

MAE143 B - Linear Control - Spring 2018
Final, June 14th

Instructions:

1. This exam is open book. You may use whatever written materials of your choice, including your notes and the book, and a calculator with no communication capabilities.
2. You have 170 minutes. Don't get stuck! Come back later if you have time.
3. Write answers on your "Blue Book" and attach the provided graph sheet.
4. Do not forget to write your name and student ID.
5. The exam has 9 questions for a total of 81 points and 2 bonus points.

Part I

The diagrams in Fig. 1 show pairs of open-loop pole-zero maps and their corresponding Nyquist plots. Assume that the associated transfer-function is of the form:

$$G(s) = G_0 \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}.$$

For each pair, answer the following questions: NOTE: Points shown are total for all 3 plots.

1. Open-loop System.

- (a) (3 points) Write down the transfer-function for each system.

Solution: Transfer-functions are:

$$G(s) = 4 \frac{1}{(s+2)(s+1-j)(s+1+j)} = 4 \frac{1}{(s+2)(s^2+2s+2)} \quad (+1 \text{ point})$$

$$G(s) = 6 \frac{(s+1)}{(s+3)(s-1)} \quad (+1 \text{ point})$$

$$G(s) = 6 \frac{(s-1)}{(s+1)(s+3)} \quad (+1 \text{ point})$$

- (b) (3 points) Is it asymptotically stable in open-loop? Explain.

Solution:

System (a) is asymptotically stable because all poles have negative real part.

(+1 point)

System (b) is not asymptotically stable because it has a pole at 1 with positive real part.

(+1 point)

System (c) is asymptotically stable because all poles have negative real part.

(+1 point)

2. Root-Locus.

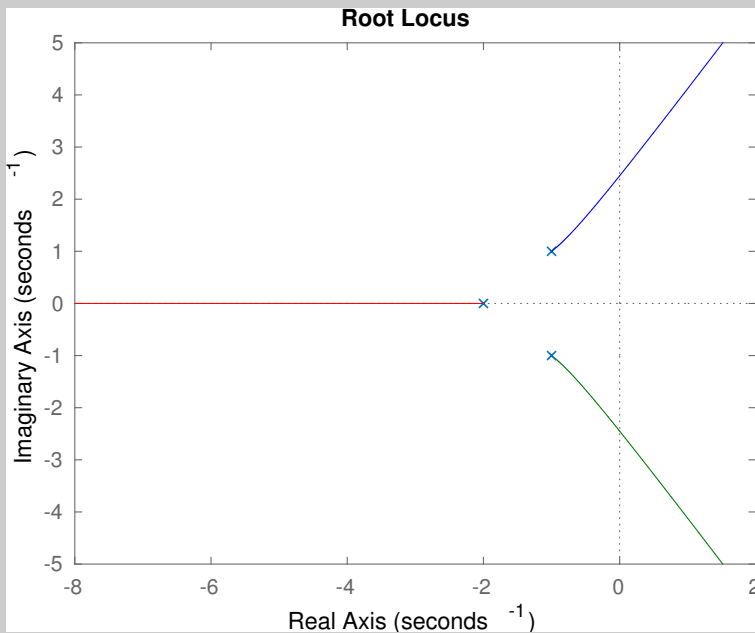
IMPORTANT: Use the plots provided!

- (a) (9 points) Sketch the corresponding root-locus diagram.
- (b) (3 points) Are there positive values of feedback gain for which the closed-loop system is asymptotically stable? Explain using your root-locus diagram.

Solution:

System (a)

- (a) The root-locus diagram should look like:



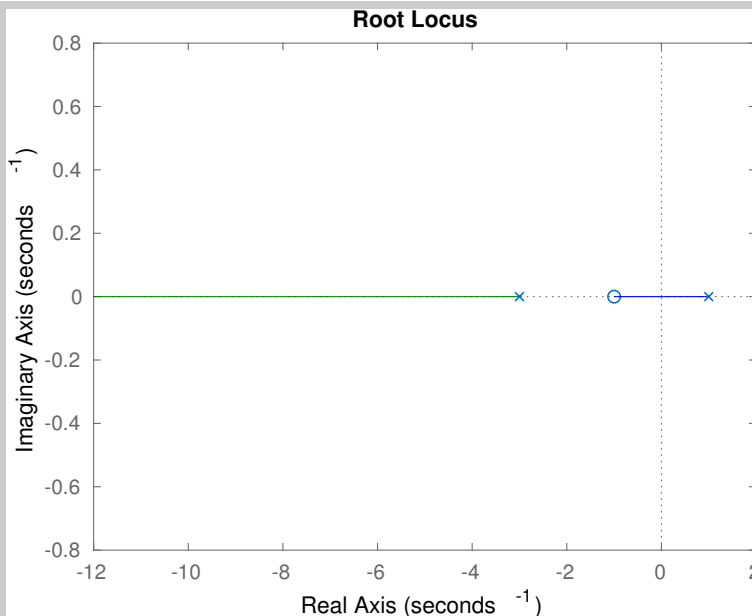
(+3 points)

- (b) All closed-loop roots remain stable for small $K > 0$. After a certain large value of $K > 0$ they become unstable.

(+1 point)

System (b)

- (a) The root-locus diagram should look like:



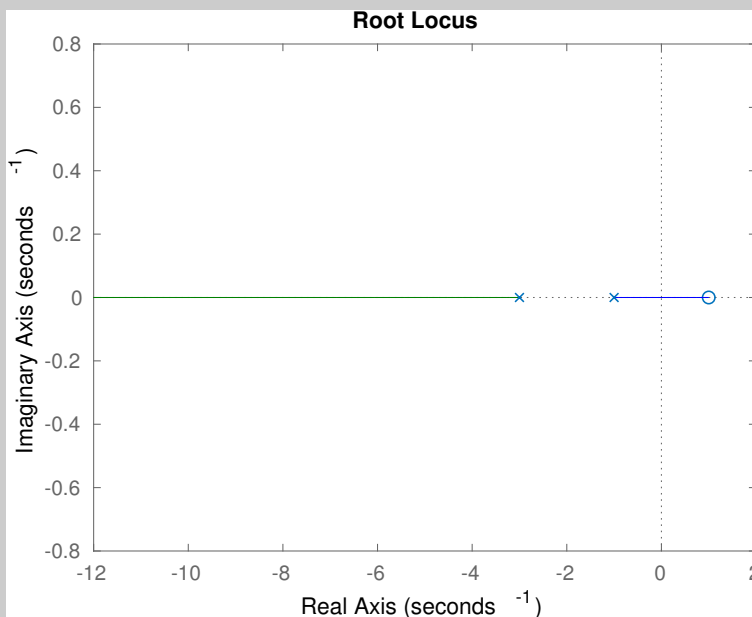
(+3 points)

(b) For small $K > 0$ there is one unstable pole. For a large enough value of $K > 0$ the two closed-loop poles become stable.

(+1 point)

System (c)

(a) The root-locus diagram should look like:



(+3 points)

(b) For small $K > 0$ both poles are stable. For a large enough value of $K > 0$ one of the closed-loop poles becomes unstable.

(+1 point)

3. Nyquist Criterion.

IMPORTANT: Use the plots provided!

- (a) (3 points) Can a proportional controller with unit gain stabilize the system in closed-loop? Explain using the Nyquist Criterion.
- (b) (3 points) What are the positive values of gain for which the closed-loop system is asymptotically stable? Approximate values are fine.
- (c) (3 points) What are the positive values of gain for which the closed-loop system is not asymptotically stable? How many unstable closed-loop poles there are?

Solution:

System (a)

- (a) Yes, because the Nyquist plot shows no encirclements of the point -1 and the number of open-loop unstable poles is $P_\Gamma = 0$. (+1 point)
- (b) For $-\infty < -1/K < -0.2$ (approximately), i.e. $K < 5$ there will be no encirclements so that the closed loop remain stable. (+1 point)
- (c) For $-0.2 < -1/K < 0$ (approximately), i.e. $K > 5$ there will be two clockwise encirclements of the point $-1/K$ which means that there will be $Z_\Gamma = 2$ unstable closed-loop poles (+1 point)

System (b)

- (a) Yes, because the Nyquist plot shows one counter-clockwise encirclements of the point -1 and the number of open-loop unstable poles is $P_\Gamma = 1$ so that the closed-loop system has $Z_\Gamma = 1 - (+1) = 0$ unstable poles. (+1 point)
- (b) For $-2 < -1/K < 0$, i.e. $K > 0.5$ there will be one counter-clockwise encirclements of the point $-1/K$ which means that there will be $Z_\Gamma = 1 - (+1) = 0$ unstable closed-loop poles (+1 point)
- (c) For $-\infty < -1/K < -2$ (approximately), i.e. $K < 0.5$ there will be no encirclements resulting in $Z_\Gamma = 1 - (0) = 1$ unstable closed-loop poles (+1 point)

System (c)

- (a) No, because the Nyquist plot shows one clockwise encirclements of the point -1 and the number of open-loop unstable poles is $P_\Gamma = 0$ so that the closed-loop system has $Z_\Gamma = 0 - (-1) = 1$ unstable pole. (+1 point)
- (b) For $-\infty < -1/K < -2$ (approximately), i.e. $K < 0.5$ there will be no encirclements resulting in $Z_\Gamma = 0 - (0) = 0$ unstable closed-loop poles (+1 point)
- (c) For $-2 < -1/K < 0$, i.e. $K > 0.5$ there will be one clockwise encirclements of the point $-1/K$ which means that there will be $Z_\Gamma = 0 - (-1) = 1$ unstable closed-loop pole (+1 point)

4. Bode Plots.

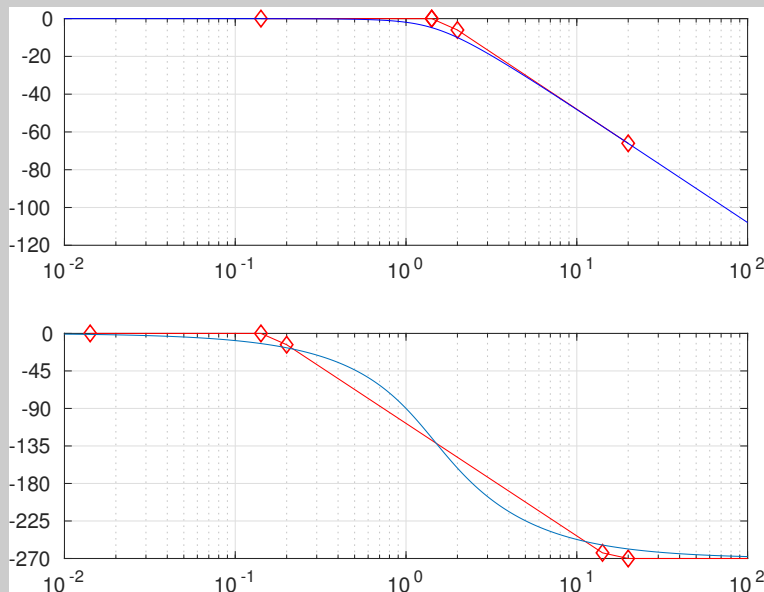
IMPORTANT: Use the plots provided!

- (a) (9 points) Sketch the straight-line approximations for the corresponding Bode plots on top of the (exact) Bode plots provided in Fig. 2.
- (b) (3 points) Annotate the plots to show the points critical for the calculation of the gain-margin and phase-margin. Provide estimates for these quantities based on your reading of the plots.

Solution:

System (a)

- (a) The Bode plot should look like:



(+3 points)

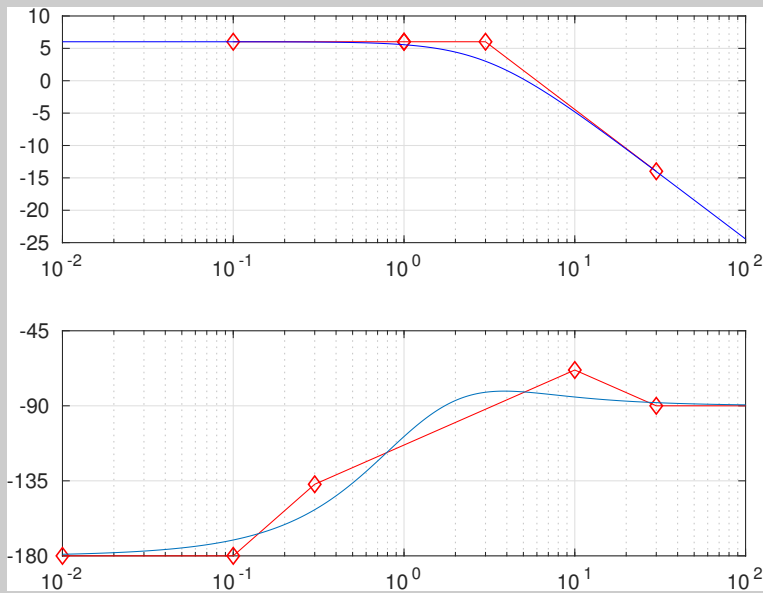
- (b) $\angle L(j\omega) = \pi$ at $\omega \approx 2.2$ with a gain of approximately -15db so the gain margin is of about $0 - (-15) = 15\text{db}$.

$|L(j\omega)| = 1$ at $\omega = 0$ so the phase margin is $0 - 180^\circ = -180^\circ$.

(+1 point)

System (b)

- (a) The Bode plot should look like:



(+3 points)

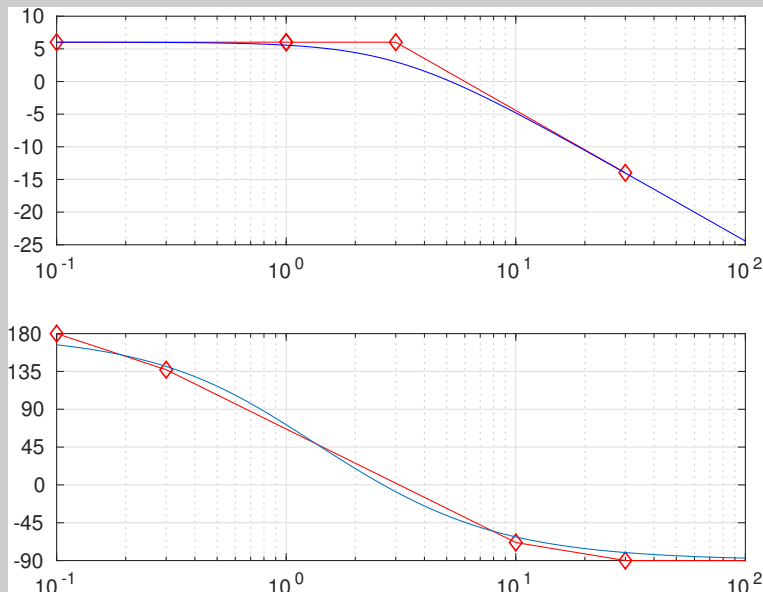
(b) $\angle L(j\omega) = \pi$ at $\omega = 0$ when the gain $20 \log_{10}(2) \approx 6\text{dB}$ so the gain margin is $0 - 6 = -6\text{dB}$.

$|L(j\omega)| = 1$ at about $\omega = 5$ where the phase is of about -80 so the phase margin is approximately $(-80 + 360) - 180 = 100^\circ$.

(+1 point)

System (c)

(a) The Bode plot should look like:



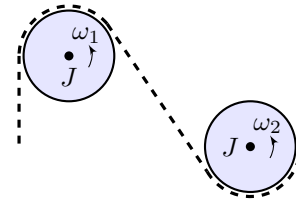
(+3 points)

(b) The closed-loop system is not stable under unit-feedback so there are no margins!

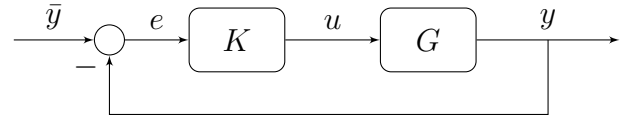
(+1 point)



(a) Paper mill[†]



(b) Two rollers



(c) Feedback diagram

[†] https://commons.wikimedia.org/wiki/File:Papermaking_machine_at_a_paper_mill_near_Pensacola.jpg

Figure 1: Figures for Part II

Part II

The motion of two identical rollers on a segment of a paper milling machine can be approximately described by the differential equations:

$$J\dot{\omega}_1(t) + b\omega_1(t) = c u_1(t) + \tau_1(t), \quad J\dot{\omega}_2(t) + b\omega_2(t) = c u_2(t) + \tau_2(t), \quad (1)$$

where u_1 and u_2 are control commands for the motors of each roller and τ_1 and τ_2 are torque disturbances. Assume that all constants are positive.

In the midterm 1 you showed that the difference between the speeds of the rollers, δ_ω , satisfy the differential equation:

$$J\dot{\delta}_\omega(t) + b\delta_\omega(t) = c(\delta_u(t) + \delta_\tau(t)), \quad (2)$$

and that the Laplace transforms of the signals in (2) are related through:

$$\Delta_\omega(s) = G_\omega(s)(\Delta_u(s) + \Delta_\tau(s)), \quad G_\omega(s) = \frac{c/J}{s + b/J}.$$

In the midterm 2 you showed that the transfer-function between the phase difference of the rollers, δ_θ , and the signals δ_u and δ_τ is given by:

$$\Delta_\theta(s) = G_\theta(s)(\Delta_u(s) + \Delta_\tau(s)), \quad G_\theta(s) = \frac{c/J}{s(s + b/J)}.$$

The signals mentioned above are

$$\delta_\omega = \omega_2 - \omega_1, \quad \delta_u = u_2 - u_1, \quad \delta_\tau = \frac{\tau_2 - \tau_1}{c}, \quad \delta_\theta = \theta_2 - \theta_1, \quad \theta_k(t) = \int_0^t \omega_k(\sigma) d\sigma, k = 1, 2 \quad (3)$$

THROUGHOUT THE REST OF THIS EXAM:

Assume that $b/J = c/J = 1$.

Consider feedback connections as in Fig. 1c

1. Open-loop System

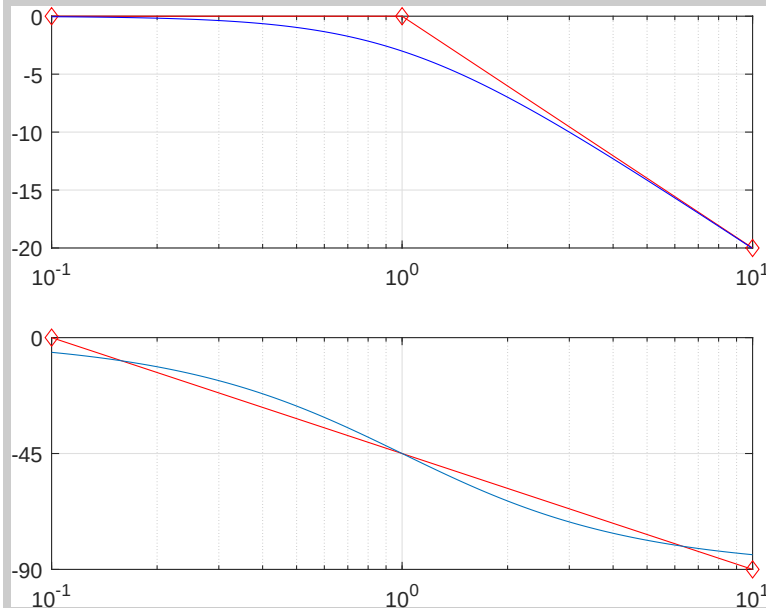
- (a) (1 point) Is $G_\omega(s)$ asymptotically stable? Explain.

Solution: $G_\omega = \frac{1}{(s+1)}$ which has all poles with negative real part, hence it is asymptotically stable. (+1 point)

- (b) (3 points) Sketch the magnitude and phase Bode plots of $G_\omega(s)$.

Solution:

Plot should look like:

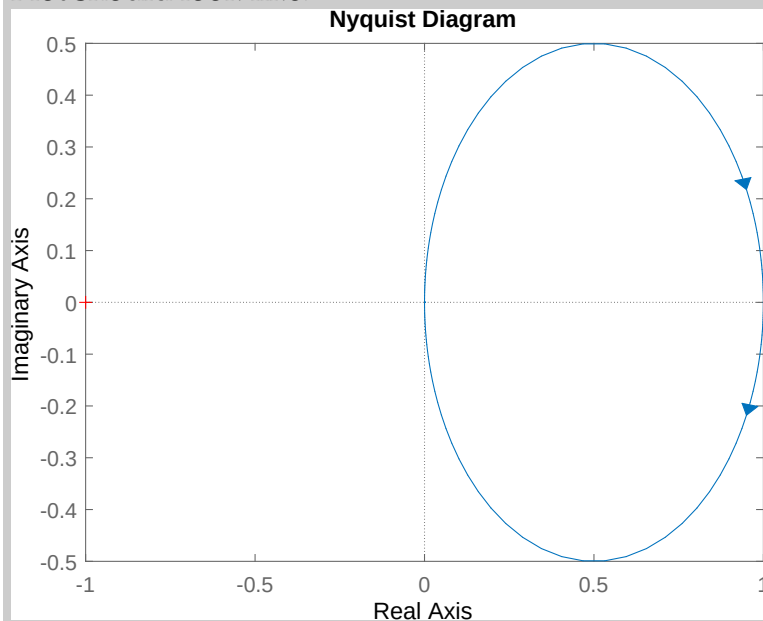


(+3 points)

- (c) (2 points) Sketch the Nyquist plot associated with $G_\omega(s)$.

Solution:

Plot should look like:



(+2 points)

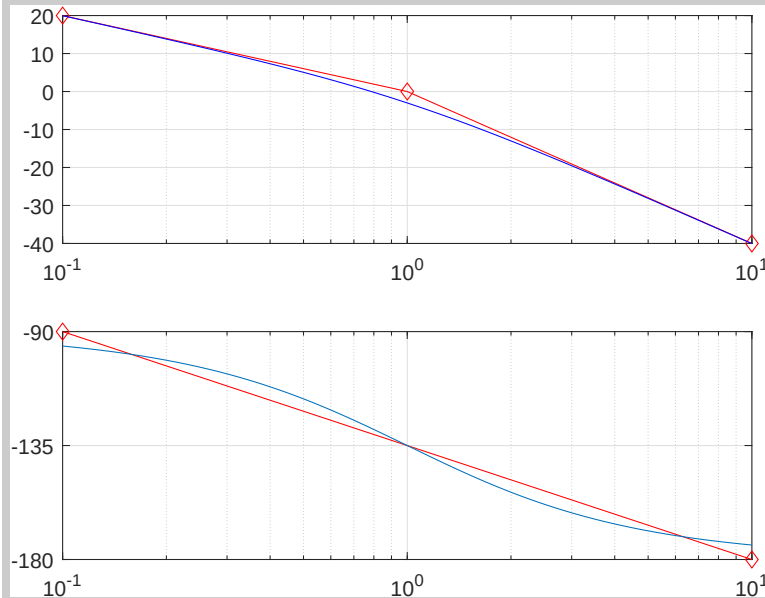
- (d) (1 point) Is $G_\theta(s)$ asymptotically stable? Explain.

Solution: $G_\omega = \frac{1}{s(s+1)}$ which has one pole at the imaginary axis, hence it is not asymptotically stable. (+1 point)

- (e) (3 points) Sketch the magnitude and phase Bode plots of $G_\theta(s)$.

Solution:

Plot should look like:

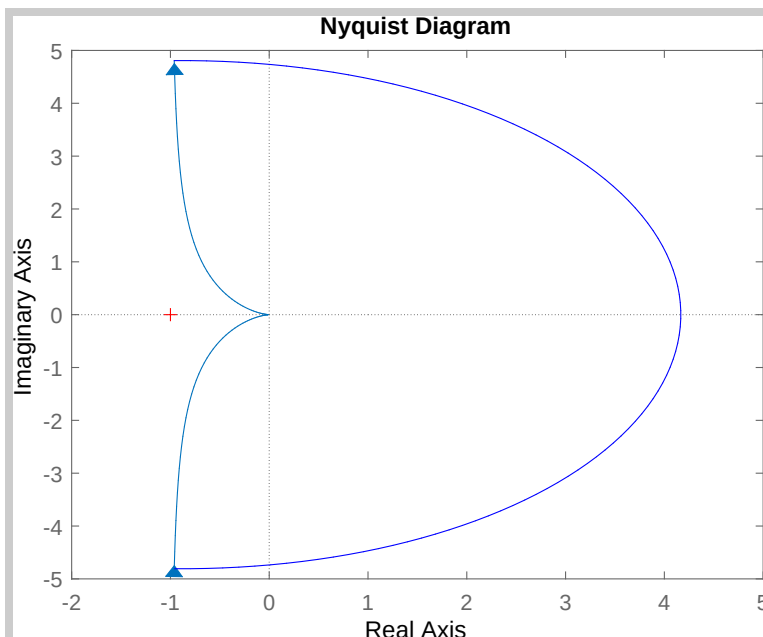


(+3 points)

- (f) (2 points) Sketch the Nyquist plot associated with $G_\theta(s)$.

Solution:

Plot should look like:



(+2 points)

2. Integral Control

Consider the feedback controller with transfer-function

$$K(s) = \frac{K_i}{s}$$

and answer the following questions:

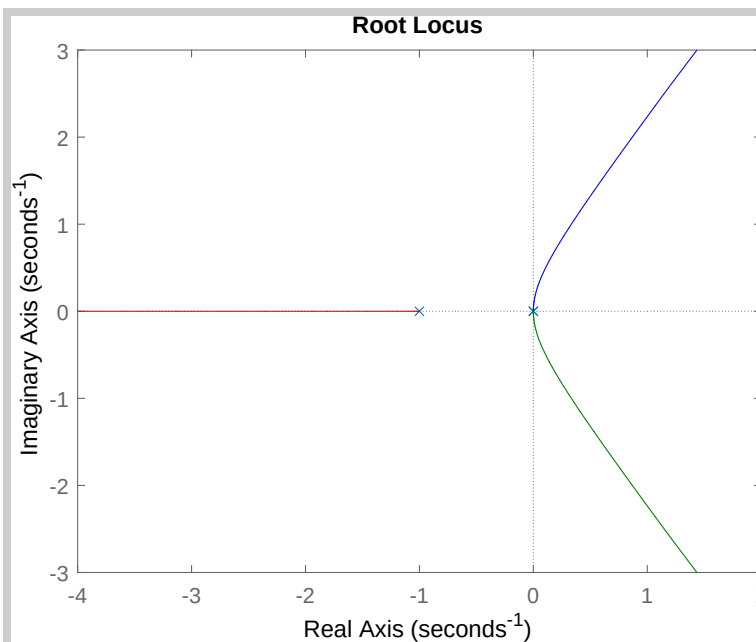
- (a) (3 points) Use a root-locus or a Nyquist diagram to show that *it is not possible* to select $K_i > 0$ so that the closed-loop connection of K and G_θ is asymptotically stable.

Solution:

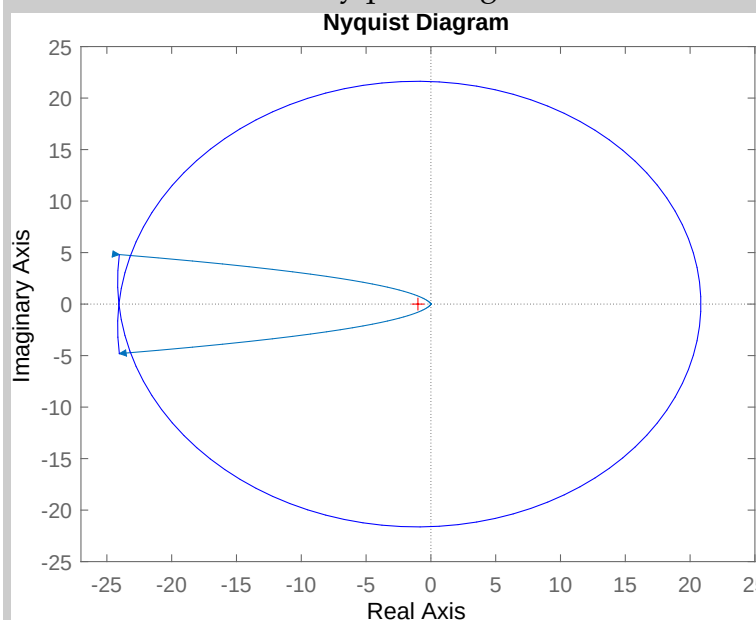
In this case

$$L(s) = \frac{1}{s} \frac{1}{s(s+1)} = \frac{1}{s^2(s+1)} \quad (+1 \text{ point})$$

A root-locus diagram should look like:



which shows two unstable poles on the right-hand side all for all $K > 0$. (+2 points)
ALTERNATIVELY a Nyquist diagram should look like:



with 2 clockwise encirclements for any $K > 0$ hence $Z_{\Gamma} = 0 - (-2) = 2$ unstable poles. (+2 points)

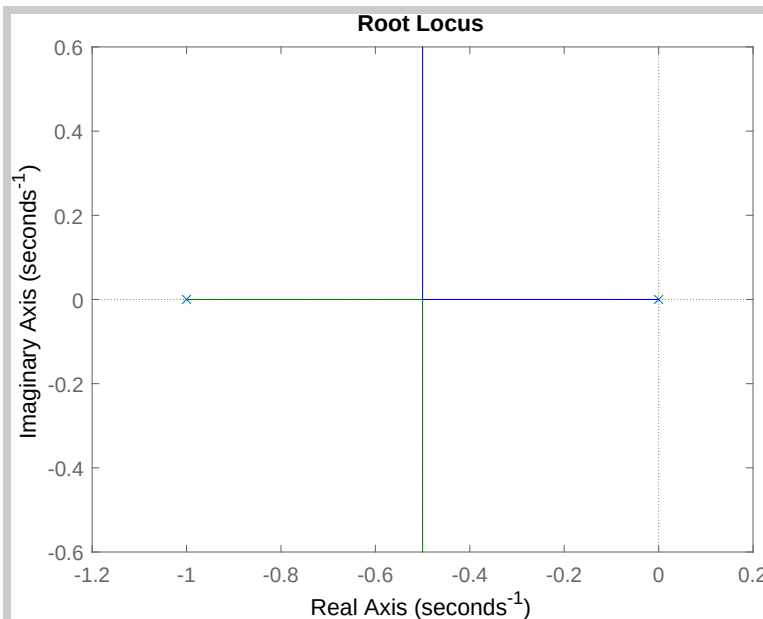
- (b) (3 points) Use a root-locus or a Nyquist diagram to show that *it is possible* to select $K_i > 0$ so that the closed-loop connection of K and G_{ω} is asymptotically stable.

Solution:

In this case

$$L(s) = \frac{1}{s} \frac{1}{s+1} = \frac{1}{s(s+1)} \quad (+1 \text{ point})$$

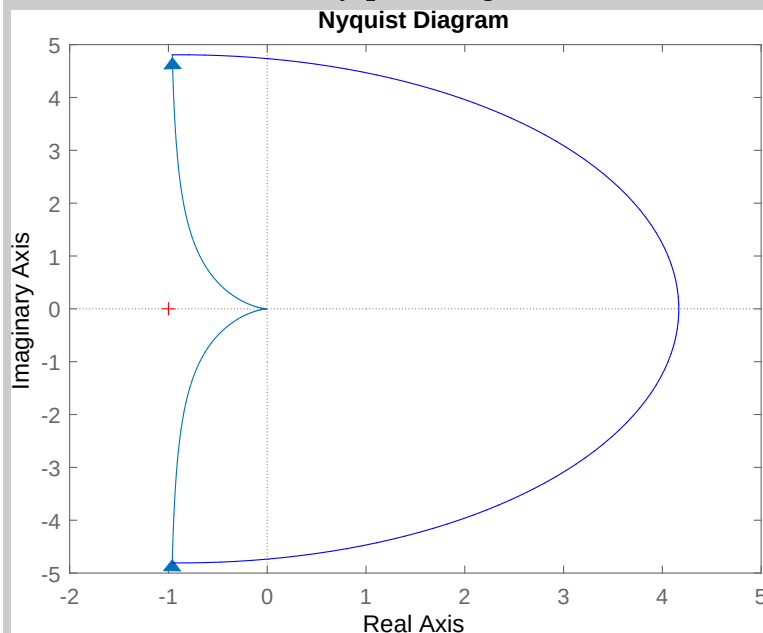
A root-locus diagram should look like:



which shows two stable poles for all $K > 0$.

(+2 points)

ALTERNATIVELY a Nyquist diagram should look like:



with 0 encirclements for any $K > 0$ hence $Z_r = 0 - (0) = 0$ unstable poles. This is the same plot as in Question II-1(f).

(+2 points)

(c) (1 point) Is the closed-loop in part (b) internally stable? Explain.

Solution: Yes because there are no pole-zero cancellations in forming $L(s)$. (+1 point)

(d) (1 point) Does the closed-loop in part (b) achieve asymptotic tracking of a constant reference angular velocity $\bar{\delta}_\omega$? Explain.

Solution: Yes because $K(s)$ has a pole at zero hence $S(s) = (1 + K(s)G_\omega(s))^{-1}$ has a zero at zero. (+1 point)

- (e) (1 point) Does the closed-loop in part (b) achieve asymptotic rejection of a constant torque disturbance δ_τ ? Explain.

Solution: Yes because $K(s)$ has a pole at zero hence $S(s) = (1 + K(s)G_\omega(s))^{-1}$ and $D(s) = K(s)S(s)$ both have zeros at zero. (+1 point)

3. Proportional-Integral Phase Control

Consider the feedback controller with transfer-function

$$K(s) = \frac{K_p s + K_i}{s} = K_p \frac{s + z}{s}, \quad z = K_i/K_p.$$

and answer the following questions:

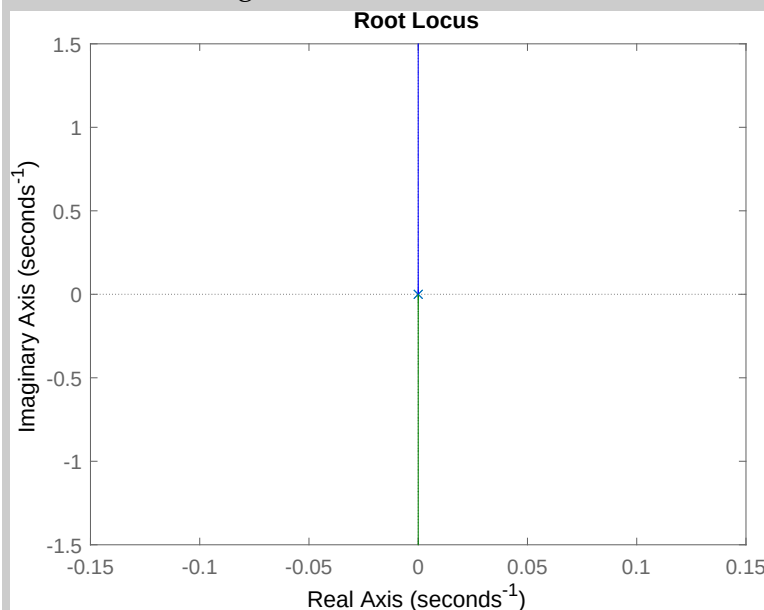
- (a) (3 points) Use a root-locus or a Nyquist diagram to show that *it is not possible* to select $K_p > 0$ and $z = b/J$ so that the closed-loop connection of K and G_θ is asymptotically stable.

Solution:

In this case

$$L(s) = \frac{s+1}{s} \frac{1}{s(s+1)} = \frac{1}{s^2} \quad (+1 \text{ point})$$

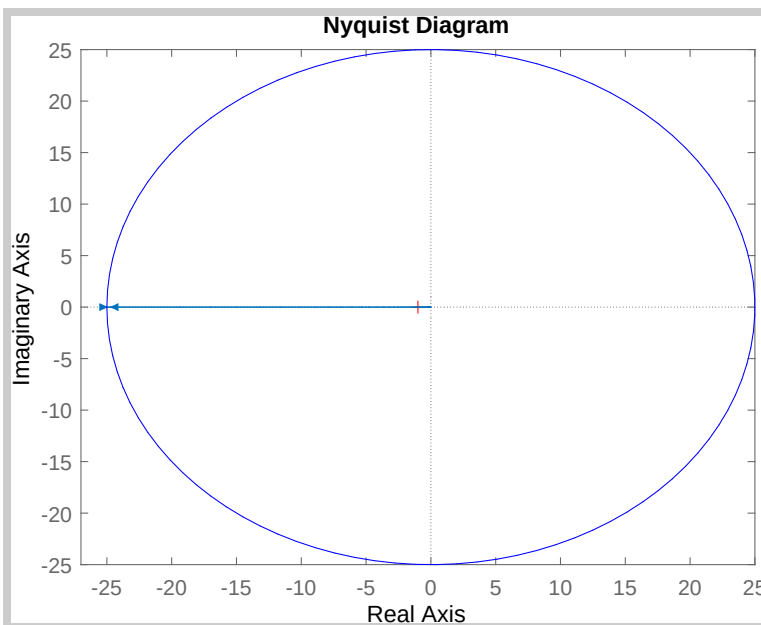
A root-locus diagram should look like:



which shows two imaginary poles on the right-hand side all for all $K > 0$.

(+2 points)

ALTERNATIVELY a Nyquist diagram should look like:



with $-1/K$ on the Nyquist diagram at all times, indicating the presence of two imaginary poles. (+2 points)

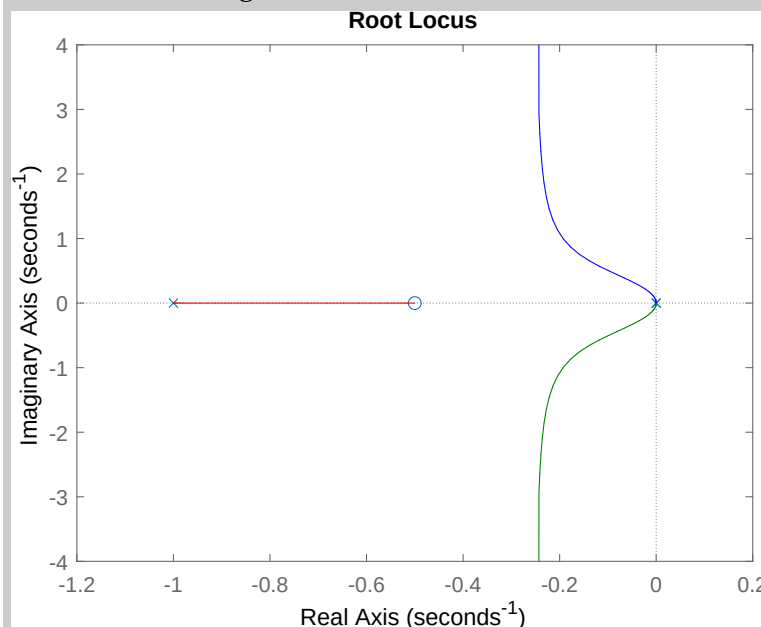
- (b) (3 points) Use a root-locus or a Nyquist diagram to show that *it is possible* to select $K_p > 0$ and $0 < z < b/J$ so that the closed-loop connection of K and G_θ is asymptotically stable.

Solution:

This is like in midterm 2. For example

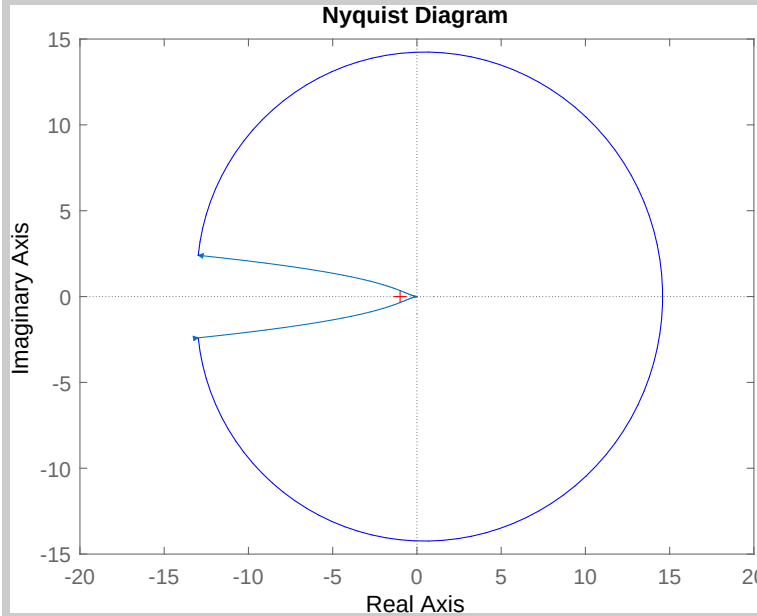
$$L(s) = \frac{s + 1/2}{s} \frac{1}{s(s + 1)} = \frac{s + 1/2}{s^2(s + 1)} \quad (+1 \text{ point})$$

A root-locus diagram should look like:



which shows two stable poles for all $K > 0$. The key is the asymptote centered at $c = \frac{0-1-(-1/2)}{2} = -0.25$. (+2 points)

ALTERNATIVELY a Nyquist diagram should look like:



with no encirclements of the $-1/K$ for all $K > 0$ hence no unstable roots because L has no poles on the right-hand side of the complex plane. (+2 points)

- (c) (1 point) Is the closed-loop in part (b) internally stable? Explain.

Solution: Yes because there are no pole-zero cancellations in forming $L(s)$. (+1 point)

- (d) (1 point) Does the closed-loop in part (b) achieve asymptotic tracking of a constant reference phase $\bar{\delta}_\theta$? Explain.

Solution: Yes because $G_\theta(s)$ and $K(s)$ have a pole at zero hence $S(s) = (1 + K(s)G_\theta(s))^{-1}$ has two zeros at zero. (+1 point)

- (e) (1 point) Does the closed-loop in part (b) achieve asymptotic rejection of a constant torque disturbance δ_τ ? Explain.

Solution: Yes because $G_\theta(s)$ and $K(s)$ have a pole at zero hence $S(s) = (1 + K(s)G_\theta(s))^{-1}$ has two zeros at zero and $D(s) = K(s)S(s)$ have a zero at zero. (+1 point)

4. Speed Control

So far you have been controlling the phase of the rollers. You also need to control the speed of the rollers.

- (a) (1 point) Show that the sum of the speed of the rollers, $\sigma_\omega = \omega_1 + \omega_2$, satisfy the differential equation:

$$J\dot{\sigma}_\omega(t) + b\sigma_\omega(t) = c(\sigma_u(t) + \sigma_\tau(t)), \quad (4)$$

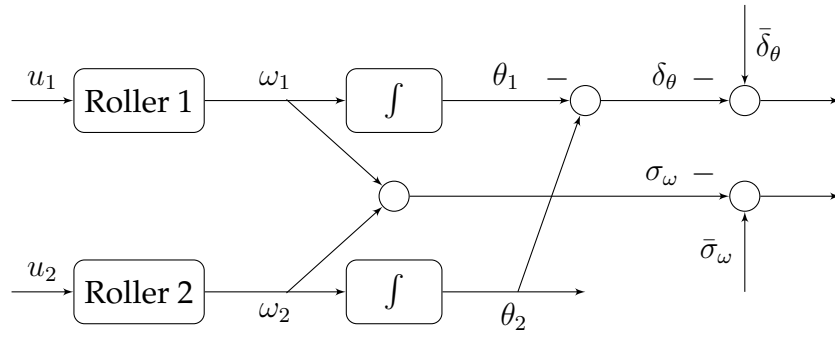


Figure 2: Block Diagram for Question II-5

where $\sigma_u = u_1 + u_2$, and $\sigma_\tau = (\tau_1 + \tau_2)/c$.

Solution: Add the two equations in (1) to obtain (4). (+1 point)

- (b) (4 points) Design a feedback controller that can achieve closed-loop asymptotic tracking of a constant reference speed $\bar{\sigma}_\omega$ and asymptotic rejection of a constant disturbance σ_τ .

Solution: The relationship between Σ_ω , Σ_τ and Σ_u is given by:

$$\Sigma_\omega(s) = G_\sigma(s)(\Sigma_u(s) + \Sigma_\tau(s)), \quad G_\sigma(s) = \frac{c/J}{(s + b/J)}. \quad (+1 \text{ point})$$

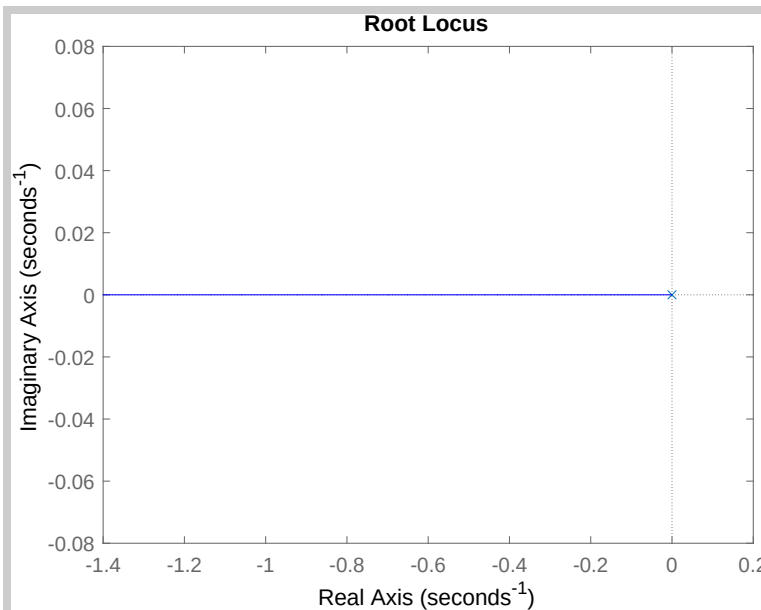
Because $G_\sigma(s) = G_\omega(s)$ one can achieve asymptotic tracking of a constant reference and asymptotic rejection of constant disturbance with integral control. For example a PI controller

$$K(s) = K_p \frac{s + z}{s}. \quad (+1 \text{ point})$$

A simple design is to choose $z = b/J = 1$ for a stable pole-zero cancellation that preserves internal stability. In this case

$$L(s) = \frac{s + 1}{s} \frac{1}{s + 1} = \frac{1}{s}. \quad (+1 \text{ point})$$

which leads to stability in closed-loop for all $K > 0$ as shown by the root-locus:



(+1 point)

5. Speed and Phase Control

Assume now that you have designed a phase controller

$$\Delta_u(s) = K_\delta(s)(\bar{\Delta}_\theta - \Delta_\theta)$$

and a speed controller

$$\Sigma_u(s) = K_\sigma(s)(\bar{\Sigma}_\omega - \Sigma_\omega)$$

(a) (2 points (bonus)) Show that

$$u_1 = \frac{1}{2}(\sigma_u - \delta_u), \quad u_2 = \frac{1}{2}(\sigma_u + \delta_u)$$

are the corresponding control inputs to the individual rollers.

Solution: Since

$$\delta_u = u_2 - u_1, \quad \sigma_u = u_1 + u_2$$

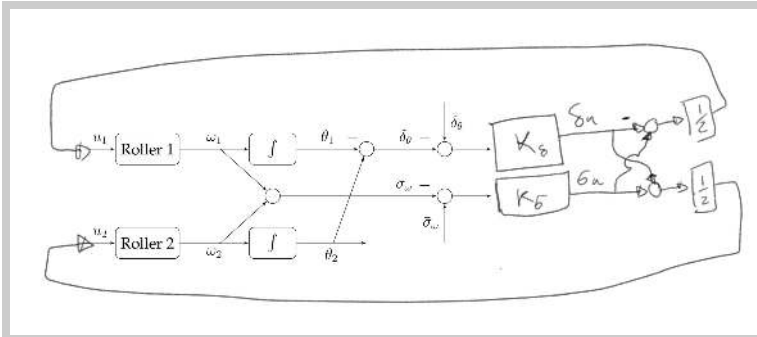
One can obtain u_1 and u_2 by summing and subtracting:

$$u_1 = \frac{1}{2}\sigma_u - \frac{1}{2}\delta_u, \quad u_2 = \frac{1}{2}\delta_u + \frac{1}{2}\sigma_u. \quad (+2 \text{ bonus points})$$

(b) (3 points) Assume $\tau_1 = \tau_2 = 0$ and complete the block-diagram in Fig. 2 to show the relationship between the signals shown in the figure and δ_u, σ_u .

Solution:

All that is needed to complete the diagram is the controllers and the calculation of the corresponding control inputs u_1 and u_2 using part (a). The resulting diagram should look like:



(+3 points)

- (c) (4 points) Explain what is necessary to ask of K_δ and K_σ if the phases of the rollers are to asymptotically converge, i.e. $\theta_1 \rightarrow \theta_2$, and the speed of the rollers are to asymptotically converge to $\omega_1 \rightarrow \omega_2 \rightarrow \bar{\omega}$. What are the corresponding references $\bar{\delta}_\theta$ and $\bar{\sigma}_\omega$?

Solution:

There is enough freedom in the control to design the phase controller and the speed controller independently. (+1 point)

If we design a controller such that $\delta_\theta = \theta_2 - \theta_1 \rightarrow 0$ then $\theta_2 \rightarrow \theta_1$. This can be achieved by asking that K_δ be stabilizing since G_θ already has a pole at the origin. Additionally, if K_δ has a pole at zero we will be able to reject constant input disturbances. The reference phase difference should be $\bar{\delta}_\theta = 0$. (+1 point)

Because the phase difference converges, that is $\delta_\theta = \theta_2 - \theta_1 \rightarrow 0$, it must also be true that the speed difference $\delta_\omega = \omega_2 - \omega_1 \rightarrow 0$ converges as well. That is, $\omega_2 \rightarrow \omega_1 \rightarrow \bar{\omega}$ where $\bar{\omega}$ is some constant speed. (+1 point)

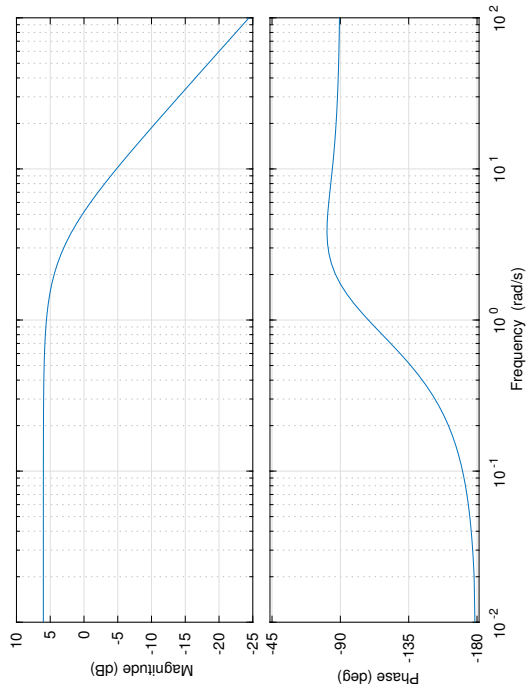
If we design a controller such that $\sigma_\omega = \omega_1 + \omega_2 \rightarrow \bar{\sigma}_\omega$ then $\omega_1 + \omega_2 \rightarrow 2\bar{\omega} \rightarrow \bar{\sigma}_\omega$. This can be achieved by asking that K_σ be stabilizing and has a pole at the origin since $G_\sigma = G_\omega$ and does not have a pole at the origin. The corresponding reference should therefore be $\bar{\sigma}_\omega = 2\bar{\omega}$. (+1 point)

Both requirements can be met by using the designs previously worked out in this exam.

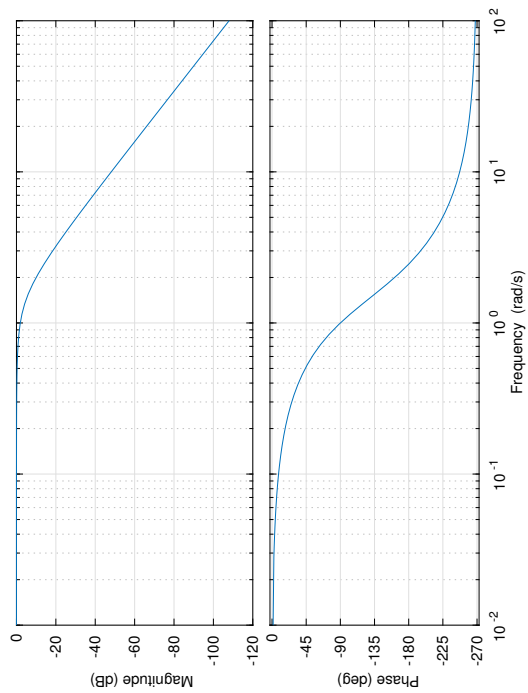
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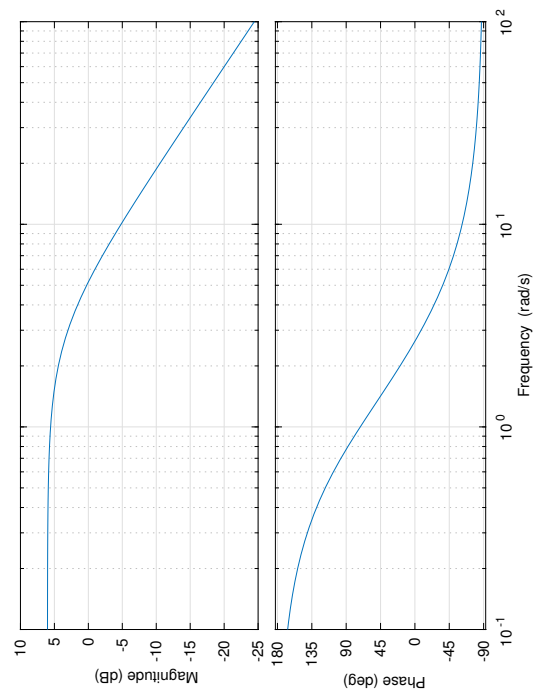
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DO NOT FORGET TO RETURN THIS PAGE WITH YOUR EXAM!



(b)



(a)



(c)

Figure 2: Diagrams for Question I-4

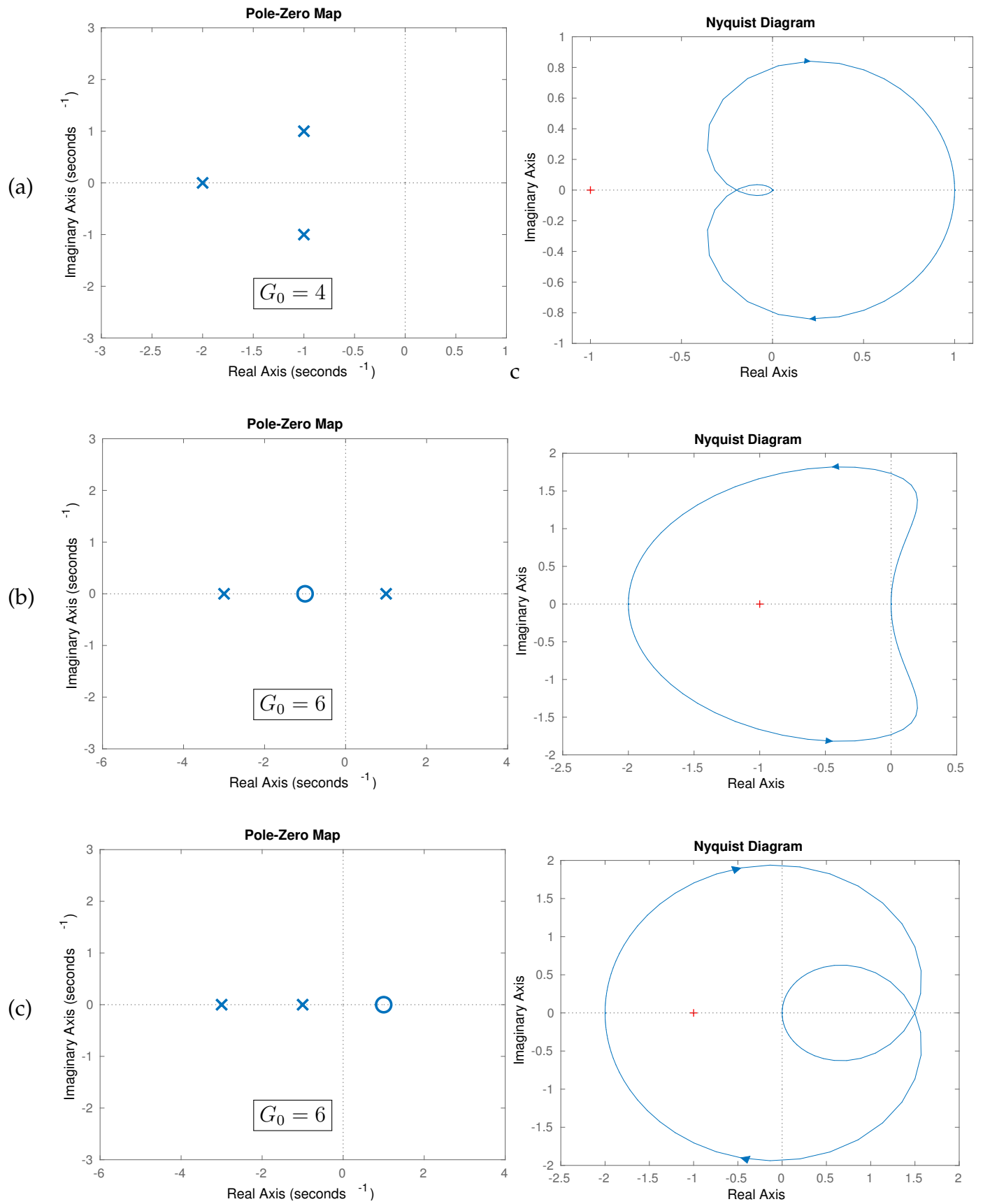


Figure 1: Diagrams for Questions I-1 through I-3