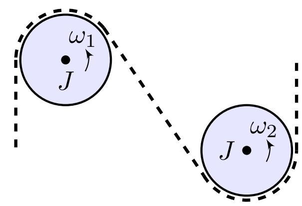
MAE143 B - Linear Control - Spring 2018
Midterm #2, May 22nd

Instructions:



1. This exam is **open book**. You can consult any printed or written material of your liking.
2. You have 70 minutes.
3. Many questions show you the answer. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book".
5. Do not forget to write your name and student ID.
6. This exam has 2 questions, 26 total points and 2 bonus points. Good luck!

Questions:

The motion of two identical rollers on a segment of a paper milling machine can be approximately described by the differential equations:

$$J\dot{\omega}_1(t) + b\,\omega_1(t) = c\,u_1(t) + \tau_1(t), \qquad J\dot{\omega}_2(t) + b\,\omega_2(t) = c\,u_2(t) + \tau_2(t), \tag{1}$$

where u_1 and u_2 are control commands for the motors of each roller and τ_1 and τ_2 are torque disturbances. Assume that all constants are positive.

In the first midterm you showed that the difference between the speeds of the rollers, $\delta_\omega = \omega_2 - \omega_1$, satisfy the differential equation:

$$J\dot{\delta}_\omega(t) + b\,\delta_\omega(t) = c\,(\delta_u(t) + \delta_\tau(t)), \tag{2}$$

where $\delta_u = u_2 - u_1$, $\delta_\tau = (\tau_2 - \tau_1)/c$, and that the Laplace transforms of the signals in (2) are related through:

$$\Delta_\omega(s) = G(s)(\Delta_u(s) + \Delta_\tau(s)), \qquad G(s) = \frac{c/J}{s + b/J}.$$

1. **Angular Velocity Control**

   (a) (2 points) Consider the proportional-integral control law

$$\delta_u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau, \quad e(t) = \bar{\delta}_\omega(t) - \delta_\omega(t) \tag{3}$$

† https://commons.wikimedia.org/wiki/File:Papermaking_machine_at_a_paper_mill_near_Pensacola.jpg

where $\bar{\delta}_\omega(t)$ is a reference velocity. Calculate the transfer-function $K(s)$ from e to δ_u and draw a block diagram of the corresponding closed-loop system showing the relationship between G, K and the signals δ_ω, δ_u, δ_τ, $\bar{\delta}_\omega$, and e.

Solution:

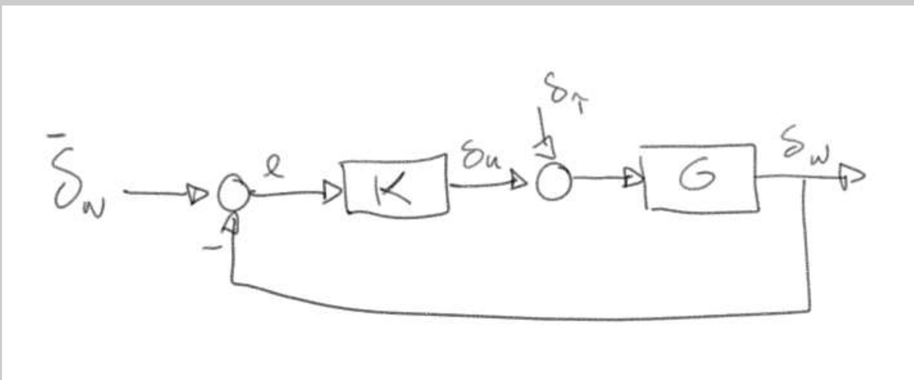
After using the Laplace transform:

$$\Delta_u(s) = K_p E(s) + \frac{K_i}{s} E(s)$$

from which the transfer-function from e to δ_u is:

$$K(s) = \frac{\Delta_{\delta_u}(s)}{E(s)} = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s} \qquad \text{(+1 point)}$$

A block-diagram should look like:



(+1 point)

(b) (2 points) Let $K_i/K_p = b/J$. Under what conditions on K_p is the closed-loop asymptotically stable?

Solution:

If $K_i/K_p = b/J$ then

$$K(s) = \frac{K_p(s + K_i/K_p)}{s} = \frac{K_p(s + b/J)}{s}$$

and the closed-loop sensitivity function is

$$S(s) = \frac{1}{1 + K(s)G(s)} = \frac{1}{1 + \frac{K_p(s+b/J)}{s}\frac{c/J}{s+b/J}} = \frac{s}{s + K_p c/J} \qquad \text{(+1 point)}$$

which is asymptotically stable for all $K_p c/J > 0$, that is all $K_p > 0$ since all model constants are assumed to be positive. (+1 point)

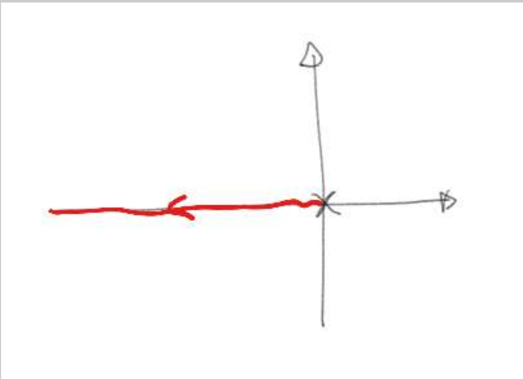
(c) (2 points) Repeat part (b) using a root-locus diagram. Clearly identify the loop transfer-function $L(s)$ and α.

Solution:

Letting $\alpha = K_p$ and

$$L(s) = \frac{s + b/J}{s} \frac{c/J}{s + b/J} = \frac{c/J}{s}$$

the root-locus diagram looks like:



(+1 point)

from which asymptotic stability follows for all $\alpha = K_p > 0$. (+1 point)

(d) (2 points) If the closed-loop is asymptotically stable is it also internally stable? Explain why or why not.

Solution:

It is also internally stable in spite of the pole-zero cancelation of $(s+b/J)$ because $b/J > 0$, that is, the cancelled pole is stable. (+2 points)

(e) (2 points) Does $\delta_\omega(t)$ converge to $\bar{\delta}_\omega \neq 0$ when $\bar{\delta}_\tau(t) = 0$ and $\bar{\delta}_\omega(t) = \bar{\delta}_\omega$, $t > 0$? Explain.

Solution:

It does because the controller has a pole at the origin, which implies asymptotic tracking of constant references. (+2 points)

(f) (2 points) Does $\delta_\omega(t)$ converge to $\bar{\delta}_\omega \neq 0$ when $\bar{\delta}_\tau(t) = \bar{\delta}_\tau \neq 0$ and $\bar{\delta}_\omega(t) = \bar{\delta}_\omega$, $t > 0$? Explain.

Solution:

It does because the controller has a pole at the origin, which implies asymptotic tracking of constant references *and* asymptotic rejection of constant disturbances. (+2 points)

2. **Phase Synchronization**

A stronger requirement is that both rollers should run at the exact same phase, that is $\theta_2 = \theta_1$, where $\theta_k(t) = \int_0^t \omega_k(\sigma) d\sigma$, $k = 1, 2$.

(a) (2 points) Represent in a block-diagram using integrators the relationship between

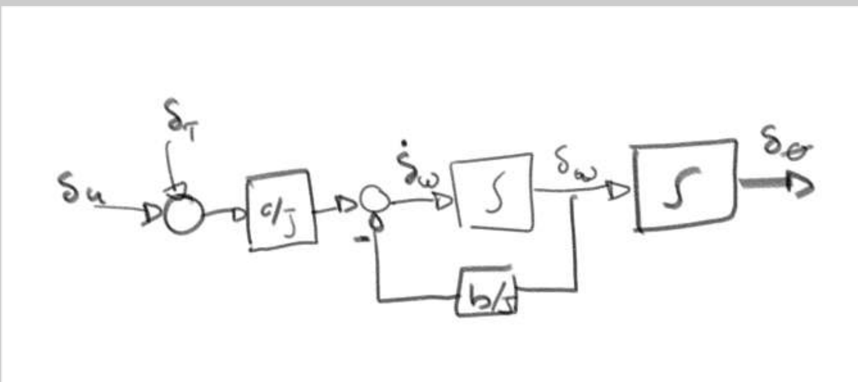
$$\delta_\theta = \theta_2 - \theta_1$$

and the signals δ_ω, δ_u, and δ_τ.

Solution:
The relationship between δ_θ and δ_ω is given by:

$$\begin{aligned}\delta_\theta(t) &= \theta_2(t) - \theta_1(t) \\ &= \int_0^t \omega_2(\sigma)\, d\sigma - \int_0^t \omega_1(\sigma)\, d\sigma \\ &= \int_0^t (\omega_2(\sigma) - \omega_1(\sigma))\, d\sigma \\ &= \int_0^t \delta_\omega(\sigma)\, d\sigma. \end{aligned} \qquad \text{(+1 point)}$$

A complete block-diagram should look like:



(+1 point)

(b) (2 points) Assume zero-initial conditions and show that the Laplace transforms of the signals in (2) and δ_θ, are related through:

$$\Delta_\theta(s) = G(s)(\Delta_u(s) + \Delta_\tau(s)), \qquad G(s) = \frac{c/J}{s(s + b/J)}.$$

Solution:
Since

$$\delta_\theta(t) = \int_0^t \delta_\omega(\sigma)\, d\sigma$$

it follows that

$$\Delta_\theta(s) = \frac{\Delta_\omega(s)}{s} \qquad \text{(+1 point)}$$

from which

$$\Delta_\theta(s) = \frac{c/J}{s(s+b/J)} \left(\Delta_u(s) + \Delta_\tau(s)\right). \qquad \text{(+1 point)}$$

(c) (2 points) Consider the proportional control law

$$\delta_u(t) = K_\theta e_\theta(t), \quad e_\theta(t) = \bar{\delta}_\theta(t) - \delta_\theta(t) \tag{4}$$

where $\bar{\delta}_\theta(t)$ is a reference phase. Calculate the closed-loop transfer-function from $\bar{\delta}_\theta$ to e_θ.

Solution: From $\bar{\delta}_\theta$ to e:

$$S(s) = \frac{1}{1+K(s)G(s)} = \frac{1}{1+K_\theta \frac{c/J}{s(s+b/J)}} = \frac{s(s+b/J)}{s^2 + sb/J + K_\theta c/J} \qquad \text{(+2 points)}$$

(d) (2 points) For what values of K_θ is the closed-loop asymptotically stable?

Solution:
Rewrite the denominator of $S(s)$ in the cannonical form:

$$s^2 + sb/J + K_\theta c/J = s^2 + 2\zeta\omega_n s + \omega_n^2$$

where

$$\omega_n = \sqrt{K_\theta c/J}, \qquad \zeta = \frac{b}{2J\sqrt{K_\theta c/J}} = \frac{b}{2\sqrt{K_\theta cJ}} \qquad \text{(+1 point)}$$

Following the analysis in Chapter 6, if $K_\theta > 0$ then $\omega_n^2 > 0$ and $\zeta > 0$ so that the closed-loop is asymptotically stable. If $K_\theta < 0$ then $\omega_n^2 < 0$ and the closed-loop is unstable. (+1 point)

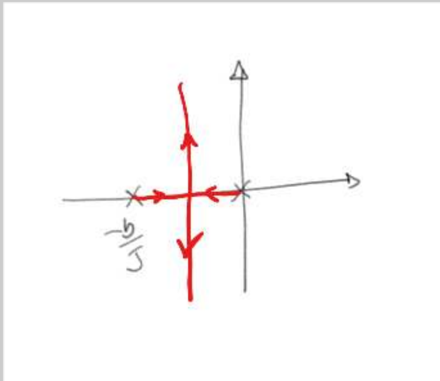
NOTE TO GRADERS: A complete analysis of the roots is also fine as an answer.

(e) (2 points) Repeat part (d) using a root-locus diagram. Clearly identify the loop transfer-function $L(s)$ and α.

Solution:
Letting $\alpha = K_\theta$ and

$$L(s) = G(s) = \frac{c/J}{s(s+b/J)}$$

the root-locus diagram looks like:



(+1 point)

from which asymptotic stability follows for all $\alpha = K_\theta > 0$. (+1 point)

(f) (2 points) Does $\delta_\theta(t)$ converge to $\bar{\delta}_\theta \neq 0$ when $\bar{\delta}_\tau(t) = 0$ and $\bar{\delta}_\theta(t) = \bar{\delta}_\theta$, $t > 0$? Explain.

Solution:

It does because the plant $G(s)$ has a pole at the origin, which implies asymptotic tracking of constant references, *and* the disturbance $\bar{\delta}_\tau = 0$. (+2 points)

(g) (2 points) How can you modify the controller so that $\delta_\theta(t)$ converge to $\bar{\delta}_\theta \neq 0$ when $\bar{\delta}_\tau(t) = \bar{\delta}_\tau \neq 0$ and $\bar{\delta}_\theta(t) = \bar{\delta}_\theta$, $t > 0$? Explain.

Solution:

In order for the closed-loop to reject the constant disturbance the controller needs to have a pole at the origin. (+1 point)

For example, the simplest controller with a pole at the origin is the PI controller

$$K(s) = \frac{K(s+z)}{s} \qquad \text{(+1 point)}$$

NOTE TO GRADERS: An I controller as an answer to this question is also fine, even though it is not stabilizing.

(h) (2 points (bonus)) Analyze closed-loop stability for the controller you proposed in part (g) using the root-locus method.

Solution:

NOTE TO GRADERS: This is a tough question. Any progress here earns marks.

Consider the PI controller

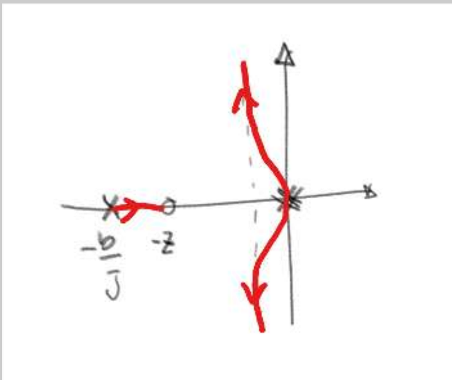
$$K(s) = \frac{K(s+z)}{s}$$

choose $\alpha = K$ so that the loop transfer-function is:

$$L(s) = \frac{s+z}{s}\frac{c/J}{s(s+b/J)} = \frac{c/J(s+z)}{s^2(s+b/J)}.$$

The asymptotes will be centered at $(z-b/J)/2$ so that choosing $0 < z < b/J$ one achieves closed-loop stability with tracking and disturbance rejection of constant inputs for a large enough gain. (+1 bonus point)

The corresponding root-locus diagram looks like:



(+1 bonus point)

Note that an I controller alone is not stabilizing!