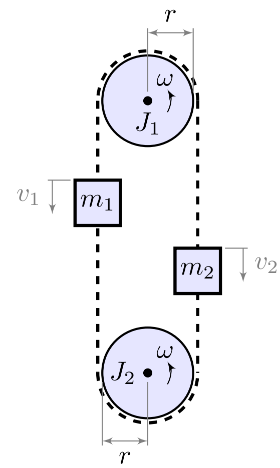
MAE143 B - Linear Control - Spring 2017
Midterm, May 5th



Instructions:

1. This exam is **not open book**. The only material you can consult is your one-page cheat sheet.
2. You have 70 minutes.
3. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book".
5. Do not forget to write your name and student number.

Questions:

1. **Modeling**

   The differential equation

   $$\big(J_1 + J_2 + r^2(m_1 + m_2)\big)\, \dot{\omega}(t) + (b_1 + b_2)\, \omega(t) = \tau(t) + g\, r(m_1 - m_2), \tag{1}$$

   is a simplified description of the motion of the elevator schematically depicted on the top of the page. In this model $\tau(t)$ is the torque applied by the driving motor on the inertia J_1, and b_1 and b_2 are viscous friction torque coefficients at the inertias J_1 and J_2. The vertical velocities are $v_1(t) = r\,\omega(t)$ and $v_2(t) = -r\,\omega(t)$.

   (a) (1 point) If a motor is used to drive the elevator then the torque can be approximated by

   $$\tau(t) = \alpha v(t) - \beta\omega(t), \tag{2}$$

   where α and β are parameters of the motors and $v(t)$ is a voltage applied to the motor. Combine (1) and (2) to show that

   $$a\,\dot{\omega}(t) + b\,\omega(t) = c\,v(t) + c\,w(t)$$

   where

   $$a = J_1 + J_2 + r^2(m_1 + m_2), \qquad b = b_1 + b_2 + \beta, \qquad c = \alpha$$

   and

   $$w(t) = \frac{g\,r}{\alpha}(m_1 - m_2)$$

   can be seen as a disturbance.

Solution: Combining (1) and (2)

$$\left(J_1 + J_2 + r^2(m_1 + m_2)\right)\, \dot{\omega}(t) + (b_1 + b_2)\, \omega(t) = \alpha\, v(t) - \beta\omega(t) + g\, r(m_1 - m_2),$$

then

$$\begin{aligned}\left(J_1 + J_2 + r^2(m_1 + m_2)\right)\, \dot{\omega}(t) + (b_1 + b_2 + \beta)\, \omega(t) &= \alpha \left(v(t) + \frac{g\, r}{\alpha}(m_1 - m_2)\right), \\ &= \alpha\, (v(t) + w(t)),\end{aligned}$$

which matches the form given in the statement. (+1 point)

(b) (2 points) Assume zero-initial conditions and show that the Laplace transforms of the voltage $v(t)$, the disturbance $w(t)$, and the vertical velocity $v_1(t)$ are related through:

$$V_1(s) = G(s)(V(s) + W(s)), \qquad G(s) = \frac{rc/a}{s + b/a}.$$

Solution: With zero-initial conditions and taking Laplace transforms:

$$as\Omega(s) + b\Omega(s) = cV(s) + cW(s), \qquad V_1(s) = r\Omega(s) \qquad \text{(+1 point)}$$

from which

$$V_1(s) = r\Omega(s) = r\frac{c}{as + b}(V(s) + W(s)) = G(s)(V(s) + W(s)). \qquad \text{(+1 point)}$$

(c) (2 points) Assume that $m_1 = m_2$ and that all constants are positive and use the frequency response method to show that if a constant voltage $v(t) = \tilde{v}$, $t \geq 0$, is applied to the motor then

$$v_1^{\text{ss}}(t) = \tilde{v}_1 = \frac{rc}{b}\tilde{v}.$$

Solution: If all constants are positive then $b/a > 0$ so that $G(s)$ is asymptotically stable (+1 point)
If $m_1 = m_2$ then $w(t) = 0$ and

$$v_1^{\text{ss}}(t) = G(0)\tilde{v} = \frac{rc/a}{b/a}\tilde{v} = \frac{rc}{b}\tilde{v}. \qquad \text{(+1 point)}$$

(d) (2 points) Assume that $m_1 = m_2$ and zero-initial conditions and use the inverse Laplace transform to show that the complete response to a constant input voltage $v(t) = \tilde{v}$, $t \geq 0$, is

$$v_1(t) = \tilde{v}_1(1 - e^{-(b/a)t}), \quad t \geq 0.$$

Solution: For the complete solution with $m_1 = m_2$ the Laplace transform of a constant input combined with the transfer-function produces:

$$V_1(s) = G(s)V(s) = \frac{rc/a}{s + b/a}\frac{\tilde{v}}{s} \qquad \text{(+1/2 point)}$$

Expanding in partial fractions:

$$V_1(s) = \frac{k_1}{s} + \frac{k_2}{s + b/a} \qquad \text{(+1/2 point)}$$

where

$$k_1 = \lim_{s \to 0} \frac{rc/a}{s + b/a}\tilde{v} = \frac{rc/a}{b/a}\tilde{v} = \tilde{v}_1, \qquad k_2 = \lim_{s \to -b/a} rc/a\frac{\tilde{v}}{s} = \frac{rc/a}{-b/a}\tilde{v} = -\tilde{v}_1$$

(+1/2 point)

and finally taking the inverse Laplace transform

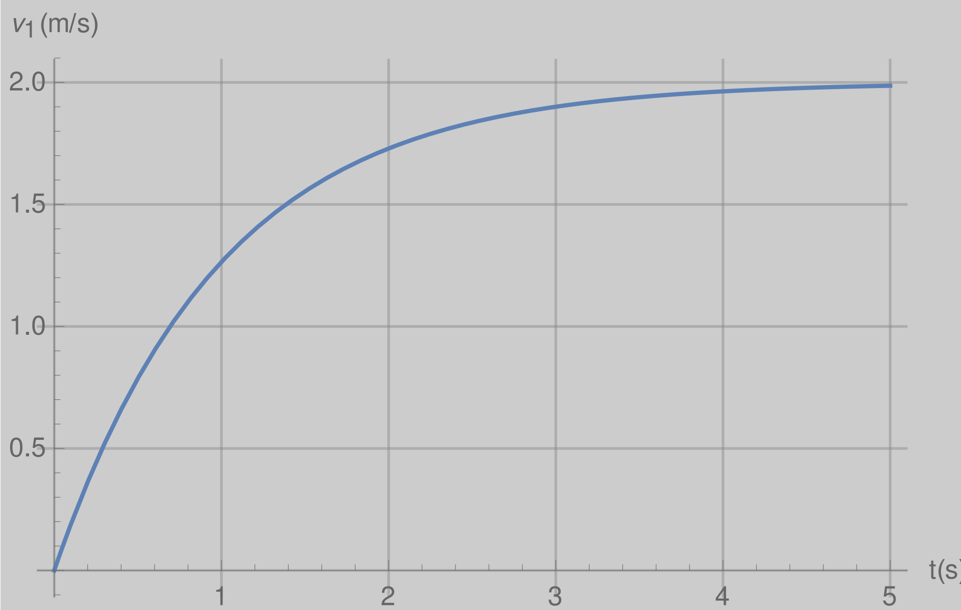
$$v_1(t) = \tilde{v}_1 - \tilde{v}_1 e^{-(b/a)t}, \quad t \geq 0. \qquad \text{(+1/2 point)}$$

(e) (2 points) Assume that $m_1 = m_2$, $r = 1$ m, $b/a = 1$ s^{-1}, $c/a = 0.02$ s^{-1}/V. Sketch the complete response to a constant input voltage $v(t) = \tilde{v} = 100$V using the formula you calculated in part (d). What is the time-constant of the system?

Solution: Substituting numeric values:

$$\tilde{v}_1 = \frac{rc}{b}\tilde{v} = \frac{rc/a}{b/a}\tilde{v} = \frac{1 \times 0.02}{1} \times 100 = 2 \text{ m/s} \qquad \text{(+1 point)}$$

The time constant is $a/b = 1$s. A sketch of the solution should look as:



(+1 point)

2. **Closed-loop control**

Consider the proportional control law

$$v(t) = Ke(t), \quad e(t) = \bar{v}_1(t) - v_1(t), \tag{3}$$

where $\bar{v}_1(t)$ is a reference velocity and answer the following questions.

(a) (3 points) Calculate the closed-loop transfer-functions from $\bar{v}_1$ to e, from w to e, and from $\bar{v}_1$ to v_1.

Solution: From $\bar{v}_1$ to e:

$$S(s) = \frac{1}{1+GK} = \frac{1}{1+K\frac{rc/a}{s+b/a}} = \frac{s+b/a}{s+b/a+Krc/a}.$$

(+1 point)

From w to e:

$$D(s) = \frac{G}{1+GK} = GS = \frac{rc/a}{s+b/a} \times \frac{s+b/a}{s+b/a+Krc/a} = \frac{rc/a}{s+b/a+Krc/a}.$$

(+1 point)

From $\bar{v}_1$ to v_1:

$$H(s) = \frac{GK}{1+GK} = GKS = K\frac{rc/a}{s+b/a} \times \frac{s+b/a}{s+b/a+Krc/a} = \frac{Krc/a}{s+b/a+Krc/a}.$$

(+1 point)

(b) (1 point) For what positive values of K is the closed-loop asymptotically stable?

Solution: Assuming all constants are positive the closed-loop is asymptotically stable if $b/a + Krc/a > 0$, which is certainly the case for all $K > 0$. (+1 point)

(c) (3 points) Select a proper K and substitute the values from Question 1(e) to calculate the steady-state error to a constant reference input $\bar{v}_1(t) = \bar{v}_1 = 1\text{m/s}$. Repeat for $\bar{v}_1(t) = \bar{v}_1 = -1\text{m/s}$. Explain any differences. Does the closed-loop system achieve asymptotic tracking?

Solution: Let's take $K = 50 > 0$ so that the closed-loop is asymptotically stable. (any positive value works!) (+1 point)

The steady-state error due to a constant reference input can be calculated from the transfer-function $S(s)$ as in:

$$e^{\text{ss}}(t) = S(0)\bar{v}_1 = \frac{b/a}{b/a + Krc/a}\bar{v}_1 \qquad \text{(+1 point)}$$

Substituting values from Question 1(e):

$$e^{\text{ss}}(t) = \frac{1}{1 + 50 \times 0.02}\bar{v}_1 = \frac{\bar{v}_1}{2} \qquad \text{(+1 point)}$$

Since $S(0) \neq 0$ we do not have asymptotic tracking. An alternative answer is that since the product $KG(s)$ does not have a pole at the origin we do not have asymptotic tracking. The magnitude of the tracking error is the same no matter the direction of the reference velocity. (+1 point)

(d) (3 points) Repeat part (c) when $m_1 = 1000\text{kg}$ and $m_2 = 800\text{kg}$. Assume $\alpha = 10$ Nm/V, $g = 10\text{kg/(m s}^2)$. Does the closed-loop system achieve asymptotic disturbance rejection?

Solution:

Since $m_1 \neq m_2$

$$e^{\text{ss}}(t) = S(0)\bar{v}_1 - D(0)\bar{w} = \frac{b/a}{b/a + Krc/a}\bar{v}_1 - \frac{rc/a}{b/a + Krc/a}\bar{w} \qquad \text{(+1 point)}$$

where $\bar{w} = \frac{gr}{\alpha}(m_1 - m_2)$.

Substituting values from Question 1(e):

$$e^{\text{ss}}(t) = \frac{1}{1 + 50 \times 0.02}\bar{v}_1 - \frac{0.02}{1 + 50 \times 0.02}\bar{w} = 0.5\bar{v}_1 + 0.01\bar{w} \qquad \text{(+1 point)}$$

where $\bar{w} = \frac{10}{10}(1000 - 800) = 200$

Since $D(0) \neq 0$ we do not have asymptotic disturbance rejection. An alternative answer is that since K does not have a pole at the origin we do not have asymptotic tracking.

The magnitude of the tracking error is also different depending on the direction of the reference velocity. For example $e^{\mathrm{ss}}(t) = 0.5+2 = 2.5\mathrm{m/s}$ when $\bar{v}_1 = 1\mathrm{m/s}$ and $e^{\mathrm{ss}}(t) = -0.5 + 2 = 1.5\mathrm{m/s}$ when $\bar{v}_1 = 1.5\mathrm{m/s}$. See that these values are higher then the reference velocity in this case and in the wrong direction!

(+1 point)

(e) (3 points) Modify the feedback law (3) so that the closed-loop can achieve asymptotic tracking of a constant reference $\bar{v}_1(t) = \bar{v}_1$, $t \geq 0$. Does the resulting closed-loop achieve asymptotic disturbance rejection of a constant disturbance? Clearly explain the requirements and your reasoning.

Solution:

For asymptotic tracking of a constant reference the controller needs to have a pole at the origin. (+1 point)

That is we need a controller of the form

$$V(s) = \frac{L(s)}{s}(\bar{V}_1(s) - V_1(s))$$

where $L(s)$ is chosen so that $S(s) = (1+s^{-1}L(s)G(s))^{-1}$ be asymptotically stable and no unstable pole-zero cancellations occur. (+1 point)

Because the controller has a pole at the origin $D(s)$ will have a zero at the origin and we will have asymptotic constant disturbance rejection. (+1 point)