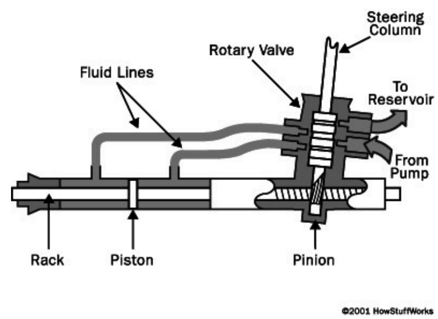
MAE143 B - Linear Control - Spring 2016
Midterm, May 3rd

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities
2. You have 70 minutes
3. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book"
5. Do not forget to write your name and student number



A hydraulic car power steering *system is composed of a hydraulic cylinder which is coupled to the vehicle's* rack *to assist steering. The hydraulic cylinder is coupled to the* steering column *which is driven by the* steering wheel *by an ingenious* valve *which is essentially a feedback loop. In a* drive-by-wire *system, this mechanical coupling is substituted by a feedback controller similar to the one you will analyze in this exam.*

Questions:

1. **Modeling the Hydraulic Cylinder**

   The behavior of the hydraulic pump + cylinder system can be approximately modeled by the following linearized differential equation model

   $$C\,\dot{p}(t) + B\,p(t) = D\,v(t) - A\,\dot{x}(t)$$

   where p is the pressure on the system, x is the cylinder's piston position, and v is the valve position.

   (a) (2 points) Assume zero initial conditions and show that

   $$P(s) = G_v(s)V(s) + G_x(s)X(s), \qquad G_v(s) = \frac{D/C}{s + B/C}, \qquad G_x(s) = \frac{-sA/C}{s + B/C}.$$

   Solution: Apply the Laplace transform assuming zero initial conditions to obtain

   $$(C\,s + B)P(s) = D\,V(s) - sAX(s) \qquad \text{(+1 point)}$$

from which

$$P(s) = \frac{D}{C\,s + B} V(s) - \frac{sA}{C\,s + B} X(s),$$
$$= \frac{D/C}{s + B/C} V(s) - \frac{sA/C}{s + B/C} X(s). \qquad \text{(+1 point)}$$

(b) (2 points) Assume that $x(t) = 0$ and that all constants are positive and use the frequency response method to show that if the valve is kept open at constant $v(t) = \tilde{v}$, $t \geq 0$, then the steady-state pressure is

$$p^{\mathrm{ss}}(t) = \tilde{p} = \frac{D\,\tilde{v}}{B}.$$

Solution: If all constants are positive then $B/C > 0$ so that $G_v(s)$ is asymptotically stable (and $G_x(s)$ as well). (+1 point)

In this case the steady state solution to a constant input can be computed by the frequency response method as:

$$p^{\mathrm{ss}}(t) = G_v(j0)\tilde{v}$$
$$= \frac{D/C}{j0 + B/C}\tilde{v} = \frac{D\tilde{v}}{B}. \qquad \text{(+1 point)}$$

(c) (5 points) Assume that $A = 10\text{cm}^2$, $B = V_0/\beta$, where $V_0 = 0.001\text{m}^3$, and $\beta \approx 1.5\text{MPa}$. Calculate the remaining parameters of the model and use the inverse Laplace transform to calculate and sketch the dynamic response to the input in part (b) knowing that $\tilde{v} = 2.5\text{cm}$, that $\tilde{p} = 10\text{MPa}$ ($\approx 1450\text{psi}$), and that the time-constant of the hydraulic cylinder is 0.03s.

Solution:

Calculate $B = 0.001/(3/2 \times 10^6) = 2/3 \times 10^{-9}$ then

$$D = \frac{B\tilde{p}}{\tilde{v}} = \frac{2/3 \times 10^{-9} \times 10 \times 10^6}{5/2 \times 10^{-2}} = 4/15. \qquad \text{(+1 point)}$$

Finally

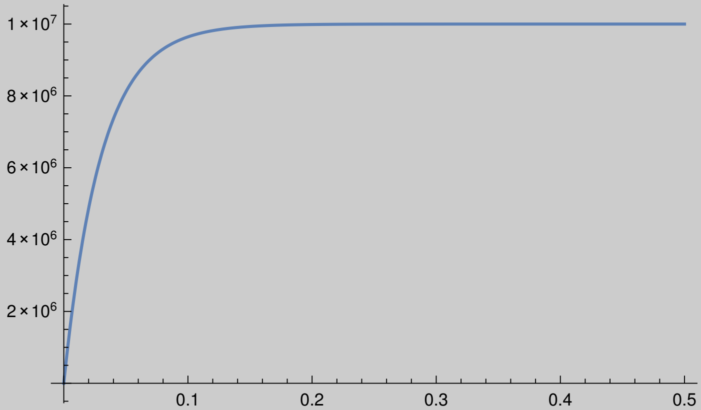
$$\frac{C}{B} = \tau = 0.03 \qquad \Longrightarrow \qquad C = 0.03B = 3 \times 10^{-2} \times 2/3 \times 10^{-9} = 2 \times 10^{-11}. \quad \text{(+1 point)}$$

The response can be calculated using the inverse Laplace transform after setting $x(t) = 0$

and calculating

$$\begin{aligned} p(t) &= \mathcal{L}^{-1}\left\{G_v(s)\frac{\tilde{v}}{s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{\tilde{v}D/C}{s\,(s+B/C)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{\tilde{v}D/C}{B/C}\left(\frac{1}{s}-\frac{1}{s+B/C}\right)\right\} \\ &= \frac{\tilde{v}D}{B}\left(1-e^{-tB/C}\right), \quad t \geq 0. \end{aligned}$$ (+2 points)

A sketch of the solution should look as:



(+1 point)

NOTE TO GRADERS: Note that one can produce the right plot from the statement even if the estimated parameters are incorrect!

NOTE TO GRADERS: Do not worry about units in the parameters, only in the sketch.

2. **Modeling the Rack**

The rack can be modeled as a mass-spring-damper. A simplified dynamic model for the rack is the differential equation

$$m\,\ddot{x}(t) + b\,\dot{x}(t) + k\,x(t) = A\,p(t) + f_w(t),$$

where x is the position of the rack which is firmly attached to the hydraulic cylinder piston, p is the pressure on the hydraulic system, and f_w represent disturbance forces.

(a) (3 points) Assume $f_w = 0$. Ignore initial conditions and show that

$$X(s) = G(s)V(s), \qquad G(s) = \frac{AD/C}{(s+B/C)(ms^2+bs+k)+sA^2/C}.$$

Solution:

Apply the Laplace transform assuming zero initial conditions to obtain

$$(ms^2+bs+k)X(s) = AP(s)$$ (+1 point)

Substitute $P(s)$ from Question 1 to obtain:

$$(ms^2 + bs + k)X(s) = \frac{AD/C}{s + B/C}V(s) - \frac{sA^2/C}{s + B/C}X(s) \qquad \text{(+1 point)}$$

Multiply by $(s + B/C)$

$$\{(s + B/C)(ms^2 + bs + k) + sA^2/C\}X(s) = (AD/C)V(s)$$

and solve for $X(s)$ to obtain

$$X(s) = \frac{AD/C}{(s + B/C)(ms^2 + bs + k) + sA^2/C}V(s). \qquad \text{(+1 point)}$$

(b) (1 point) Assume that $f_w = 0$ and that the system is asymptotically stable and use the frequency response method to show that if the valve is kept open at constant $v(t) = \tilde{v}$, $t \geq 0$ then the steady-state rod displacement is

$$x^{\text{ss}}(t) = \tilde{x} = \frac{A\,\tilde{p}}{k},$$

where $\tilde{p}$ is as in Question 1(c).

Solution:
If the system is asymptotically stable the steady state solution to a constant input can be computed by the frequency response method as:

$$\begin{aligned} x^{\text{ss}}(t) &= G(j0)\tilde{v} \\ &= \frac{AD/C}{Bk/C} = \frac{A}{k}\frac{D\tilde{v}}{B} = \frac{A\tilde{p}}{k} \end{aligned} \qquad \text{(+1 point)}$$

3. **Closing the Loop**

Let θ denote the angle of the *steering wheel* and consider the power-steering valve command

$$v(t) = L(r\,\theta - x), \qquad (1)$$

where $r > 0$ is equivalent to the radius of a standard rack-and-pinion steering column.

(a) (2 points) Explain what requirements need to be imposed on the feedback gain L for the feedback loop to be internally stable. You do not need to explicitly calculate L!

Solution:
Since there is no pole-zero cancellation between the plant, $G(s)$, and controller, $K(s) = L$, we need only to assert that the sensitivity transfer-function, $S(s) = (1 + LG(s))^{-1}$ be asymptotically stable for internal stability. (+1 point)

That is all poles of

$$\begin{aligned} S(s) &= \frac{1}{1 + LG(s)} \\ &= \frac{1}{1 + L\frac{AD/C}{(s+B/C)(ms^2+bs+k)+sA^2/C}} \\ &= \frac{(s+B/C)(ms^2+bs+k)+sA^2/C}{(s+B/C)(ms^2+bs+k)+sA^2/C+LAD/C} \end{aligned}$$

or all roots of the characteristic polynomial equation

$$(s+B/C)(ms^2+bs+k)+sA^2/C+LAD/C = 0$$

need to have negative real part. (+1 point)

(b) (2 points) Assume that all constants are positive and that the closed-loop is asymptotically stable. Explain why the closed-loop system does not achieve asymptotic tracking of a constant reference $\theta(t) = \tilde{\theta}$, $t \geq 0$ and show that the expected steady-state error when $f_w = 0$ satisfies

$$|e^{\mathrm{ss}}(t)| \leq \frac{Bk\,r\,\tilde{\theta}}{Bk + LAD}$$

Answer without calculating the closed-loop transfer-function!

Solution:
If all constants are positive, $K(s) = L$ and $G(s)$ do not have poles at $s = 0$ and therefore the closed-loop system cannot achieve asymptotic tracking of a constant reference.
(+1 point)

In this case, the steady-state error is bounded by

$$\begin{aligned} |e^{\mathrm{ss}}(t)| &\leq |S(j0)| r\,\tilde{\theta} \\ &= \frac{1}{|1 + LG(j0)|} r\,\tilde{\theta} \\ &= \frac{1}{|1 + L\frac{AD/C}{Bk/C}|} r\,\tilde{\theta} \\ &= \frac{Bk\,r\,\tilde{\theta}}{Bk + LAD} \end{aligned}$$ (+1 point)

(c) (2 points) Modify the feedback law (1) so that the closed-loop can potentially achieve asymptotic tracking of a constant reference $\theta(t) = \tilde{\theta}$, $t \geq 0$. Explain the requirements.

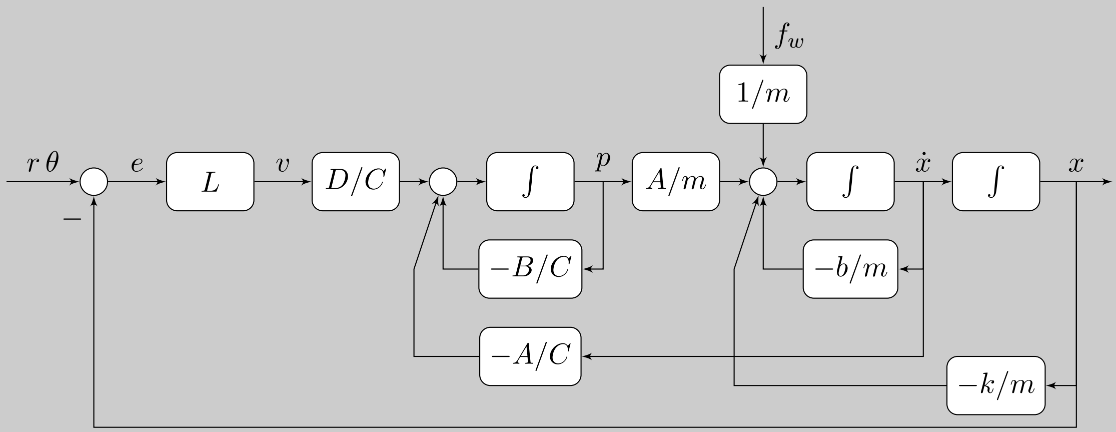
Solution:
For asymptotic tracking of a constant reference the controller needs to have a pole at the origin. (+1 point)
That is we need a controller of the form

$$V(s) = \frac{L(s)}{s}(r\Theta(s) - X(s))$$

where $L(s)$ is chosen so that $S(s) = (1 + L(s)G(s))^{-1}$ be asymptotically stable and no unstable pole-zero cancellations occur. (+1 point)

(d) (3 points) Draw a simulation diagram showing all components and all signals of the closed-loop system using only gains, summers and integrators.

Solution:



(+3 points)

NOTE TO GRADERS: Diagram is not unique, make sure all signals are represented.

(e) (2 points (bonus)) Does the closed-loop system achieve asymptotic rejection of a constant disturbance $f_w(t) = \tilde{f}_w$, $t \geq 0$?

Solution:

Because of the additional feedback between $\dot{x}$ and p we need to calculate the transfer-function from f_w to e.

As in Question 2:

$$(ms^2 + bs + k)X(s) = \frac{AD/C}{s + B/C}V(s) - \frac{sA^2/C}{s + B/C}X(s) + F_w(s)$$

Multiply by $(s + B/C)$

$$\{(s + B/C)(ms^2 + bs + k) + sA^2/C\}X(s) = (AD/C)V(s) + (s + B/C)F_w(s)$$

and solve for $X(s)$ to obtain

$$X(s) = G(s)V(s) + H(s)F_w(s), \qquad H(s) = \frac{(s + B/C)}{(s + B/C)(ms^2 + bs + k) + sA^2/C}$$

(+1 point)

From here

$$\begin{aligned} E(s) &= r\Theta(s) - X(s) \\ &= r\Theta(s) - G(s)V(s) + H(s)F_w(s), \\ &= r\Theta(s) - LG(s)E(s) + H(s)F_w(s) \end{aligned}$$

and

$$E(s) = S(s)r\Theta(s) + H(s)S(s)F_w(s).$$

Evaluating

$$H(s)S(s) = \frac{(s + B/C)}{(s + B/C)(ms^2 + bs + k) + sA^2/C + LAD/C}$$

we see that $H(s)S(s)$ does not have a zero at $s = 0$ and therefore the closed loop does not achieve asymptotic disturbance rejection of a constant disturbance. (+1 point)