MAE143B HW#6 SOLUTIONS.

6.19

We have the second order differential equation:

$$m\ddot{x} + b\dot{x} + kx = ky + b\dot{y}, \qquad (1)$$

and the change of variable $z = x - y$, rearranging this gives us, $x = z + y$, and by taking derivatives we get:

$$z = x - y$$
$$\dot{z} = \dot{x} - \dot{y}$$
$$\ddot{z} = \ddot{x} - \ddot{y}$$

Substitute these into (1) to get,

$$m(\ddot{z} + \ddot{y}) + b(\dot{z} + \dot{y}) + K(z + y) = ky + b\dot{y}$$

and cancel out common terms,

$$M\ddot{z} + b\dot{z} + kz = -M\ddot{y} \qquad (2)$$

which is the desired expression

6.20) Let us compute the transfer function. Our input is $y(t)$, the road profile and the output is $z(t)$. We take the Laplace transformation of (2) from above assuming that we have zero initial conditions:

$$ms^2 Z(s) + bs\,Z(s) + kZ(s) = -Ms^2 Y(s)$$

$$H(s) = \frac{Z(s)}{Y(s)} = \frac{-ms^2}{ms^2 + bs + k}$$

$$= \frac{-s^2}{s^2 + \frac{b}{m}s + \frac{k}{M}}$$

We note the general form of the characteristic equation for a second order system:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0,$$

Comparing this to the transfer function that we derived, we see that:

$$\omega_n^2 = \frac{k}{M} \quad \Rightarrow \quad k = \omega_n^2 \cdot M$$
$$= (2\pi f_n)^2 \cdot M$$
$$= \underline{157913.7}$$

We also need to compute the value of the damping coefficient, b.

$$\frac{b}{M} = 2\zeta\omega_n$$

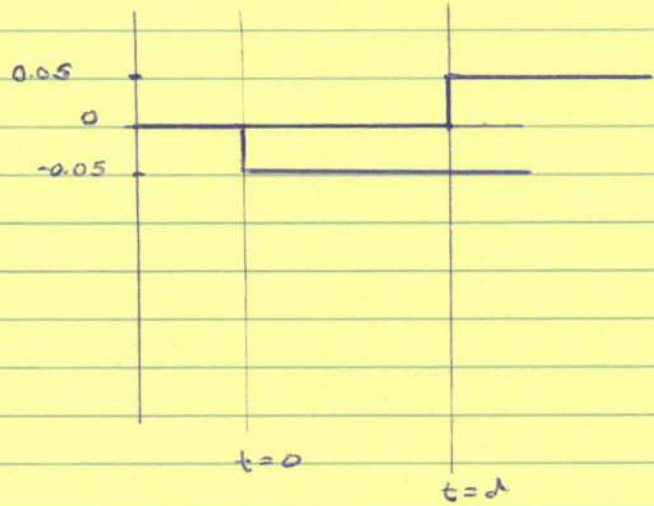
$$\underline{b = 1608.5}$$

We can also compute the roots of the characteristic equation. We have that,

$$s_1 = -\omega_n(\zeta + \sqrt{\zeta^2 - 1}) \qquad s_2 = -\omega_n(\zeta - \sqrt{\zeta^2 - 1})$$

Using the values from above we get that $s_1 = -1.257 - 15.658j$ and $s_2 = -1.257 + 15.658j$. We note that the system also has two zeros at zero.

(6.21) First, we observe that the road profile in question is actually the sum of two step functions



i.e. $y(t) = -0.05\,1(t) + 0.05\,1(t-d)$ where d is the end of the profile. Given the speed of the car, we can actually compute the value of d.

CASE #1

$t = \frac{d}{v}$

$= 1/(10 \times 1000/3600) = \underline{0.36 \text{ seconds}}$

CASE #2

$1/(100 \times 1000/3600) = \underline{0.036 \text{ seconds}}$

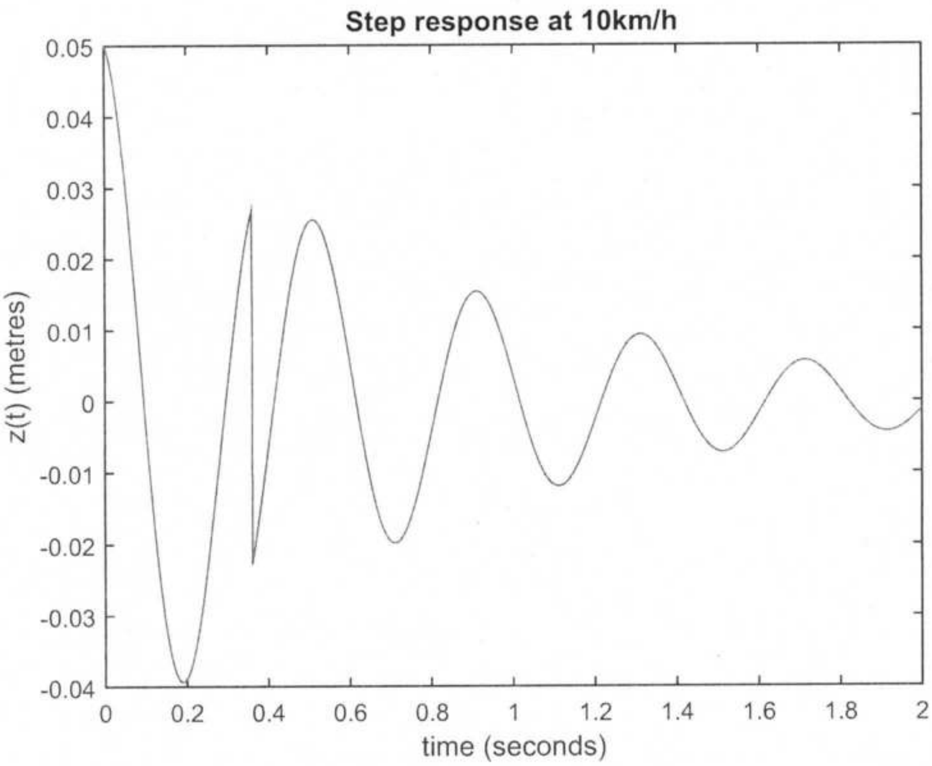
Now we are ready to plot in Matlab. We note that our system is linear, so we can look at the response to the two step functions seperately and sum the responses. and we also note the system is time invariant, so we can just shift the output response by d in the second case.

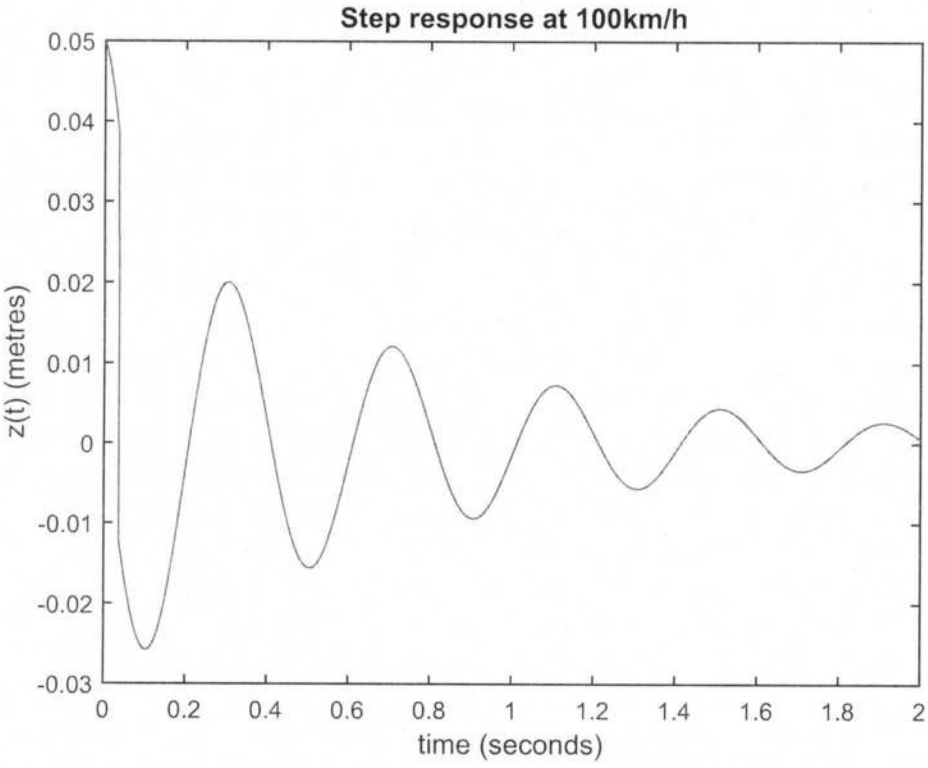
```
% Problem 6.21

% Parameters
wn    = 2.5*2*pi;
eta = 0.08;
% Define transfer function
num = [-1 0 0];
den = [1 2*eta*wn wn^2];
sys = tf(num,den);
% Determine step response
T = 0:0.001:2;
[y,t] = step(-0.05*sys,T);
[y1,~]= step(0.05*sys,T);
% Shift second response
delta = 0.036;
index = 1 + delta/0.001;

y1_shift = [zeros(1,index) y1(1:end-index)'];
plot(t,y'+y1_shift);
title('Step response at 100km/h');
xlabel('time (seconds)');
ylabel('z(t) (metres)');
set(gcf,'color','w');
% Alternative formulation
hold on
i1 = 0.05*heaviside(t-delta);
i2 = -0.05*heaviside(t);
is = i1+i2;
lsim(sys,is,t)

plot(t,is)
```


Step response at 10km/h
z(t) (metres)
time (seconds)
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


Step response at 100km/h
z(t) (metres)
time (seconds)
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

(6.22)

We expect the worst case frequency to be $\omega_d = \omega_n \sqrt{1-\zeta^2}$

$$= (2.5)(2\pi)\cdot\sqrt{1-0.08^2}$$

$$= 15.658 \text{ rad/s.}$$

which is the frequency corresponding to the peak in the magnitude plot.

We can now express the velocity as a function of this worst-case frequency

$$\frac{2\pi v}{\lambda} = \omega_d \Rightarrow v = \frac{\omega_d \cdot \lambda}{2\pi}$$

$$\cong 2.5\lambda$$

(6.10) we have, $J_r \dot{\omega}_1 + b_r \omega_1 = r_2^2 \tau$, $\omega_2 = \left(\frac{r_2}{r_1}\right)\omega_1$

where $J_r = J_1 r_2^2 + J_2 r_1^2$ $\quad b_r = b_1 r_2^2 + b_2 r_1^2$.

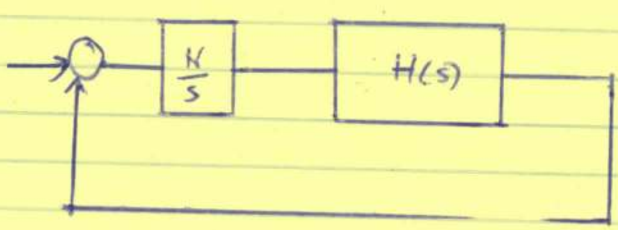
Let us start by expressing our differential equation in terms of ω_2.

$$J_r \frac{r_1}{r_2} \dot{\omega}_2 + b_r \frac{r_2}{r_1} \omega_2 = r_2^2 \tau,$$

Now we can take the Laplace transform to get the transfer function.

$$H(s) = \frac{W_2(s)}{\tau(s)} = \frac{r_1 r_2 / J_r}{s + \frac{b_r}{J_r}}$$

The form of our controller is $\tau(t) = K \int_0^t e(\sigma)\, d\sigma$, which we observe is an integral (I) controller with a gain K. Let us draw the block diagram of the entire process.



We want to design K such that both closed loop poles are real and as negative as possible. To do this we look at the roots of the characteristic equation.

$$1 + K(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{K}{s} \cdot \frac{r_1 r_2 / J_r}{s + \frac{b_r}{J_r}} = 0$$

$$\Rightarrow s\left(s + \frac{b_r}{J_r}\right) + \frac{K r_1 r_2}{J_r} = 0$$

$$\Rightarrow s^2 + \frac{s b_r}{J_r} + \frac{K r_1 r_2}{J_r} = 0$$

We can now solve for the roots of this expression which will be the closed-loop poles of our system.

$$s = \frac{-\frac{b_r}{J_r} \pm \sqrt{\left(\frac{b_r}{J_r}\right)^2 - \frac{4 K r_1 r_2}{J_r}}}{2}$$

To make the poles real and as negative as possible, we require that the term within the square root is equal to zero. This will give us two poles at $s = -\frac{b_r}{2J_r}$, so

$$\left(\frac{b_r}{J_r}\right)^2 = \frac{4 K r_1 r_2}{J_r} \Rightarrow K = \left(\frac{b_r}{J_r}\right)^2 \frac{J_r}{4 r_1 r_2} = \frac{b_r^2}{4 J_r r_1 r_2}$$

$$\hat{=}\ 12.3$$

We see that $K(s)H(s)$ has a pole at $s=0$, and given that we designed K such that the poles are negative, we are able to asymptotically track a constant reference