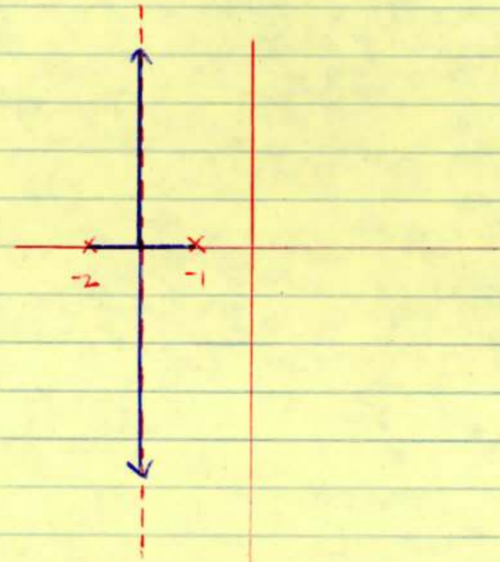
(1) MAE143B HW#7 SOLUTIONS

(6.4a)



We know root locus must exist between the pole at −1 and the pole at −2, because it is to the left off an odd-numbered singularity. We can also compute the asymptote angles and the centroid point.

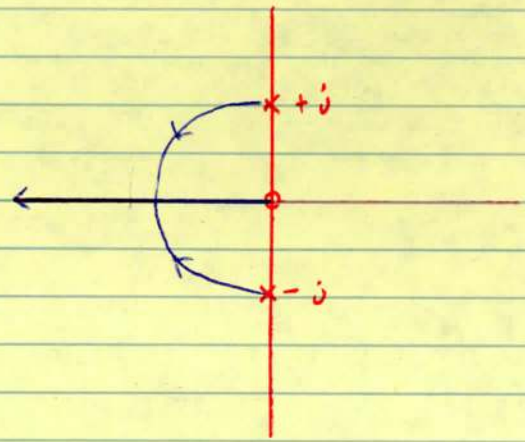
$$c = \frac{-1-2}{2} = -1.5$$

$$\phi_N = \frac{\pi + 2\pi k}{n-m}, \quad k = 0, \ldots 1$$

$$= \pm \frac{\pi}{2}$$

(6.4f)

(2)
(6.41)



We know that the root locus will exist to the left of the zero at zero. To determine the nature of the branches going to infinity, we can compute the centroid point and the asymptote angles.

$$c = 0,$$

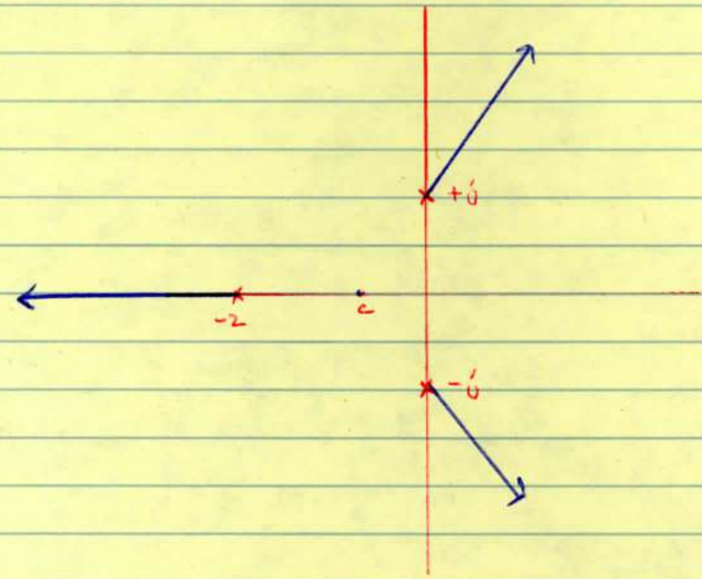
$$\phi_N = \frac{\pi + 2\pi N}{1}, \quad N = 0,$$

$$= \pi$$

So we have a single asymptote at 180°. (π radians). Thus, the branches from the two imaginary poles need to meet at the real axis to the left of the pole at zero. One pole will go to the zero and the other pole will go out to infinity.

(3)

(6.49)



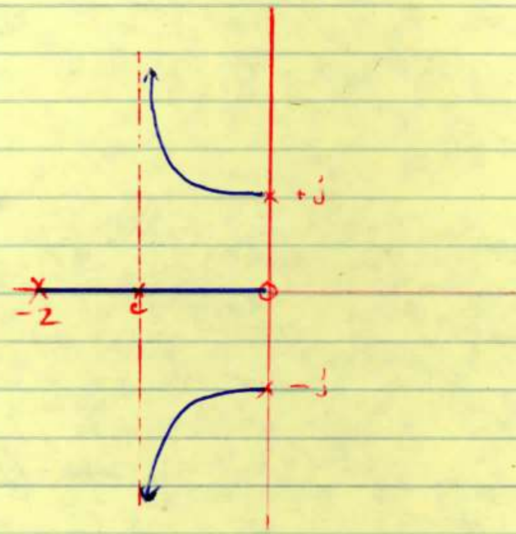
Again, we note that the root locus will exist to the left of the pole at s = 2. Let us compute the centroid and asymptote angle to determine the remaining behavior.

$c = -2/3$ (we know that only the real component of imaginary poles will contribute to the centroid location.

$$\phi_N = \frac{\pi + 2\pi N}{n-m} = \pm\frac{\pi}{3}, \pi$$

(4)

(6.4h)

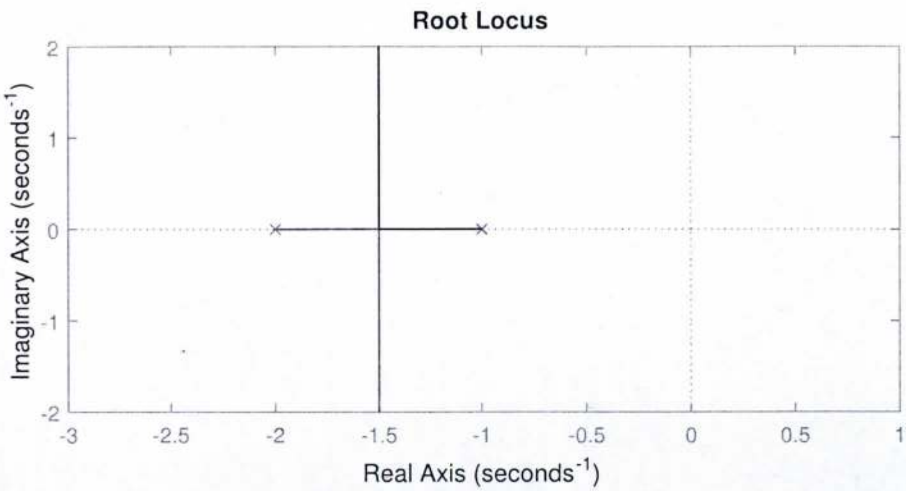


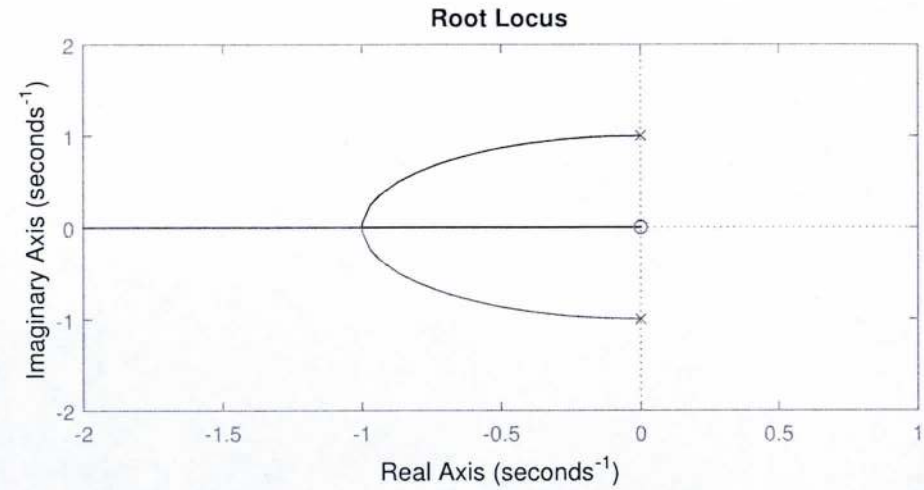
Locus will exist between the zero at $s=0$ and the pole at $s=-2$.

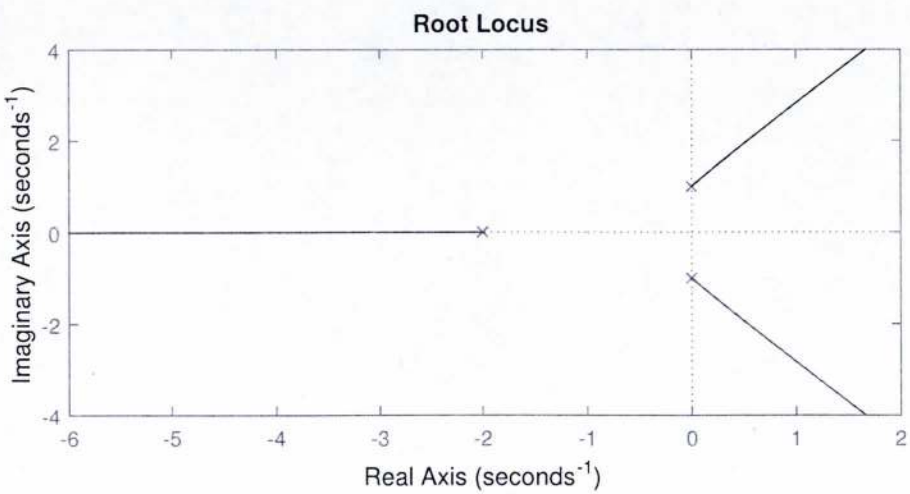
$$c = \frac{-2}{2} = -1$$

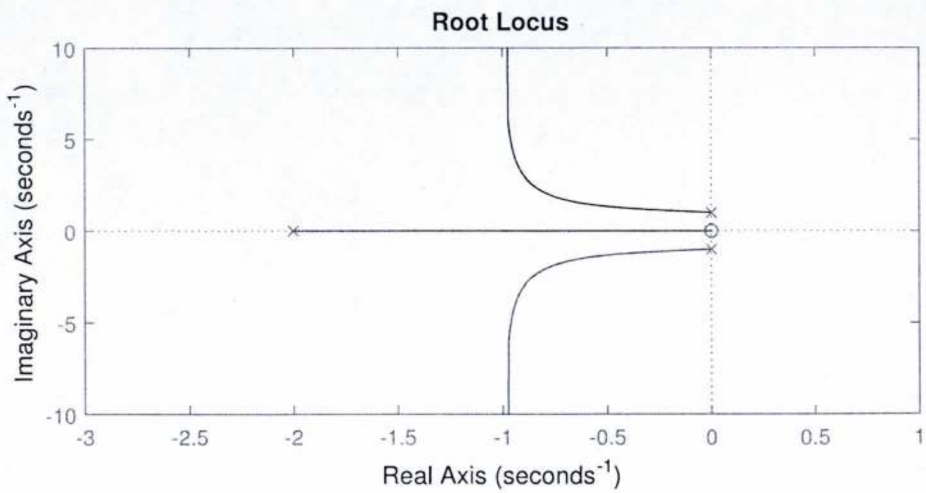
$$\phi_N = \frac{\pi + 2\pi N}{n-m} = \frac{\pi + 2\pi N}{2} \qquad N=0,1$$

$$= \underline{\pm\frac{\pi}{2}},$$

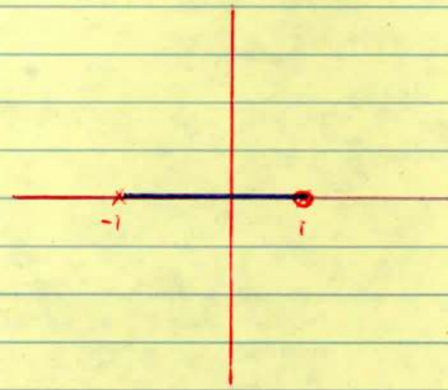

Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})

(6.5b)



Here if $K(s) = K$, the root locus lies entirely on the real axis. We can see immediately that for small values of K, the ~~root locus will lie~~ closed loop pole will lie on the left hand side. We can compute the maximal value of K for which our system is asymptotically stable.

$$S = \underbrace{\frac{1}{1 + K \cdot \frac{s+1}{s+1}}}_{1 + KH(s)}$$

We solve $1 + KH(s) = 0$ to determine the location of the closed loop poles:

$$1 + K\,\frac{s-1}{s+1} = 0$$

$$\Rightarrow \quad s + 1 + Ks - K = 0$$

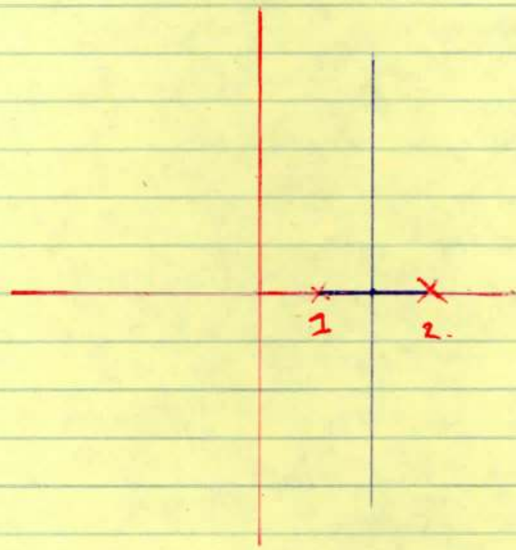
$$s(1+K) = K - 1$$

$$\underline{s = \frac{K-1}{K+1}}$$

so for $K < 1$, the value of our pole will be negative and we will have asymptotic stability.

(65c)

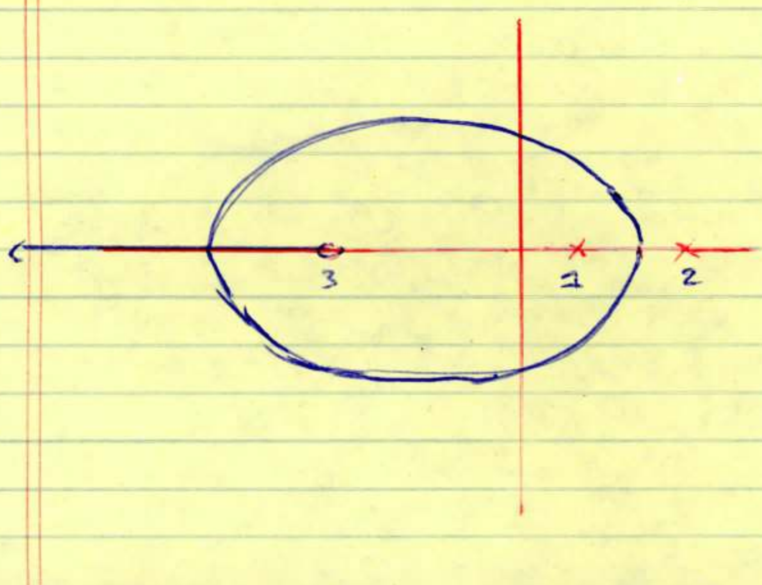
Root locus with a proportional controller only.



H(s) has two positive poles. We know that we cannot stabilize with a proportional controller K only because the root locus branches will never go to the left hand side of the complex plane.

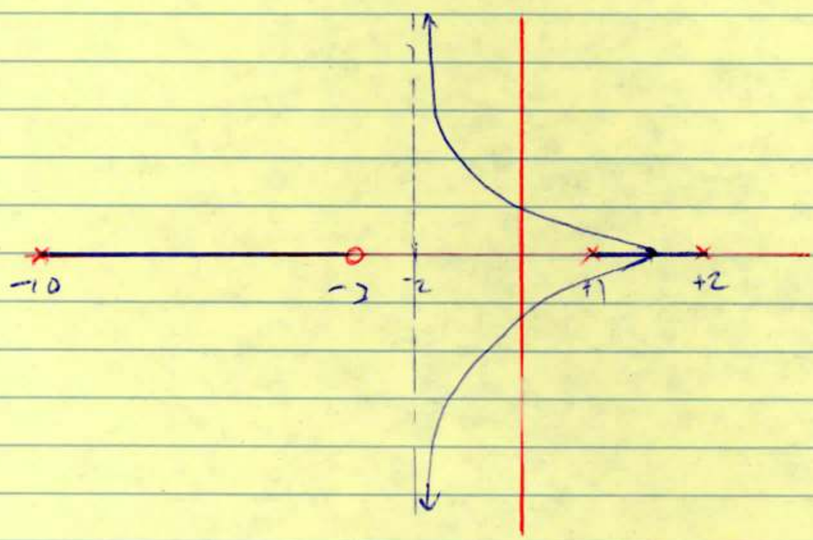
So, we need to add poles/zeros to our controller to be able to stabilize the system. Let us start by adding a zero on the LHS. We do this because we know that open loop poles will go toward open loop zeros, so this is an easy way to get the branches to go to the left hand plane.

$$K(s) = K \cdot (s+3)$$



This controller is stabilizing but it is not proper. Inorder to make it proper we need to add a pole.

$$K(s) = K \cdot \frac{(s+3)}{(s+10)}$$



To draw the root locus, we know that branches will exist to the left of the 1st pole and also to the left of the zero. We can also compute the asymptotes and centroid.

$$c = \frac{-10+1+2-(-3)}{2} = \underline{-2}$$

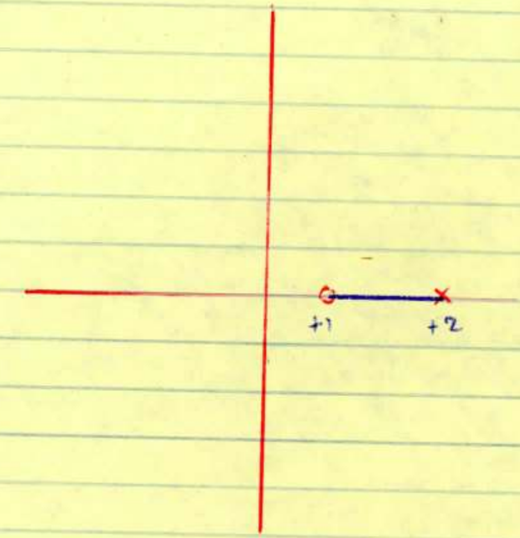
$$\phi_N = \frac{\pi + 2\pi N}{n-m} \qquad N = 0, 1, \ldots n-m-1$$

$$= \frac{\pi + 2\pi N}{2} \qquad N = 0, 1$$

$$= \pm\frac{\pi}{2}$$

We see here that even with this pole added, the branches all go to the LHS, so we can find K such that the ~~controller~~ system is asymptotically stable with a proper controller.

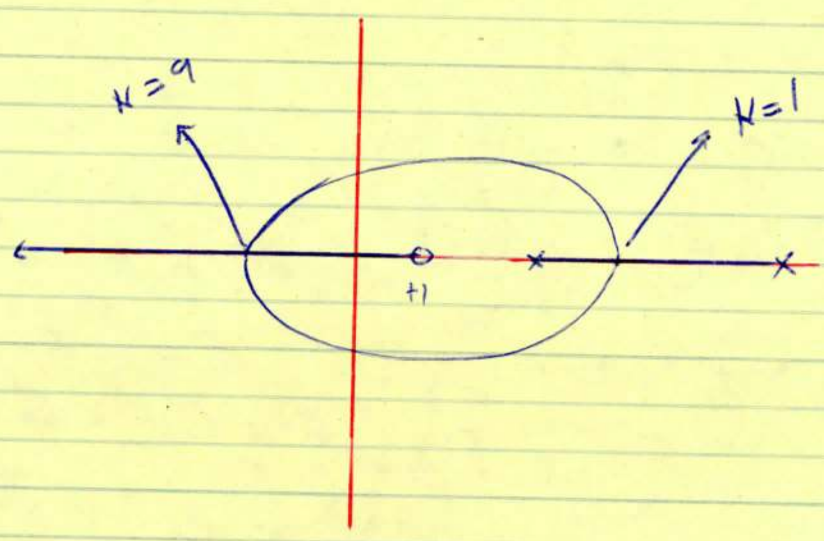
(6.5d)



We see immediately that a proportional controller alone cannot make our closed loop system asymptotically stable → the pole just goes toward the zero on the RHS.

By playing around, we can also see that adding poles and zeros on the LHS do not help, they cannot cause the branches to split and go toward the LHS of the complex plane. Let us try adding an unstable pole.

$$H(s) = K \cdot \frac{1}{(s-5)}$$



We can see here that there will exist a K which will make the closed loop system asymptotically stable

We note that the controller is proper but unstable.

What value of K should we choose?

$$1 + \frac{K}{(s-5)} \cdot \frac{(s-1)}{(s-2)} = 0$$

$$\Rightarrow (s-5)(s-2) + K(s-1) = 0$$

$$\Rightarrow s^2 - 7s + 10 + Ks - K = 0$$

$$\Rightarrow s^2 + s(K-7) + (10-K) = 0$$

Using the quadratic formula:

$$s = \frac{7-K \pm \sqrt{(K-7)^2 - 4(10-K)}}{2}$$

$$(K-7)^2 - 4(10-K) = 0$$

$$K^2 - 14K + 49 - 40 + 4K = 0$$

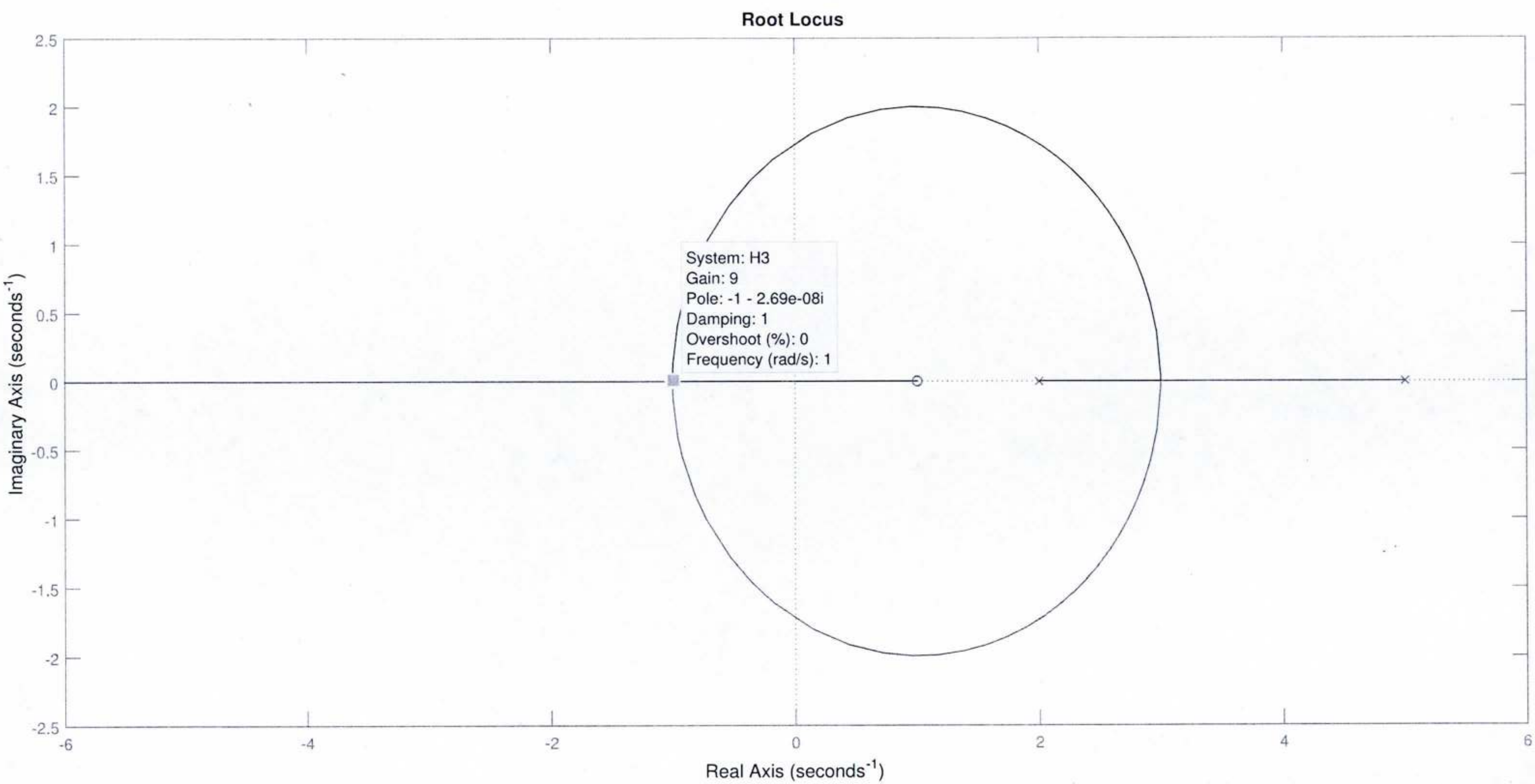
$$K^2 - 10K + 9 = 0$$

$$\Rightarrow K = 1, K = 9$$

One possible value of K is such that we have two repeated roots.

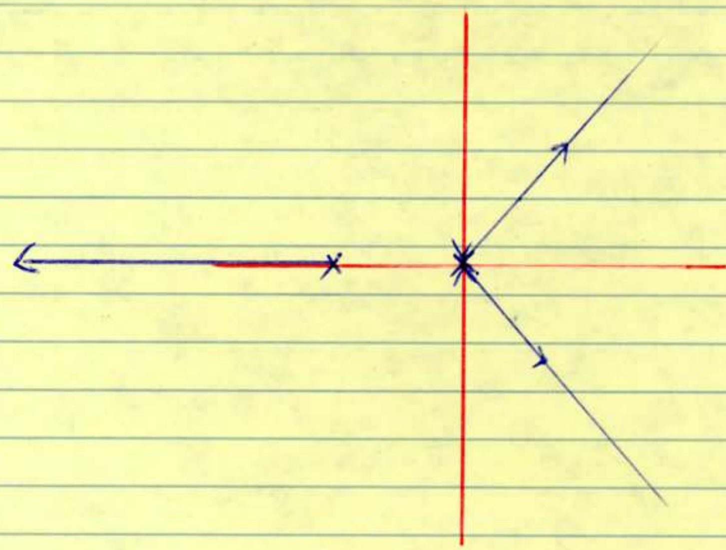
We see that if $K=1$, or $K=9$ we will have two repeated poles. However, only the choice $K=9$ will give us two negative real poles, so we make this selection.

This controller is strictly proper. If we add an additional zero our controller would be proper.



Root Locus
System: H3
Gain: 9
Pole: -1 - 2.69e-08i
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 1
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-6
-4
-2
0
2
4
6

(6.5e)



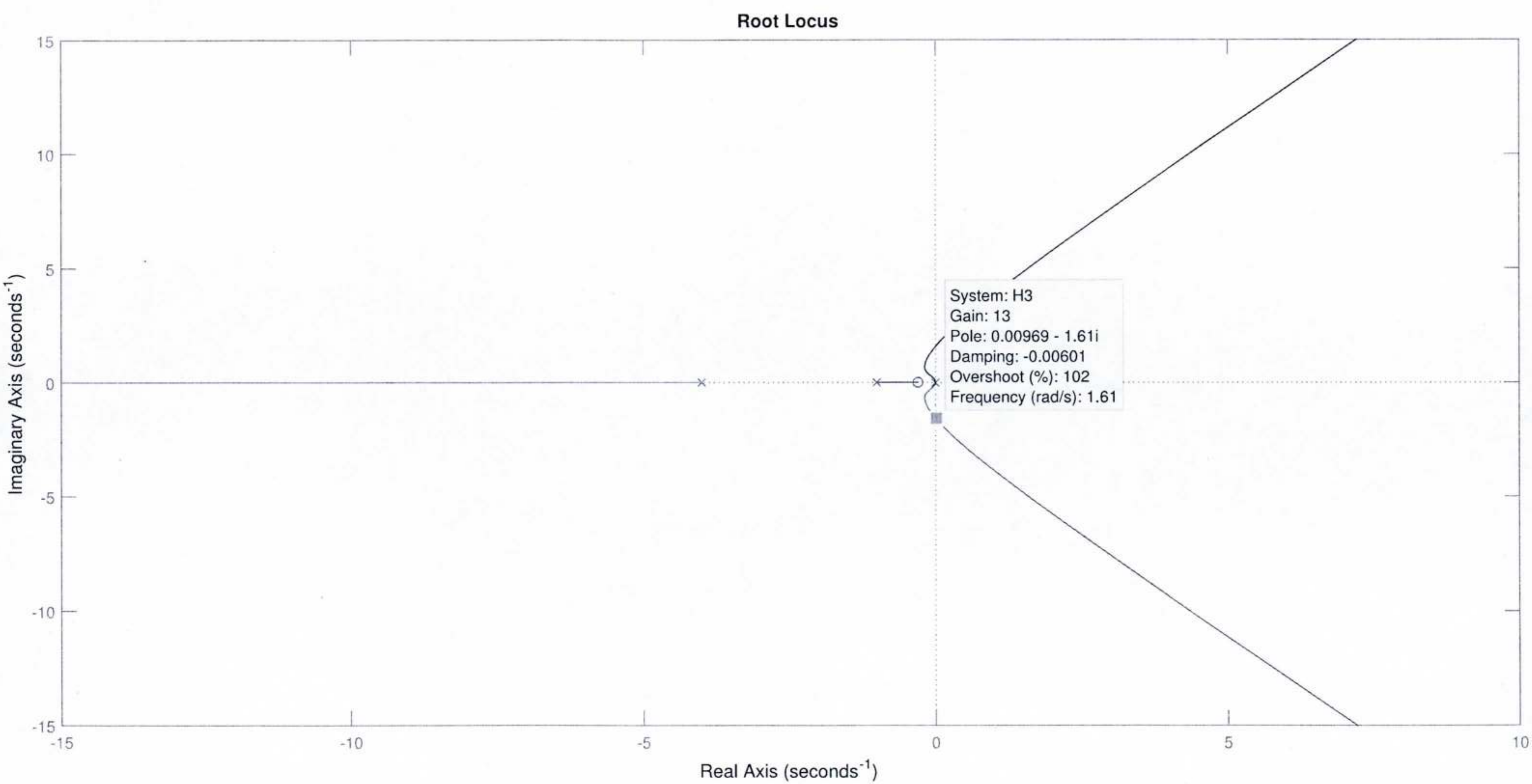
Here we have a double pole at $s=0$ and a pole at $s=-1$. The root locus is as indicated above.

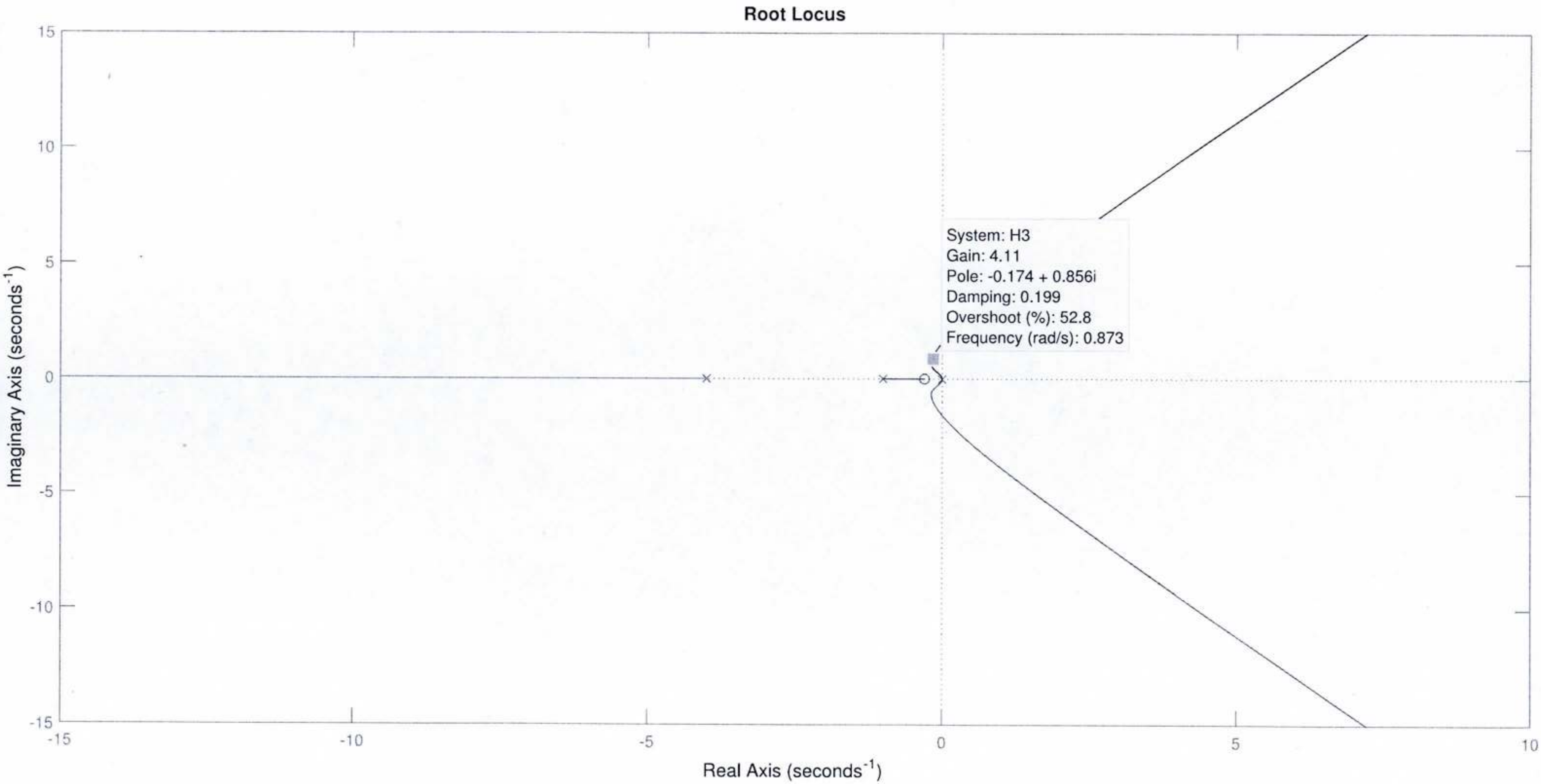
$$c = -\frac{1}{3} \qquad \phi_k = \pm\frac{\pi}{3}, \ +\pi$$

So, we see that no proportional controller will be able to make the closed loop system asymptotically stable. As we have done in a previous example, let us add a pole at $s=-4$ and a zero at $s=\cancel{1/2}\,-0.3$ Again, we are adding the pole so that the system controller transfer function is proper.

We indicate the root locus on the next page with an appropriate value of K.

We can see from our diagram that $K=4$ is appropriate. However, we need to be careful that we do not set K too high (above 13).


Root Locus
System: H3
Gain: 13
Pole: 0.00969 - 1.61i
Damping: -0.00601
Overshoot (%): 102
Frequency (rad/s): 1.61
Imaginary Axis (seconds-1)
Real Axis (seconds-1)
15
10
5
0
-5
-10
-15
-15
-10
-5
0
5
10


Root Locus
System: H3
Gain: 4.11
Pole: -0.174 + 0.856i
Damping: 0.199
Overshoot (%): 52.8
Frequency (rad/s): 0.873
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)
15
10
5
0
-5
-10
-15
-15
-10
-5
0
5
10

(6.10)

This question is similar to what we considered last week. The transfer function of our plant is given by

$$H(s) = \frac{r_1 r_2 / J_r}{s + \frac{b_r}{J_r}},$$

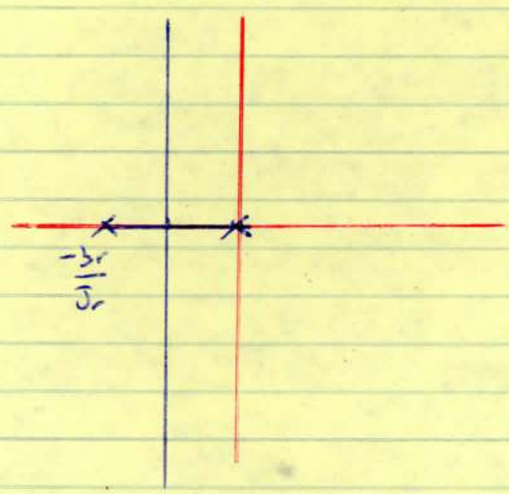
which we see has a single pole at $s = \frac{-b_r}{J_r}$.

$$K(s) = \frac{K}{s}.$$

So, our open loop transfer function is given by $K(s)H(s)$

$$= \frac{r_1 r_2 / J_r}{s\left(s + \frac{b_r}{J_r}\right)},$$

We can easily draw the root locus of this:



Our root locus indicates that the integral controller would be sufficient to make the closed loop system asymptotically stable.

We want to choose K such that the poles are real and are as negative as possible. To do this, we follow the same procedure from HW#6

$$1 + K(s)H(s) = 0$$

$$\Rightarrow \quad s = \frac{-\frac{br}{Jr} \pm \sqrt{\left(\frac{br}{Jr}\right)^2 - \frac{4Kr_1r_2}{Jr}}}{2}$$

to get desired pole locations, set:

$$\left(\frac{br}{Jr}\right)^2 = \frac{4Kr_1r_2}{Jr}$$

$$\Rightarrow K = \frac{br^2}{4Jr_1r_2}$$

(More detail on this procedure can be found in the solutions to HW#6)

Because we design our closed loop system to be AS and K(S)H(S) has a pole at $s=0$, we are capable of asymptotically tracking a constant reference.

(6.11)

Now we want to use a more complicated controller:

$$K(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s} = \frac{K_p\left(s + \frac{K_i}{K_p}\right)}{s}$$

We see that this controller has a pole at $s=0$ (which is what we saw in 6.10 as well) but we also have a zero at

$$K_p s + K_i = 0$$

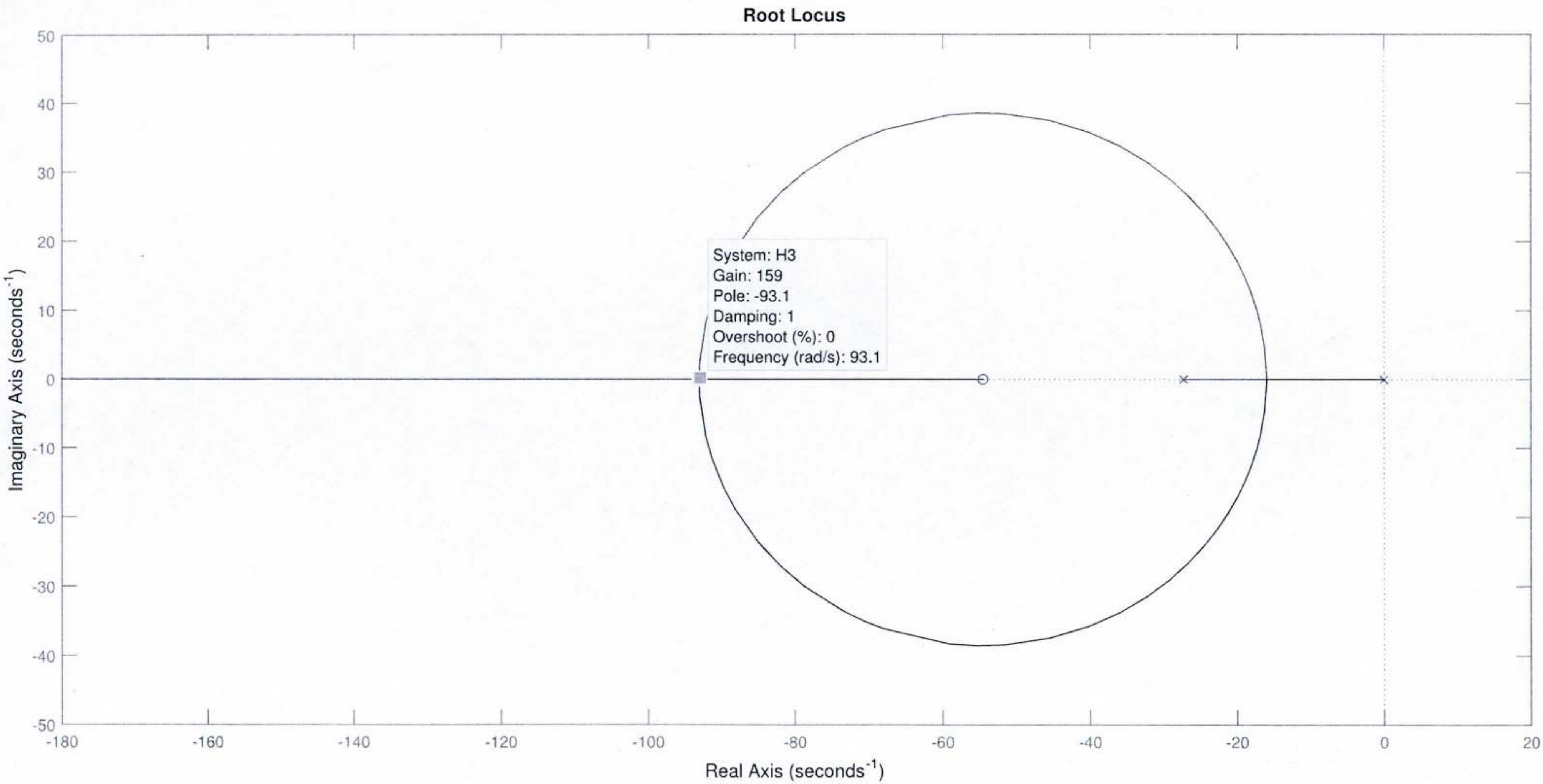
$$\Rightarrow \quad s = \frac{-K_i}{K_p}$$

We recall that our plant also has a pole at $s = -\frac{b_r}{J_r}$.

So, to design using the root locus method, let us assume $K_i, K_p > 0$ and place the zero, $s = -\frac{K_i}{K_p}$ at a location which meets the specifications in the question. Then we can select the appropriate values of K_i & K_p to ensure that we have a zero in the correct location.

Using the rules of root locus, we see that the only way to meet our specification is to ensure that the open loop zero has a real part smaller than both open loop poles.

So, lets place our zero at $s = \frac{-2b_r}{J_r}$, which gives the following root locus


Root Locus
System: H3
Gain: 159
Pole: -93.1
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 93.1
Real Axis (seconds-1)
Imaginary Axis (seconds-1)
50
40
30
20
10
0
-10
-20
-30
-40
-50
-180
-160
-140
-120
-100
-80
-60
-40
-20
0
20

So we require $\frac{K_i}{K_p} = \frac{2br}{Jr}$ and $K_p \approx 159$. to achieve

the desired specification with both closed loop poles being located on the real axis and to the left of the machine pole.

We can show this another way.

$$1 + K(s)H(s) = 0$$

$$\Rightarrow \quad 1 + \frac{K_p}{s} \cdot \left(s + \frac{K_i}{K_p}\right) \cdot \left(\frac{1}{s + \frac{br}{Jr}}\right) = 0$$

$$\Rightarrow \quad 1 + \frac{K_p}{s}\left(s + \frac{2br}{Jr}\right) \cdot \left(\frac{1}{s + \frac{br}{Jr}}\right) = 0$$

$$\Rightarrow \quad 1 + \frac{K_p}{s} \cdot \frac{sJr + 2br}{sJr + br} = 0$$

$$\Rightarrow \quad s(sJr + br) + K_p(sJr + 2br) = 0$$

$$\Rightarrow \quad Jrs^2 + sbr + sJrK_p + 2brK_p = 0$$

We want ~~real~~ imaginary part of solution to equal zero, so we require

$$(br + JrK_p)^2 - 8JrbrK_p = 0$$

$$\Rightarrow \quad br^2 - 6brJrK_p + Jr^2K_p^2 = 0$$

solve for K_p to get $\quad K_p = 158.9571$

$K_p = 4.6793$

Of which only the $K_p = 158.9571$ is appropriate.