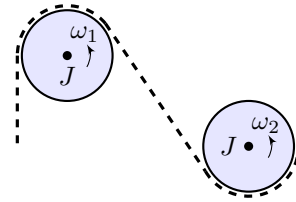


MAE143 B - Linear Control - Spring 2018
Midterm, May 3rd

Instructions:

1. This exam is **open book**. You can consult any printed or written material of your liking.
2. You have 70 minutes.
3. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book".
5. Do not forget to write your name and student number.
6. This exam has 2 questions, 25 total points and 2 bonus points.
7. Good luck!



Questions:

The motion of two identical rollers on a segment of a paper milling machine can be approximately described by the differential equations:

$$J\dot{\omega}_1(t) + b\omega_1(t) = cu_1(t) + \tau_1(t), \quad J\dot{\omega}_2(t) + b\omega_2(t) = cu_2(t) + \tau_2(t), \quad (1)$$

where u_1 and u_2 are control commands for the motors of each roller and τ_1 and τ_2 are torque disturbances. Assume that all constants are positive. A basic requirement is that both rollers should run at the exact same velocity, that is $\omega_2 = \omega_1$. In this exam you will design controllers that attempt to achieve that goal.

1. Modeling and Open-Loop Response

- (a) (1 point) Show that the difference between the speeds of the rollers, $\delta_\omega = \omega_2 - \omega_1$, satisfy the differential equation:

$$J\dot{\delta}_\omega(t) + b\delta_\omega(t) = c(\delta_u(t) + \delta_\tau(t)), \quad (2)$$

where $\delta_u = u_2 - u_1$, and $\delta_\tau = (\tau_2 - \tau_1)/c$.

Solution: Subtract the two equations in (1) to obtain (2). (+1 point)

[†] https://commons.wikimedia.org/wiki/File:Papermaking_machine_at_a_paper_mill_near_Pensacola.jpg

- (b) (3 points) Assume zero-initial conditions and show that the Laplace transforms of the signals in (2) are related through:

$$\Delta_\omega(s) = G(s)(\Delta_u(s) + \Delta_\tau(s)), \quad G(s) = \frac{c/J}{s + b/J}.$$

Draw a block diagram of the corresponding open-loop system showing the relationship between G and the signals δ_ω , δ_u and δ_τ .

Solution: Assume zero initial conditions and take the Laplace transform:

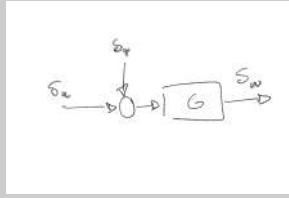
$$sJ \Delta_\omega(s) + b \Delta_\omega(s) = c(\Delta_u(s) + \Delta_\tau(s)), \quad (+1 \text{ point})$$

which can be rearranged as

$$(s + b/J) \Delta_\omega(s) = c/J(\Delta_u(s) + \Delta_\tau(s)), \quad (+1 \text{ point})$$

from which the given relationship follows. (+1 point)

A block-diagram should look like:



(+1 point).

- (c) (3 points) Assume zero initial conditions and that $\delta_u(t) = 0$, and use the Laplace transform to calculate the response $\delta_\omega(t)$ to a disturbance $\delta_\tau(t) = \bar{\delta}_\tau \neq 0, t \geq 0$. Sketch the response and identify the transient and steady-state components of $\delta_\omega(t)$.

Solution: Set $\Delta_u(s) = 0$, and use the fact that a constant signal has the Laplace transform $\Delta_\tau(s) = \bar{\delta}_\tau/s$, to calculate

$$\begin{aligned} \Delta_\omega(s) &= G(s)(\Delta_u(s) + \Delta_\tau(s)) \\ &= \frac{c/J}{s + b/J} \frac{\bar{\delta}_\tau}{s} \end{aligned}$$

Expand in partial fractions:

$$\Delta_\omega(s) = \frac{k_1}{s} + \frac{k_2}{s + b/J}$$

where

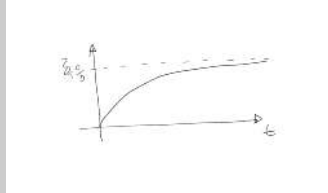
$$k_1 = \lim_{s \rightarrow 0} s \Delta_\omega(s) = \bar{\delta}_\tau \frac{c}{b}, \quad k_2 = \lim_{s \rightarrow -b/J} (s + b/J) \Delta_\omega(s) = -\bar{\delta}_\tau \frac{c}{b} \quad (+1 \text{ point})$$

from which

$$\delta_\omega(t) = \underbrace{\bar{\delta}_\tau \frac{c}{b}}_{\delta_\omega^{ss}(t)} - \underbrace{\bar{\delta}_\tau \frac{c}{b} e^{-(b/J)t}}_{\delta_\omega^{tr}(t)}, \quad t \geq 0 \quad (+1 \text{ point})$$

where we have also identified the steady-state and the transient parts of the response.

A sketch should look like:



(+1 point)

- (d) (2 points) If possible, use the frequency response method to calculate the steady-state component of $\delta_\omega(t)$ you calculated in part (c). If not possible, explain why.

Solution: Because all constants are positive, $-b/J < 0$ and the system is asymptotically stable. (+1 point)

Therefore it is possible to calculate the steady-state part of the response using the frequency-response method:

$$\delta_\omega^{ss}(t) = G(0)\bar{\delta}_\tau = \bar{\delta}_\tau \frac{c}{b}. \quad (+1 \text{ point})$$

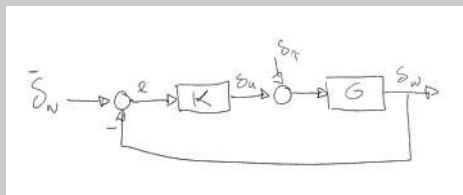
2. Controller Design

- (a) (2 points) Consider the proportional control law

$$\delta_u(t) = Ke(t), \quad e(t) = \bar{\delta}_\omega(t) - \delta_\omega(t) \quad (3)$$

where $\bar{\delta}_\omega(t)$ is a reference velocity. Draw a block diagram of the corresponding closed-loop system showing the relationship between G , K and the signals δ_ω , δ_u , $\bar{\delta}_\tau$, $\bar{\delta}_\omega$, and e .

Solution: A block-diagram should look like:



(+2 points)

- (b) (3 points) Calculate the closed-loop transfer-functions from $\bar{\delta}_\omega$ to e , from $\bar{\delta}_\tau$ to e , and from $\bar{\delta}_\omega$ to δ_ω .

Solution: From $\bar{\delta}_\omega$ to e :

$$S(s) = \frac{1}{1 + GK} = \frac{1}{1 + K \frac{c/J}{s+b/J}} = \frac{s + b/J}{s + b/J + Kc/J}. \quad (+1 \text{ point})$$

From δ_τ to e :

$$D(s) = \frac{G}{1 + GK} = GS = \frac{c/J}{s + b/J} \times \frac{s + b/J}{s + b/J + Kc/J} = \frac{c/J}{s + b/J + Kc/J} \quad (+1 \text{ point})$$

From $\bar{\delta}_\omega$ to δ_ω :

$$H(s) = \frac{GK}{1 + GK} = KD = \frac{Kc/J}{s + b/J + Kc/J}. \quad (+1 \text{ point})$$

(c) (3 points) Assume that $\delta_\tau(t) = 0$ and $\bar{\delta}_\omega(t) = \bar{\delta}_\omega \neq 0, t \geq 0$, and show that

$$\delta_\omega(t) = H(0)(1 - e^{-\lambda t})\bar{\delta}_\omega + e^{-\lambda t}\delta_\omega(0), \quad t \geq 0, \quad \lambda = \frac{b}{J} + K\frac{c}{J}, \quad (4)$$

where $H(s)$ is the transfer function from $\bar{\delta}_\omega$ to δ_ω you calculated in part (b).

Solution: The closed-loop response when $\delta_\tau = 0$ is described by the equations:

$$\dot{\delta}_\omega + \frac{b}{J}\delta_\omega = \frac{c}{J}\delta_u, \quad \delta_u = K(\bar{\delta}_\omega - \delta_\omega)$$

which can be combined to produce

$$\dot{\delta}_\omega + \left(\frac{b}{J} + K\frac{c}{J} \right) \delta_\omega = K\frac{c}{J}\bar{\delta}_\omega \quad (+1 \text{ point})$$

whose solution to a constant $\bar{\delta}_\omega$ is given by

$$\delta_\omega(t) = \frac{K\frac{c}{J}}{\frac{b}{J} + K\frac{c}{J}}(1 - e^{-\lambda t})\bar{\delta}_\omega + e^{-\lambda t}\delta_\omega(0), \quad t \geq 0. \quad (+1 \text{ point})$$

Note that

$$H(s) = \frac{K\frac{c}{J}}{s + \frac{b}{J} + K\frac{c}{J}}$$

so that

$$H(0) = \frac{K\frac{c}{J}}{\frac{b}{J} + K\frac{c}{J}} \quad (+1 \text{ point})$$

NOTE TO GRADERS: Many solutions here. Note that it is not important to *derive* the solution. Identifying a formulations, e.g. ODE, Laplace, etc, from which a solution can be calculated is more important.

- (d) (1 point) Identify the transient and steady-state components of $\delta_\omega(t)$ in part (c).

Solution: The transient is the part of the solution that converges to zero, while the steady-state solution persists over time. In this case:

$$\delta_\omega(t) = \delta_\omega^{tr}(t) + \delta_\omega^{ss}(t)$$

where the transient is:

$$\delta_\omega^{tr}(t) = (\delta_\omega(0) - H(0)\bar{\delta}_\omega)e^{-\lambda t} \quad (+1/2 \text{ point})$$

and the steady-state is:

$$\delta_\omega^{ss}(t) = H(0)\bar{\delta}_\omega \quad (+1/2 \text{ point})$$

- (e) (2 points) Under what conditions in K does the closed-loop signal $\delta_\omega(t)$ in part (c) converge to $\bar{\delta}_\omega$ when $\bar{\delta}_\omega = 0$?

Solution: Because $\bar{\delta}_\omega = 0$:

$$\delta_\omega(t) = e^{-\lambda t}\delta_\omega(0)$$

all that is needed for convergence is that $\lambda > 0$, that is asymptotic stability.

(+1 point)

Assuming all constants are positive the closed-loop is asymptotically stable if $\lambda = b/J + Kc/J > 0$, which is the case for all $K > -b/c$.

(+1 point)

- (f) (2 points) Assume that $\delta_\tau(t) = \bar{\delta}_\tau \neq 0$, select a proper K and use the frequency response method to calculate the steady-state error to a constant reference input $\bar{\delta}_\omega(t) = \bar{\delta}_\omega \neq 0$. Does the closed-loop signal $\delta_\omega(t)$ converge to $\bar{\delta}_\omega$?

Solution: Let's take any $K > 0$ so that the closed-loop is asymptotically stable. (+1 point)

The steady-state error due to a constant reference input can be calculated from the transfer-function $S(s)$ as in:

$$e^{ss}(t) = S(0)\bar{\delta}_\omega = \frac{b/J}{b/J + Kc/J}\bar{\delta}_\omega = \frac{b}{b + Kc}\bar{\delta}_\omega \quad (+1 \text{ point})$$

Since $S(0) \neq 0$, hence $e^{ss}(t) \neq 0$, $\delta_\omega(t)$ does not converge to $\bar{\delta}_\omega$. (+1 point)

- (g) (3 points) Modify the feedback law (3) so that $\delta_\omega(t)$ converges to zero when $\bar{\delta}_\omega(t) = 0$ and $\delta_\tau(t) = \bar{\delta}_\tau \neq 0$, $t \geq 0$. Make sure to select a feedback gain so that the closed-loop is asymptotically stable and clearly explain the requirements and your reasoning.

Solution: In order for the closed-loop to converge to zero in response to a non-zero input constant disturbance K must have a pole at zero. (+1 point)

The simplest controller that can achieve that is one with transfer-function:

$$K(s) = \frac{K}{s} \quad (+1 \text{ point})$$

For closed-loop stability one needs to have the poles of

$$S(s) = \frac{1}{1 + KG} = \frac{1}{1 + \frac{K}{s} \frac{c/J}{s+b/J}} = \frac{s}{s^2 + b/Js + Kc/J}$$

have negative real part. (+1 point)

This happens if $K > 0$ since $b/J < \sqrt{\Delta}$, when $\Delta = (b/J)^2 - 4Kc/J > 0$. When $\Delta < 0$ the real part is always $-b/(2J)$ which is always negative.

- (h) (1 point (bonus)) Does your controller in part (g) perform a pole-zero cancellation? Explain why your closed-loop system is internally stable.

Solution: My controller did not perform a pole-zero cancellation, hence all that is needed for internal stability is that the poles of S have negative real part. (+1 bonus point)

ALTERNATIVELY: If your controller performed a cancellation of the pole $-b/J$ than all is fine if S is asymptotically stable. You cannot cancel the pole at 0. (+1 bonus point)

- (i) (1 point (bonus)) What happens in part (g) if $\bar{\delta}_\omega(t) = \bar{\delta}_\omega \neq 0, t \geq 0$?

Solution: Because a pole at zero in K means also a pole at zero in KG the close-loop has asymptotic tracking of constant references so that the response will track the non-zero reference asymptotically. (+1 bonus point)