MAE143 B - Linear Control - Spring 2019
Midterm, April 26th

Instructions:

1. This exam is **open book**. You can consult any printed or written material of your liking.
2. You have 50 minutes.
3. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book".
5. Do not forget to write your name and student number.
6. This exam has 3 questions, 24 total points and 2 bonus points.
7. Good luck!

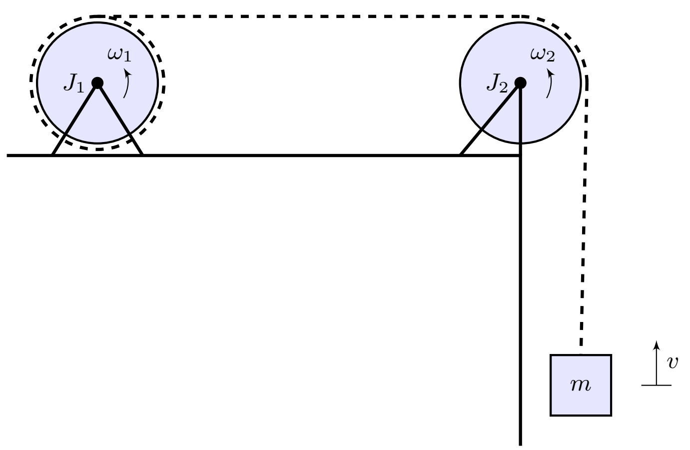

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Figure 1: Winch/pulley/mass systems

Questions:

In this exam you will be concerned with the motion of the winch/pulley/mass system in Fig. 1. In this system, consider that the winch, with inertia J_1, angular speed ω_1, and radius r_1, is connected to a pulley, with inertia J_2, angular speed ω_2, and radius r_2, which is connected to the mass m by an inextensible and massless cable that is never slack. Consider also that $r_1 = r_2 = r$.

1. **Modeling and Open-Loop Response**
   (a) (2 points (bonus)) Show that the ordinary differential equation:

$$J_r\,\dot{\omega}_1(t) + b_r\,\omega_1(t) = \tau(t) - mgr, \qquad v(t) = r\,\omega_1(t) = r\,\omega_2(t) \tag{1}$$

   where $J_r = J_1 + J_2 + mr^2$, $b_r = b_1 + b_2$, and b_1 and b_2 represent viscous damping at the winch and pulley, is a simplified description of the motion of the system on the left of Fig. 1.

Solution:
Free-body diagram relationships for each mass are as follows:

$$\begin{aligned} J_1\dot{\omega}_1 + b_1\,\omega_1 &= \tau - f_1 r, && \text{(+1 point)} \\ J_2\dot{\omega}_2 + b_2\,\omega_2 &= f_1 r - f_2 r, \\ m\dot{v} &= -mg + f_2, && v = \omega_1 r = \omega_2 r \end{aligned}$$

where f_1 and f_2 are the string tensions. Multiplying the last equation by r and using the relationship between ω_1, ω_2 and v

$$\begin{aligned} J_1\dot{\omega}_1 + b_1\,\omega_1 &= \tau - f_1 r, \\ J_2\dot{\omega}_1 + b_2\,\omega_1 &= f_1 r - f_2 r, \\ mr^2\dot{\omega}_1 &= -mgr + f_2 r \end{aligned}$$

then summing up to eliminate f_1 and f_2:

$$(J_1 + J_2 + mr^2)\,\dot{\omega}_1 + (b_1 + b_2)\,\omega_1 = \tau - mgr \qquad \text{(+1 point)}$$

leads to equation (1).

(b) (2 points) Use part (a) to show that

$$J_r\,\dot{v}(t) + b_r v(t) = r(\tau(t) + w(t)), \qquad w(t) = -mgr, \tag{2}$$

where $w(t)$ can be seen as a disturbance. Calculate the transfer-function $G(s)$ from τ to v and draw a block diagram showing the relationship between G and the signals τ, w, and v.

Solution:
Multiply (1) by r on both sides

$$J_r\,r\,\dot{\omega}_1 + b_r\,r\,\omega_1 = r\tau - mgr^2$$

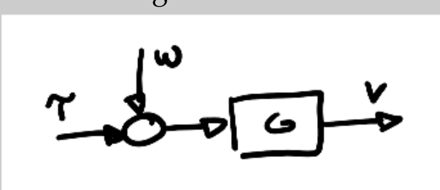
then use $v = r\,\omega_1$, $\dot{v} = r\,\dot{\omega}_1$, to obtain

$$J_r\,\dot{v} + b_r\,v = r\tau - mgr^2 = r(\tau - w) \qquad \text{(+0.5 point)}$$

Taking Laplace transforms

$$V(s) = G(s)(T(s) - W(s)), \qquad G(s) = \frac{\frac{r}{J_r}}{s + \frac{b_r}{J_r}} \qquad \text{(+1 point)}$$

A block-diagram should look like:



(c) (2 points) Show that when $w(t) = -mgr$ the response from a zero-initial velocity to a constant input torque $\tau(t) = \bar{\tau}$, $t \geq 0$, is given by

$$v(t) = \frac{r(\bar{\tau} - mgr)}{b_r}\left(1 - e^{-(b_r/J_r)t}\right)$$

HINT: You do not have to "derive" the solution if you can quote the book or the lecture notes.

Solution:

Put in standard form

$$\dot{v}(t) + \frac{b_r}{J_r} v(t) = \frac{r\tau(t) - mgr^2}{J_r} \qquad \text{(+1 point)}$$

from which for a constant $\tau(t) = \bar{\tau}$

$$v(t) = v(0)e^{-(b_r/J_r)t} + \frac{r\bar{\tau} - mgr^2}{b_r}\left(1 - e^{-(b_r/J_r)t}\right) \qquad \text{(+1 point)}$$

after setting $v(0) = 0$.

(d) (2 points) In order to identify the parameters J_r and b_r a student performed the following experiment: starting from zero initial velocity the student let the winch loose and observed that the mass achieved a steady state velocity of -1m/s, having reached approximately 63% of that velocity in 1s. Use this information to calculate J_r and b_r knowing that the winch and pulley radius are $r = 0.1\text{m}$ and that $m = 10\text{kg}$. Use $g = 10\text{m/s}^2$ and do not forget to state the units of J_r and b_r.

Solution:
With the given terminal velocity and $\bar{\tau} = 0$ (because the winch is loose) we obtain

$$\frac{-mgr^2}{b_r} = -1\text{ m/s} \quad \Longrightarrow \quad b_r = mgr^2 = \frac{10\text{ kg} \times 10\text{ m/s}^2 \times 10^{-2}\text{ m}^2}{1\text{ m/s}} = 1\text{ kg m}^2/\text{s} \quad \text{(+1 point)}$$

and from the time-constant

$$\frac{J_r}{b_r} \approx 1\text{ s} \Longrightarrow \qquad J_r \approx b_r \times 1\text{ s} = 1\text{ Kg m}^2 \qquad \text{(+1 point)}$$

2. **Proportional Control**

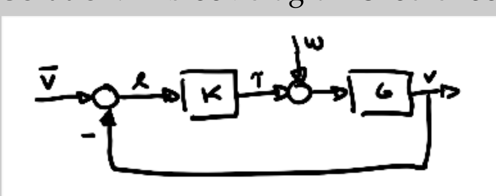
In this question, assume that $J_r = b_r = 1$, $m = g = 10$, and $r = 0.1$.

(a) (2 points) Consider the proportional control law

$$u(t) = Ke(t), \quad e(t) = \bar{v}(t) - v(t) \qquad (3)$$

where $\bar{v}(t)$ is a reference velocity. Draw a block diagram of the corresponding closed-loop system showing the relationship between G, K and the signals $\bar{v}$, e, τ, w, and v.

Solution: A block-diagram should look like:



(+2 points)

(b) (2 points) Calculate the closed-loop transfer-functions from $\bar{v}$ to e and from w to e.

Solution: From $\bar{v}$ to e:

$$S(s) = \frac{1}{1 + GK} = \frac{1}{1 + K\frac{r/J_r}{s + b_r/J_r}} = \frac{s + b_r/J_r}{s + b_r/J_r + Kr/J_r} \qquad \text{(+1 point)}$$

From w to e:

$$D(s) = \frac{G}{1+GK} = GS = \frac{r/J_r}{s+b_r/J_r} \times \frac{s+b_r/J_r}{s+b_r/J_r+Kr/J_r} = \frac{r/J_r}{s+b_r/J_r+Kr/J_r} \qquad \text{(+1 point)}$$

(c) (2 points) What are the values of K for which the closed-loop system is asymptotically stable?

Solution:
For asymptotic stability

$$-(b_r/J_r + Kr/J_r) < 0 \qquad \text{(+1 point)}$$

hence

$$K > -b_r/r \qquad \text{(+1 point)}$$

(d) (2 points) Does the signal $v(t)$ converge to $\bar{v}$, when $w = 0$ and $\bar{v}$ is a non-zero constant? Explain.

Solution:
No because with $w = 0$

$$e = S\bar{y} \qquad \text{(+1 point)}$$

and

$$S(0) = \frac{b_r/J_r}{b_r/J_r + Kr/J_r} \neq 0 \qquad \text{(+1 point)}$$

Another explanation: G does not have a pole at the origin gets full points.

(e) (2 points) Does the signal $v(t)$ converge to $\bar{v}$, when both $\bar{v}$ and w are non-zero constants? Explain.

Solution:
No because

$$e = S\bar{y} + Dw \qquad \text{(+1 point)}$$

and

$$S(0) = \frac{b_r/J_r}{b_r/J_r + Kr/J_r} \neq 0, \qquad D(0) = \frac{r/J_r}{b_r/J_r + Kr/J_r} \neq 0 \qquad \text{(+1 point)}$$

Another explanation: nor G nor K have a pole at the origin gets full points.

3. **Proportional-Integral Control**

In this question, assume that $J_r = b_r = 1$, $m = g = 10$, and $r = 0.1$.

Consider the proportional plus integral control law

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau, \quad e(t) = \bar{v}(t) - v(t) \tag{4}$$

where $\bar{v}(t)$ is a reference velocity. In the following questions consider that $K_i = K_p\, b_r/J_r$.

(a) (2 points) Calculate the closed-loop transfer-functions from $\bar{v}$ to e and from w to e.

Solution:
Because

$$K(s) = \frac{K_p s + K_i}{s} = \frac{K_p(s + b_r/J_r)}{s}, \qquad G(s)K(s) = \frac{K_p(s + b_r/J_r)}{s} \times \frac{r/J_r}{s + b_r/J_r} = K_p \frac{r/J_r}{s}$$

(+1 point)

From $\bar{v}$ to e:

$$S(s) = \frac{1}{1 + GK} = \frac{1}{1 + K_p \frac{r/J_r}{s}} = \frac{s}{s + K_p r/J_r} \qquad \text{(+1 point)}$$

From w to e:

$$D(s) = \frac{G}{1 + GK} = GS = \frac{r/J_r}{s + b_r/J_r} \times \frac{s}{s + K_p r/J_r} = \frac{r/J_r s}{(s + b_r/J_r)(s + K_p r/J_r)} \qquad \text{(+1 point)}$$

(b) (2 points) What are the values of K_p for which the closed-loop system is asymptotically stable?

Solution:
For asymptotic stability

$$-K_p r/J_r < 0 \qquad \text{(+1 point)}$$

hence

$$K_0 > 0 \qquad \text{(+1 point)}$$

(c) (2 points) Does the signal $v(t)$ converge to $\bar{v}$, when $w = 0$ and $\bar{v}$ is a non-zero constant? Explain.

Solution:
Yes because with $w = 0$

$$e = S\bar{y} \qquad \text{(+1 point)}$$

and

$$S(0) = 0 \qquad \text{(+1 point)}$$

Another explanation: G has a pole at the origin gets full points.

(d) (2 points) Does the signal $v(t)$ converge to $\bar{v}$, when both $\bar{v}$ and w are non-zero constants? Explain.

Solution:
Yes because

$$e = S\bar{y} + Dw \qquad \text{(+1 point)}$$

and

$$S(0) = D(0) = 0 \qquad \text{(+1 point)}$$

Another explanation: K has a pole at the origin gets full points.