrium points at $x_1 = 0$ and $x_2 = k$. Writing $f(x(t)) = r_0(1 - x/k)x$, we have

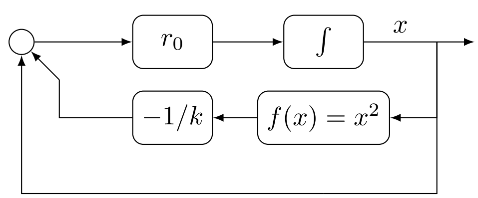
$$\frac{\partial f}{\partial x} = r_0(1 - 2x/k),$$

and

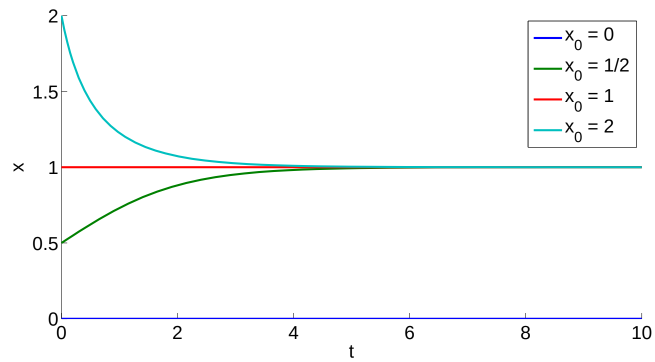
$$a_1 = \left.\frac{\partial f}{\partial x}\right|_{x_1} = r_0, \qquad a_2 = \left.\frac{\partial f}{\partial x}\right|_{x_2} = -r_0.$$

That is, by positivity of r_0, the equilibrium point at $x_1 = 0$ is unstable and the one at $x_2 = k$ asymptotically stable. Your block diagram for this problem could look as in Figure 3a. Simulation results are displayed in Figure 3b. The plots are generated by the following lines of code.

```
x0 = [0,1/2,1,2];
for kk = 1:length(x0)
    [t{kk},y{kk}] = ode45(@verhulst,[0 10],x0(kk));
end;

figure
hold all;
for kk = 1:length(x0)
    plot(t{kk},y{kk},'LineWidth',2);
end;
xlabel('t');
ylabel('x');
legend('x_0 = 0', 'x_0 = 1/2', 'x_0 = 1', 'x_0 = 2');

function dxdt = verhulst(t,x);

r0  = 1;
k   = 1;

dxdt = r0*(1-x/k)*x;
```



(a) Block Diagram



(b) Simulation results

Figure 3: Graphics for 5.27

5.28

This nonlinear model has equilibrium points at

$$\begin{bmatrix} x_{1,a} \\ x_{2,a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_{1,b} \\ x_{2,b} \end{bmatrix} = \begin{bmatrix} m/(ea) \\ r/a \end{bmatrix}.$$

Writing

$$f(x(t)) = \begin{bmatrix} rx_1 - ax_1x_2 \\ eax_1x_2 - mx_2 \end{bmatrix},$$

we have

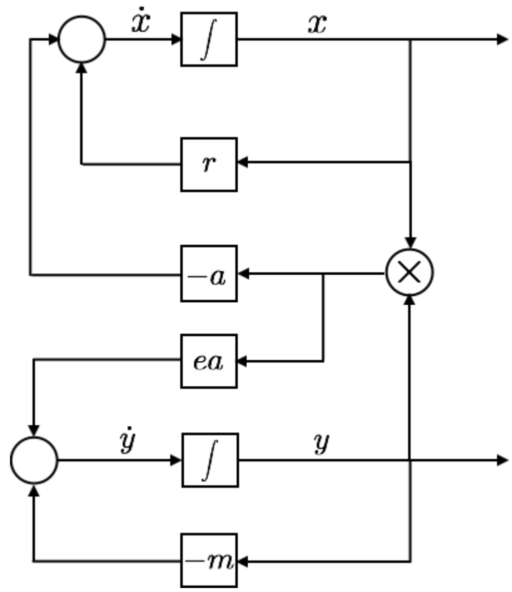
$$\frac{\partial f}{\partial x} = \begin{bmatrix} r - ax_2 & -ax_1 \\ eax_2 & eax_1 - m \end{bmatrix}$$

and

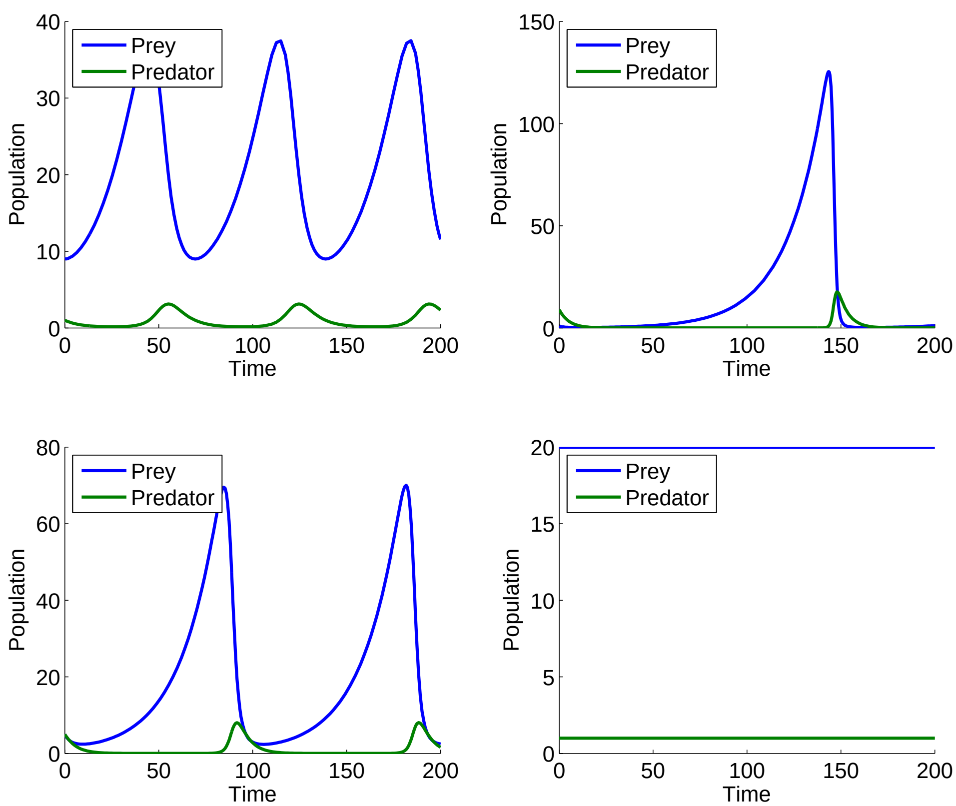
$$A_a = \left.\frac{\partial f}{\partial x}\right|_{x_a} = \begin{bmatrix} r & 0 \\ 0 & -m \end{bmatrix}, \qquad A_b = \left.\frac{\partial f}{\partial x}\right|_{x_b} = \begin{bmatrix} 0 & -m/e \\ re & 0 \end{bmatrix}.$$

A_a always has a positive eigenvalue at $s = m > 0$, implying instability of the first equilibrium point, x_a. However, A_b has complex conjugate eigenvalues on the imaginary axis, which means that we cannot assess stability of the second equilibrium point, x_b, using our linearized model. Check out Figure 4a for a block diagram, where $x \to x_1$ and $y \to x_2$. Simulations are displayed in Figure 4b and generated based on the following code.

```
x0{1} = [9;1];
x0{2} = [1;9];
x0{3} = [5;5];
x0{4} = [20;1];
for kk = 1:size(x0,2)
    [t{kk},y{kk}] = ode45(@LotkaVolterra,[0 200],x0{kk});
end;

figure
for kk = 1:size(x0,2)
    subplot(2,2,kk)
    hold on;
    plot(t{kk},[y{kk}(:,1),y{kk}(:,2)],'LineWidth',2);
    xlabel('Time');
    ylabel('Population');
    legend('Prey','Predator','Location','NorthWest');
end;

function dxdt = LotkaVolterra(t,x);

r = 0.05;
a = 0.05;
m = 0.2;
e = 0.2;

dxdt = [r*x(1) - a*x(1)*x(2);e*a*x(1)*x(2) - m*x(2)];
```



(a) Block Diagram



(b) Simulation results

Figure 4: Graphics for 5.28