MAE143 B - Linear Control - Spring 2019
Final, June 12th

Instructions:

1. This exam is open book. You may use whatever written materials of your choice, including your notes and the book, and a calculator with no communication capabilities.
2. You have 170 minutes. Don't get stuck! Come back later if you have time.
3. Write answers on your "Blue Book" and attach the provided graph sheet.
4. Do not forget to write your name and student ID.
5. The exam has 8 questions for a total of 71 points and 2 bonus points.

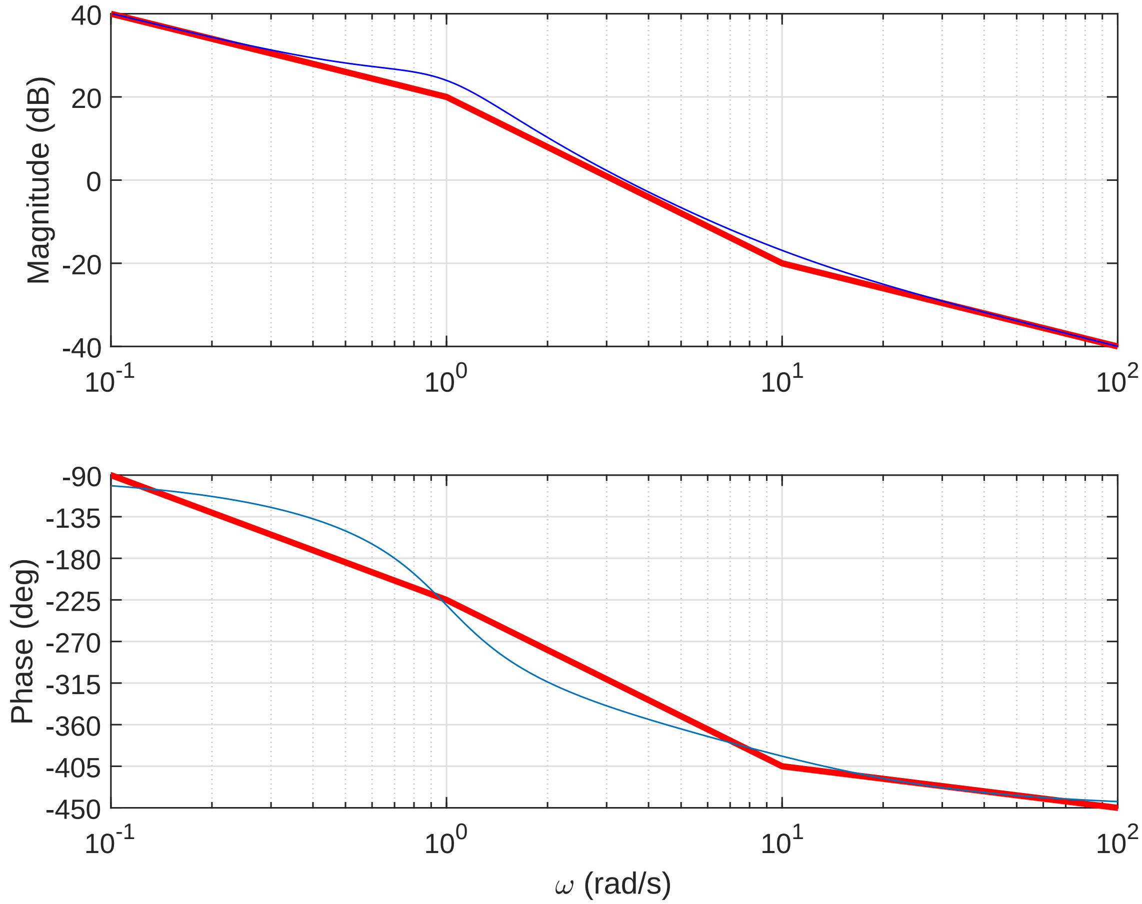


Figure 1: Bode diagrams for Part I

1. **Frequency Domain Analysis and Design.**

   Answer the following questions based on the Bode magnitude and phase diagrams and straight-line approximations shown in Figure 1.

   (a) (2 points) Segment the plots and calculate the slope of the magnitude and phase of the diagrams in each segment.

Solution: Magnitude:

1. $0 < \omega \leq 1$: -20 dB/dec
2. $1 \leq \omega \leq 10$: -40 dB/dec
3. $\omega \geq 10$: -20 dB/dec

(+1 point)

Phase:

1. $0 < \omega \leq 0.1$: 0 deg/dec
2. $0.1 \leq \omega \leq 1$: -135 deg/dec

3. $1 \le \omega \le 10$: -180 deg/dec

4. $\omega \ge 10$: -45 deg/dec

(+1 point)

(b) (5 points) Use the slopes from part (a) to identify poles and zeros of the corresponding transfer-function and indicate whether they are real, complex, or imaginary, and the sign of the corresponding real part. Explain your reasoning.

Solution: There's a pole at zero because the magnitude slope starts at -20dB. (+1 point)

There's a stable complex pole with $0 < \zeta < 1$, $\omega_n = 1$ because the magnitude plot exceeds the straight-line approximation and a zero at $s = 1$ because the magnitude slope is $-20 - 40 + 20 = -40$dB/dec. The zero must be on the right-hand side of the complex plane because the phase slope at $\omega = 0.1$ is equal to $-90 - 45 = -135$deg/dec. (+3 points)

There's a zero at $s = 10$ because the magnitude slope is at $\omega = 10$ becomes $-40 + 20 = -20$dB/dec. The zero is on the right-hand side because the phase slope increases to $-135 - 45 = -180$deg/dec at $\omega = 1$. (+1 point)

(c) (3 points) Use parts (a) and (b) to write down the corresponding transfer-function.

Solution: Transfer-function should be of the form:

$$G(s) = \tilde{\beta}\frac{(1-s)(1-s/10)}{s(1+2\zeta s+s^2)}, \qquad 0 < \zeta < 1, \qquad \text{(+2 points)}$$

where the value of $\tilde{\beta}$ can be inferred from

$$20\log_{10}|\tilde{\beta}/(j1)| = 20 \qquad \Longrightarrow \qquad \tilde{\beta} = 10 \qquad \text{(+1 point)}$$

Any reasonable $0 < \zeta < 1$ is fine.

(d) (1 point) Is the transfer-function you obtained in part (c) asymptotically stable in open-loop? Explain.

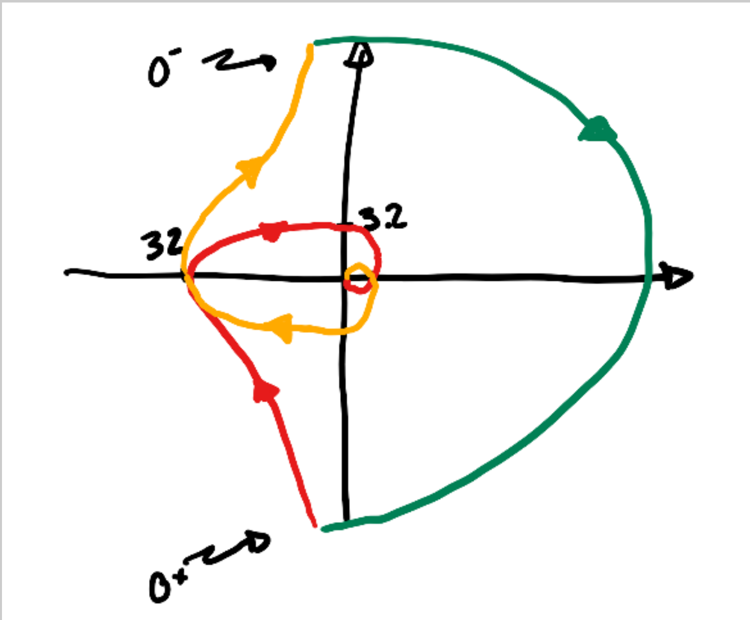
Solution:

No because of the pole at zero. (+1 point)

(e) (4 points) Sketch the Nyquist diagram corresponding to the transfer-function you obtained in part (c). Based on this diagram, are there positive values of gain for which this transfer-function is asymptotically stable in closed loop? Explain.

Solution:

Reading from the Bode plot



in which I am guestimating that it crosses -180 at around 30dB and then -270 at around 10dB and -360 at -10db. Any other reasonable guesses are fine!

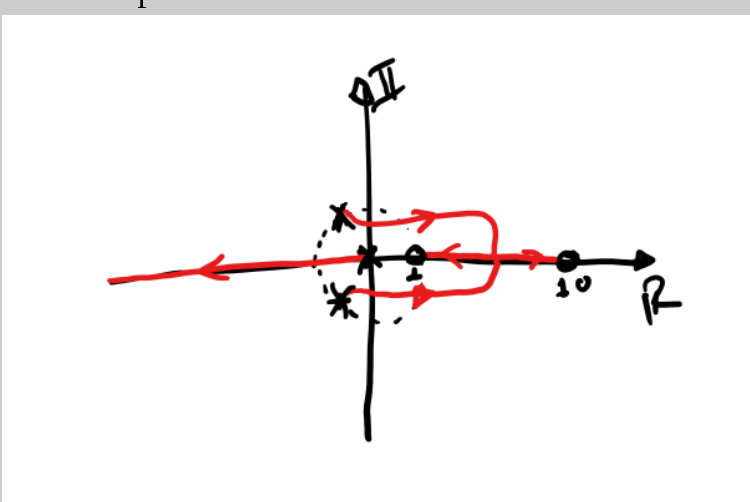
(+3 points)

Because $P_\Gamma = 0$, it will remain stable for any value of $-1/\alpha$ to the left of the first crossing of the negative real axis. In my estimates, for any $1/\alpha > 32$ or $0 < \alpha < 1/32$. It is unstable with two unstable poles for any other value due to the two clockwise encirclements.

(+1 point)

(f) (4 points) Sketch the root-locus diagram corresponding to the transfer-function you obtained in part (c). Based on this diagram, are there positive values of gain for which this transfer-function is asymptotically stable in closed loop? Explain.

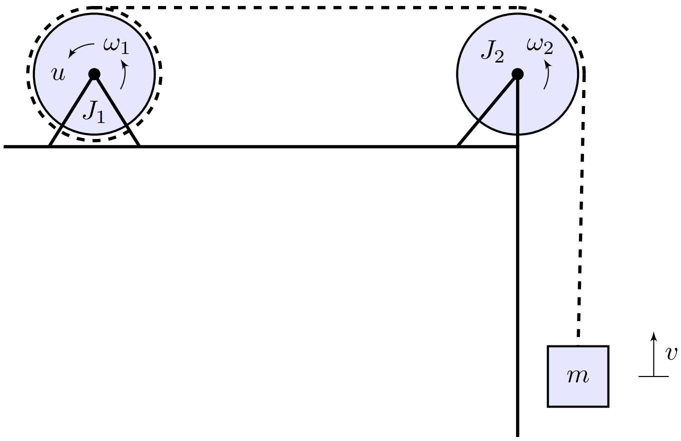
Solution:
From the poles and zeros on the transfer-fuction



(+3 points)

As in the Nyquist diagram, all poles will remain stable for a small value of gain before two roots depart to the right-hand side of the complex plane.

(+1 point)



Source: `www.cranestodaymagazine.com`

Figure 2: Winch/pulley/mass systems

For the rest of this exam, consider the winch/pulley/mass system in Fig. 2, in which the winch, with inertia J_1, angular speed ω_1, and radius r_1, is connected to a pulley, with inertia J_2, angular speed ω_2, and radius r_2, which is connected to the mass m by an inextensible and massless cable that is never slack. Consider also that $r_1 = r_2 = r$ and that a torque u is applied at the winch as shown.

In the Midterm 1 you showed that the motion of this system could be approximately modeled by the ordinary differential equation

$$J_r\,\dot{v}(t) + b_r v(t) = r(u(t) + w(t)), \qquad w(t) = -mgr, \qquad v(t) = r\,\omega_1(t) = r\,\omega_2(t), \tag{1}$$

where $J_r = J_1 + J_2 + mr^2$, $b_r = b_1 + b_2$, and b_1 and b_2 represent viscous damping at the winch and pulley.

$\boxed{\text{For the rest of the exam assume that } J_r = b_r = 1,\ m = g = 10, \text{ and } r = 0.1.}$

You have also shown that the Laplace transform of the above signals are related by

$$V(s) = G_v(s)(U(s) + W(s)), \qquad G_v(s) = \frac{r/J_r}{s + b_r/J_r} = \frac{0.1}{s+1},$$

and, in the Midterm 2, that

$$X(s) = G_x(s)(U(s) + W(s)), \qquad G_x(s) = \frac{r/J_r}{s(s + b_r/J_r)} = \frac{0.1}{s^2+s},$$

where

$$x(t) = x(0) + \int_0^t v(\tau)\,d\tau.$$

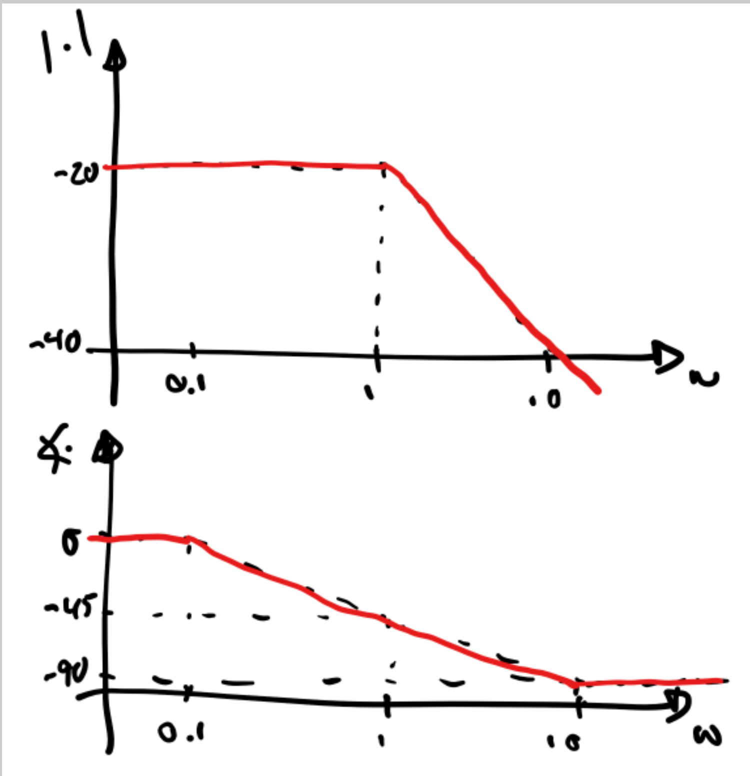
2. **Open-loop System for Speed Control.**

   (a) (1 point) Is $G_v(s)$ asymptotically stable? Explain.

Solution: $G_v = \frac{0.1}{(s+1)}$ which has all poles with negative real part, hence it is asymptotically stable. (+1 point)

(b) (3 points) Sketch the magnitude and phase Bode plots of $G_v(s)$.

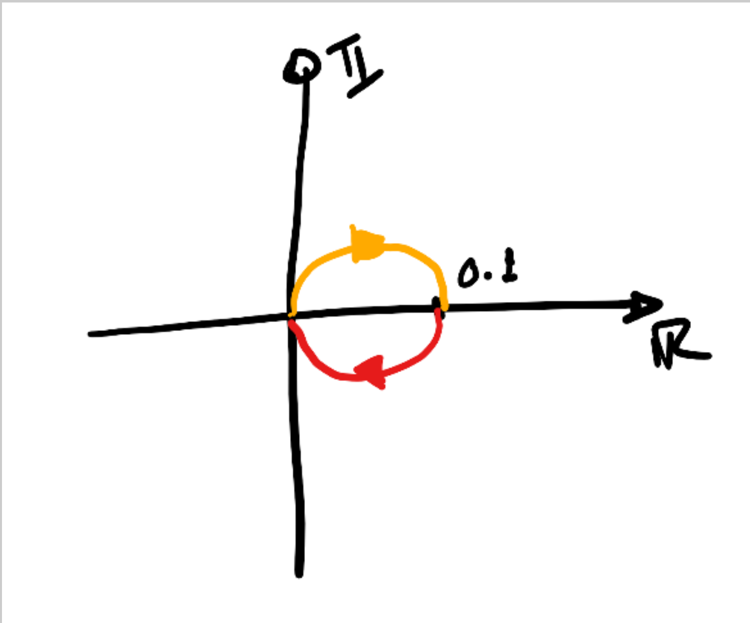
Solution:
A sketch should look like:



(+3 points)

(c) (2 points) Sketch the Nyquist plot associated with $G_v(s)$.

Solution:
A sketch should look like:



(+2 points)

3. **Speed Control I**

One student suggested that in order to design a feedback controller for controlling the speed of the mass, one can simply compensate for the gravitational force by offsetting the winch torque to be

$$u(t) = \tilde{u}(t) + mgr. \tag{2}$$

From this perspective, answer the following questions.

(a) (1 point) Show that the model (1) with (2) is equivalent to

$$J_r\,\dot{v}(t) + b_r v(t) = r\tilde{u}(t).$$

Solution: Just plug in (2) into (1) to obtain

$$J_r\dot{v} + b_r v = r(\tilde{u} + mgr - mgr) = r\tilde{u}.$$

(+1 point)

(b) (2 points) Is it possible to design a proportional controller

$$\tilde{u}(t) = K_p\,e(t), \qquad e(t) = \bar{v}(t) - v(t), \tag{3}$$

that would be able to asymptotically stabilize the closed-loop system and ensure that

$$\lim_{t\to\infty} v(t) = \bar{v} \tag{4}$$

when $\bar{v}(t) = \bar{v}$, $t \geq 0$? If not possible, explain why. If possible, calculate a suitable gain.

Solution: Since

$$v = G_v \tilde{u}, \qquad G_v = \frac{0.1}{s+1}$$

and

$$S = \frac{1}{1 + K_p G_v} = \frac{s+1}{s+1+0.1K_p}$$

for instance, any $K_p > 0$ is such that S is asymptotically stable. (+1 point)

However, it will not achieve asymptotic tracking because S does not have a zero at the origin. (+1 point)

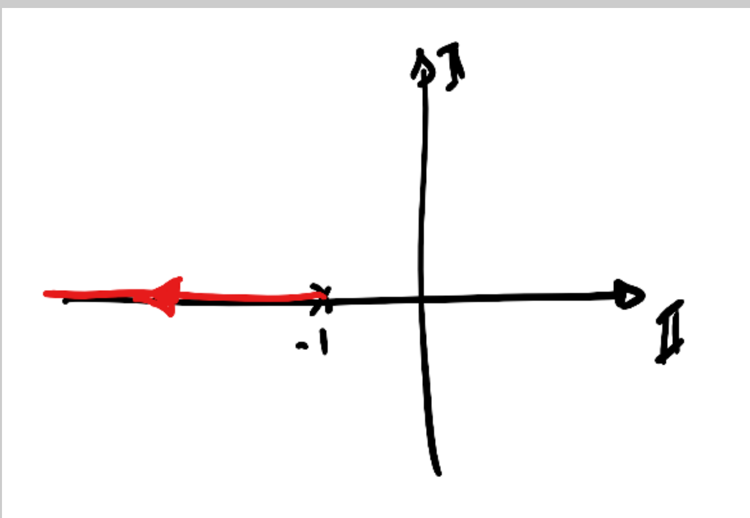
Alternatively, because nor G_v nor K have a pole at the origin.

(c) (2 points) Use a root-locus or a Nyquist diagram to support your argument about asymptotic closed-loop stability in part (b). If you have already used one such argument then skip to the next question.

Solution:

One solution is via Nyquist which you already drew on a previous question: $P_\Gamma = 0$ and there are no encirclements of $-1/\alpha$, $\alpha > 0$ so any positive $K_p = \alpha$ will work. (+2 points)

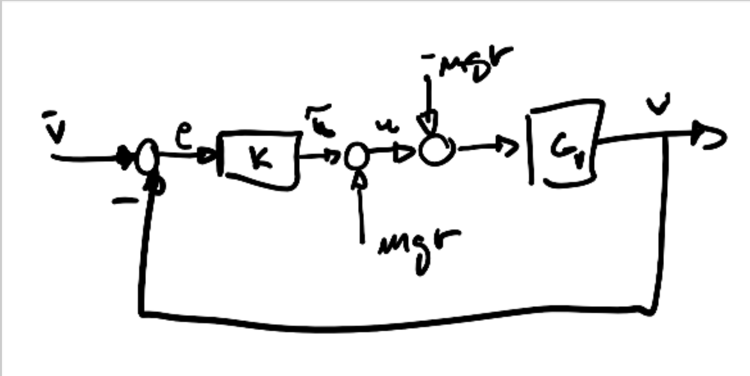
Alternatively, a root-locus plot would looke like:



which shows that any $K_p = \alpha > 0$ works.

(d) (2 points) Draw a block-diagram for the closed-loop system resulting from using the controller in part (b) in term of the signals v, $\bar{v}$, e, u, and $\tilde{u}$.

Solution: Should look like:



(+2 points)

(e) (1 point) Would your answer to part (b) change if the value of g deviated from the value used to design the controller? Explain.

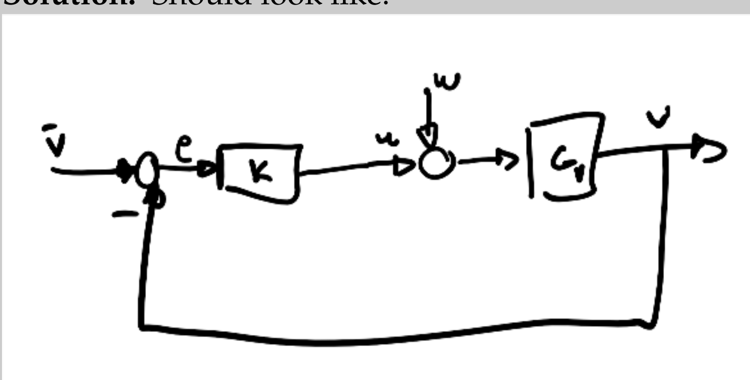
Solution:
In this case the $\tilde{u}$ would not compensate the gravity exactly anymore and one would need to take into account a non-zero disturbance. (+1 point)

4. **Speed Control II**

Another student suggested that in order to design a feedback controller for controlling the speed of the mass, one should leave model (1) as is and treat the gravitational force as a constant disturbance $w(t) = \bar{w} = -mgr = -10$ Nm. From this perspective, answer the following questions.

(a) (2 points) Draw a block-diagram for a standard closed-loop system in which $u = Ke$, $e = \bar{v} - v$, in terms of the signals v, $\bar{v}$, e, u, and w.

Solution: Should look like:



(+2 points)

(b) (2 points) Is it possible to design a proportional controller

$$u(t) = K_p\, e(t), \qquad e(t) = \bar{v}(t) - v(t), \tag{5}$$

that would be able to asymptotically stabilize the closed-loop system and ensure that

$$\lim_{t\to\infty} v(t) = \bar{v} \tag{6}$$

when $\bar{v}(t) = \bar{v}$, $t \geq 0$? If not possible, explain why. If possible, calculate a suitable gain.

Solution: In this case

$$v = G_v(u + w), \qquad G_v = \frac{0.1}{s+1}$$

As in the previous problem

$$S = \frac{1}{1 + K_p G_v} = \frac{s+1}{s + 1 + 0.1K_p}$$

for instance, any $K_p > 0$ is such that S is asymptotically stable. (+1 point)

As before, it will not achieve asymptotic tracking because nor S nor

$$D = \frac{G_v}{1 + K_p G_v} = \frac{0.1}{s + 1 + 0.1K_p}$$

have a zero at the origin. (+1 point)

Alternatively, because nor G_v nor K have a pole at the origin.

(c) (2 points) Use a root-locus or a Nyquist diagram to support your argument about asymptotic closed-loop stability in part (b). If you have already used one such argument then skip to the next question.

Solution: Same solution as in Question 2 (d) (+2 points)

(d) (1 point) Would your answer to part (b) change if the value of g deviated from the value used to design the controller? Explain.

Solution: A change in g would affect only the value of the disturbance, which would continue to be nonzero. Stability analysis would be the same. (+1 point)

(e) (1 point) Compare and constrast the solution from Question 3 with the solution from Question 4.

Solution: The main difference between the two solutions is that in Question 2 the controller does not have to deal with the constant gravitational disturbance which is been compensated by the offset in the control signal. None achieve asymptotic tracking nor asymptotic disturbance rejection but can be keep the closed-loop stable. The first solution will achieve better performance because of the disturbance compensation. (+1 point)

5. **Speed Control III**

Consider the same setup as in Question 4 and answer the following questions.

(a) (2 points) Is it possible to design a proportional-integral controller

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau, \quad e(t) = \bar{v}(t) - v(t), \tag{7}$$

that would be able to asymptotically stabilize the closed-loop system and ensure that

$$\lim_{t \to \infty} v(t) = \bar{v} \tag{8}$$

when $\bar{v}(t) = \bar{v}$, $t \geq 0$? If not possible, explain why. If possible, calculate suitable gains.

Solution:
In this case

$$v = G_v(u + w), \qquad u = Ke, \qquad G_v = \frac{0.1}{s+1}, \qquad K = \frac{K_p s + K_i}{s},$$

and

$$S = \frac{1}{1 + KG_v} = \frac{1}{1 + \frac{K_p s + K_i}{s} \frac{0.1}{s+1}} = \frac{s^2 + s}{s^2 + (1 + 0.1K_p)s + 0.1K_i}$$

for instance, any $K_p > 0$, $K_i > 0$ is such that S is asymptotically stable. (+1 point)

It will achieve asymptotic tracking and asymptotic disturbance rejection because S and

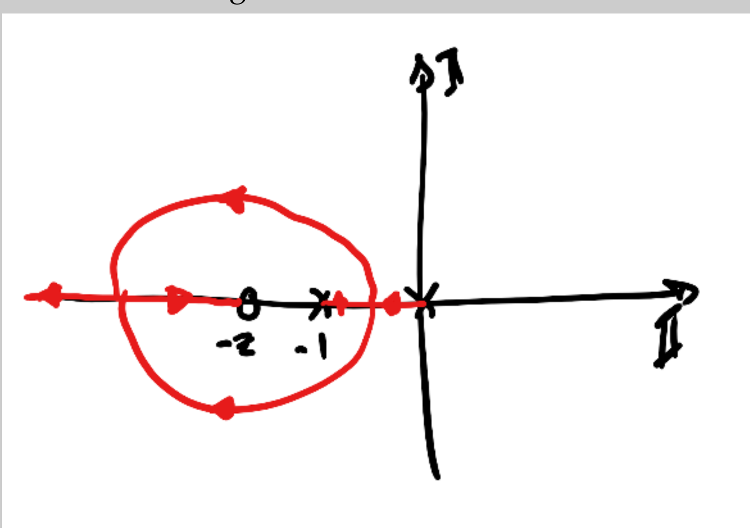
$$D = \frac{G_v}{1 + KG_v} = \frac{0.1s}{s^2 + (1 + 0.1K_p)s + 0.1K_i}$$

have a zero at the origin. (+1 point)

Alternatively, because K has a pole at the origin.

(b) (2 points) Use a root-locus or a Nyquist diagram to support your argument about asymptotic closed-loop stability in part (b). If you have already used one such argument then skip to the next question.

Solution:
A root-locus diagram should look like:



which shows that for any $z = K_i/K_p > 1$ and $K_p > 0$ the closed-loop system will be asymptotically stable. Any other choice of $z > 0$ would also work but with a different diagram. (+2 points)

(c) (1 point) Would your answer to part (b) change if the value of g deviated from the value used to design the controller? Explain.

Solution: A change in g would affect only the value of the disturbance which would continue to be rejected if that change was also a constant. (+1 point)

(d) (1 point) Compare and constrast the solution from Question 4 with the solution from Question 5.

Solution: In Question 5, the addition of the integrator in the controller ensures asymptotic tracking and disturbance rejection, whereas in Question 4 both were lacking.

(+1 point)

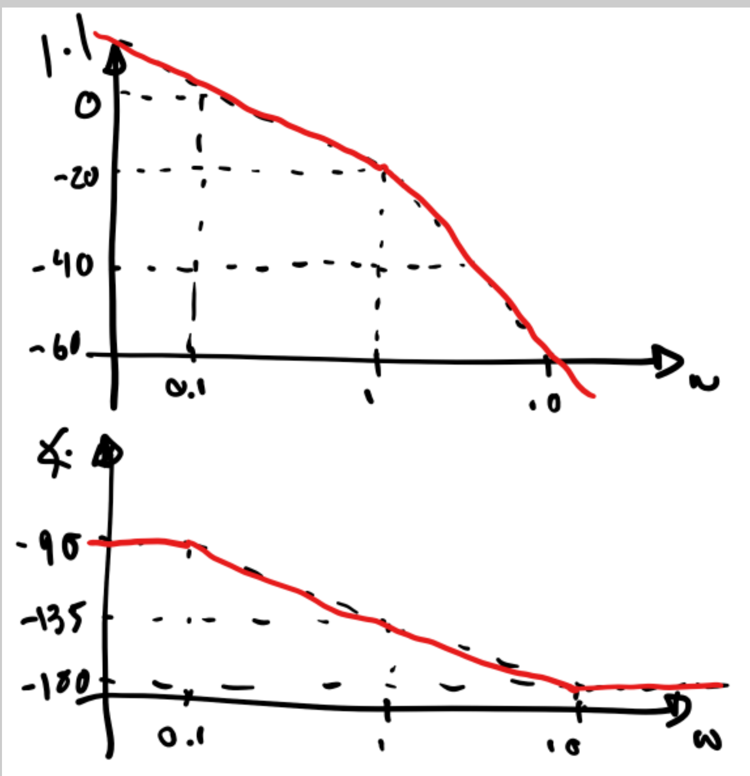
6. **Open-loop System for Position Control.**

(a) (1 point) Is $G_x(s)$ asymptotically stable? Explain.

Solution: $G_x = \frac{0.1}{s(s+1)}$ which has one pole at the imaginary axis, hence it is not asymptotically stable. (+1 point)

(b) (3 points) Sketch the magnitude and phase Bode plots of $G_x(s)$.

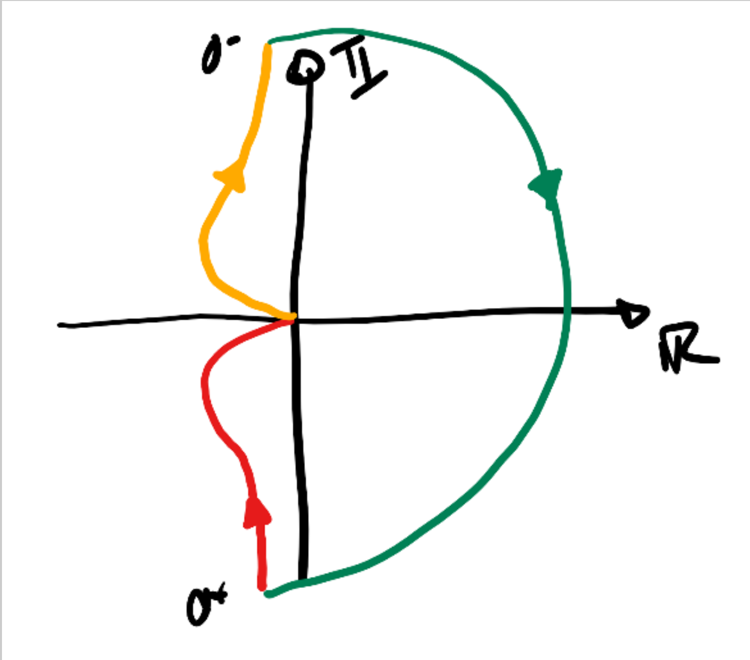
Solution:
A sketch should look like:



(+3 points)

(c) (2 points) Sketch the Nyquist plot associated with $G_x(s)$.

Solution:
A sketch should look like:



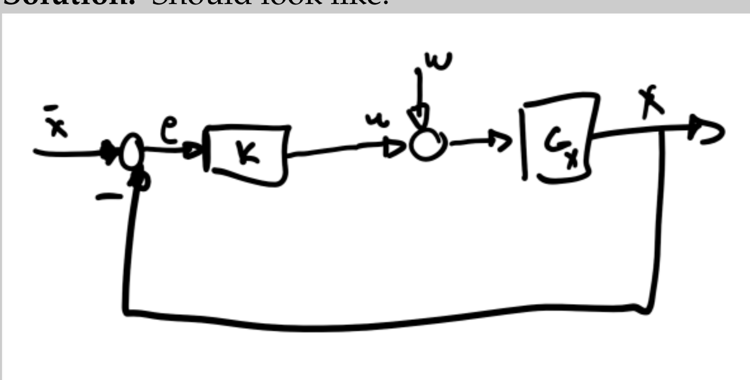
(+2 points)

7. **Position Control I**

   A third student interjected that in many cases it is the position of the mass and not the velocity that might need to be controlled. From this perspective, answer the following questions.

   (a) (2 points) Draw a block-diagram for a standard closed-loop system in which $u = Ke$, $e = \bar{x} - x$, in terms of the signals x, $\bar{x}$, e, u, and w.

Solution: Should look like:



(+2 points)

   (b) (2 points) Is it possible to design a proportional controller

$$u(t) = K_p\, e(t), \qquad\qquad e(t) = \bar{x}(t) - x(t), \tag{9}$$

that would be able to asymptotically stabilize the closed-loop system and ensure that

$$\lim_{t\to\infty} x(t) = \bar{x} \tag{10}$$

when $\bar{x}(t) = \bar{x}$, $t \geq 0$? If not possible, explain why. If possible, calculate a suitable gain.

Solution:
In this case

$$x = G_x(u + w), \qquad u = Ke, \qquad G_x = \frac{0.1}{s^2 + s}, \qquad K = K_p$$

and

$$S = \frac{1}{1 + KG_x} = \frac{1}{1 + K_p \frac{0.1}{s^2+s}} = \frac{s^2 + s}{s^2 + s + 0.1K_p}$$

for instance, any $K_p > 0$ is such that S is asymptotically stable. (+1 point)

It will achieve asymptotic tracking because S but not asymptotic disturbance rejection because

$$D = \frac{G_x}{1 + KG_x} = \frac{0.1}{s^2 + s + 0.1K_p}$$

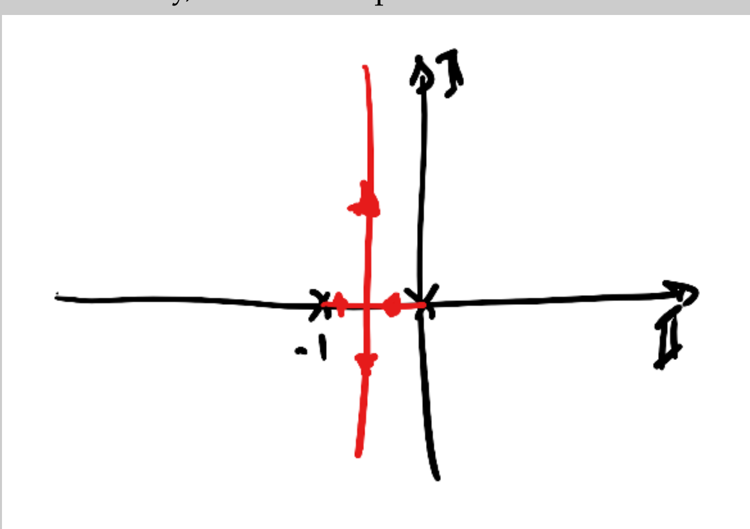
does not have a zero at the origin. Therefore, $\lim_{t\to\infty} x(t) \neq \bar{x}$ because of the gravitational disturbance. (+1 point)

Alternatively, because G_x has a pole at the origin but K does not.

(c) (2 points) Use a root-locus or a Nyquist diagram to support your argument about asymptotic closed-loop stability in part (b). If you have already used one such argument then skip to the next question.

Solution:
One solution is via Nyquist diagram that you already drew for another question: $P_\Gamma = 0$ and there are no encirclements of $-1/\alpha$, $\alpha > 0$ so any positive $K_p = \alpha$ will work. (+2 points)

Alternatively, a root-locus plot would looke like:



which shows a stable closed-loop system for any $K_p = \alpha > 0$.

(+2 points)

(d) (1 point) Would your answer to part (b) change if the value of g deviated from the value used to design the controller? Explain.

Solution: A change in g would affect only the value of the disturbance, which would continue to be nonzero. All other analysis would be the same. (+1 point)

8. **Position Control II**

After implementing a proportional controller as in Problem 7, a student found out that starting from $x(0) = \dot{x}(0) = 0$ and for values of reference position $0 \leq \bar{x} \leq 10$m that a) the system response display no oscillations, b) the mass would not reach the desired reference position $\bar{x}$, and c) the mass would actually come down instead of coming up toward $\bar{x}$. With that in mind, assume that $\bar{x}(t) = \bar{x}$, $t > 0$, and answer the following questions.

(a) (1 point (bonus)) Show that

$$x = H\bar{x} + Dw, \qquad u = Q\bar{x} - Hw,$$

where

$$H = \frac{0.1K_p}{s^2 + s + 0.1K_p}, \qquad D = \frac{0.1}{s^2 + s + 0.1K_p}, \qquad Q = \frac{K_p(s^2 + s)}{s^2 + s + 0.1K_p}.$$

Solution:
NOTE TO GRADERS: Students get points for just verifying the formulas.
Follows from

$$S = \frac{1}{1 + KG_x} = \frac{s^2 + s}{s^2 + s + 0.1K_p}$$

and the fact that

$$H = G_x K S = \frac{0.1K_p}{s^2 + s + 0.1K_p},$$
$$D = G_x S = \frac{0.1}{s^2 + s + 0.1K_p},$$
$$Q = KS = \frac{K_p(s^2 + s)}{s^2 + s + 0.1K_p}.$$

(+1 point)

(b) (2 points) Calculate the range of $K_p > 0$ for which the closed-loop response to a constant reference position displays no oscillations.

Solution: Because

$$\omega_n^2 = 0.1K_p, \qquad 2\zeta\omega_n = 1 \qquad \text{(+1 point)}$$

from which for stability and no oscillations

$$\zeta = \frac{1}{2\omega_n} = \frac{1}{2\sqrt{0.1K_p}} \geq 1$$

therefore

$$\omega_n^2 = 0.1K_p, \qquad 2\zeta\omega_n = 10 \qquad < 0.1K_p \leq 1/4 \qquad \Longrightarrow 0 < K_p < 2.5 \qquad \text{(+1 point)}$$

(c) (2 points) Use the frequency-response method to show that

$$\lim_{t\to\infty} x(t) = \bar{x} + \bar{w}/K_p = \bar{x} - 10/K_p, \qquad \text{and} \qquad \lim_{t\to\infty} u(t) = -\bar{w} = 10.$$

Solution: Because both the reference as well as the disturbance are constants

$$\begin{aligned}\lim_{t\to 0^+} x(t) &= H(0)\bar{x} + D(0)\bar{w} \\ &= \bar{x} + \frac{\bar{w}}{K} = \bar{x} - 10/K_p\end{aligned}$$

(d) (2 points) Use the initial-value theorem

$$\lim_{t\to 0^+} f(t) = \lim_{|s|\to\infty} sF(s)$$

to show that

$$\lim_{t\to 0^+} u(t) = K_p\,\bar{x}.$$

Solution: First calculate

$$\begin{aligned}U(s) &= \frac{K_p(s^2+s)}{s^2+s+0.1K_p} \times \frac{\bar{x}}{s} - \frac{0.1K_p}{s^2+s+0.1K_p} \times \frac{\bar{w}}{s} \\ &= \frac{K_p(s^2+s) + 0.1K_p\bar{w}}{s(s^2+s+0.1K_p)}\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 0^+} u(t) &= \lim_{|s|\to\infty} sU(s) \\ &= \lim_{|s|\to\infty} \frac{K_p(s^2+s)\bar{x} + 0.1K_p\bar{w}}{s^2+s+0.1K_p} = K_p\bar{x}\end{aligned}$$

(e) (2 points) Use the above facts to infer what was the student's gain. Explain what is happening to the student's closed-loop control system.

Solution:
Because starting at $\bar{x} = 10$m the mass would actually move down, it must be that

$$0 \leq \bar{x} - 10/K_p = 10(1 - 1/K_p)$$

or $1/K_p \geq 1$ or $K_p \leq 1$. Indeed, $K_p = 1$ would explain why that starts happeningn at $\bar{x} = 10$m and is compatible with no oscillations, $K_p = 1 < 2.5$ and of course not reaching the desired target. (+2 points)

(f) (3 points) Knowing that the maximum torque available at the winch is 400 Nm, can you propose a value of gain that: a) would not lead to closed-loop oscillations, b) the mass would not come down when $\bar{x} \geq 1$ m, and c) would not saturate the motor when $\bar{x} < 10$m?

Solution: For no oscillation: $K_p < 2.5$.
Based on (c), the mass will not come down for $\bar{x} \geq 1$ if $K_p > 10$.
The largest value of torque will occur with the largest error, that is at the initial time so

$$K_p \bar{x} < 400 \text{ Nm}$$

so that for $\bar{x} < 20$m

$$K_p < 20 = \frac{400}{20} < \frac{400}{\bar{x}}.$$

It is not possible to satisfy all three requirements simultaneously. (+3 points)

(g) (1 point (bonus)) Can you add to the requirements of part (f) that the closed-loop system d) would have a steady-state error of less than 1% when 1 m $\leq \bar{x} \leq 100$ m?

Solution:
Of course not since (f) already has no solution but, for perspective, the steady state-error

$$\frac{|\bar{x} - \bar{x} + 10/K_p|}{\bar{x}} < 0.01$$

is such that for $\bar{x} \geq 1$

$$K_p > \frac{1000}{\bar{x}} > 1000.$$

To achieve that level of performance would need to deal with severe oscillations!
(+1 bonus point)