MAE143 B - Linear Control - Spring 2019
Midterm, May 24th

Instructions:

1. This exam is **open book**. You can consult any printed or written material of your liking.
2. You have 50 minutes.
3. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book".
5. Do not forget to write your name and student number.
6. This exam has 2 questions, 19 total points and 2 bonus points.
7. Good luck!

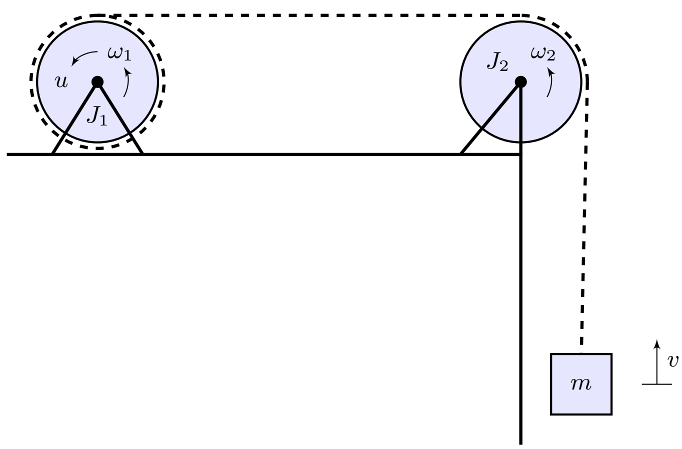

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Figure 1: Winch/pulley/mass systems

Consider the winch/pulley/mass system in Fig. 1, in which the winch, with inertia J_1, angular speed ω_1, and radius r_1, is connected to a pulley, with inertia J_2, angular speed ω_2, and radius r_2, which is connected to the mass m by an inextensible and massless cable that is never slack. Consider also that $r_1 = r_2 = r$ and that a torque u is applied at the winch as shown.

Questions:

In the first midterm you showed that the motion of this system could be approximately modeled by the ordinary differential equation

$$J_r\,\dot{v}(t) + b_r v(t) = r(u(t) + w(t)), \qquad w(t) = -mgr, \qquad v(t) = r\,\omega_1(t) = r\,\omega_2(t), \tag{1}$$

where $J_r = J_1 + J_2 + mr^2$, $b_r = b_1 + b_2$, and b_1 and b_2 represent viscous damping at the winch and pulley.

For the rest of the exam assume that $J_r = b_r = 1$, $m = g = 10$, and $r = 0.1$.

1. **Modeling**

   (a) (1 point) Show that

   $$G(s) = \frac{r/J_r}{s + b_r/J_r} = \frac{0.1}{s+1} \tag{2}$$

   is the open-loop transfer-function associated with (1) from u to v and draw a block-diagram showing the relationship between G and the signals u, w, and v.

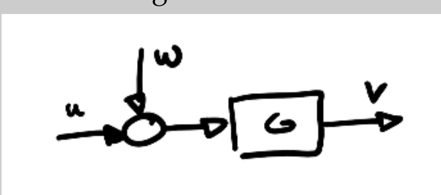
   Solution:
   Divide by J_r and apply the Laplace transform

   $$(s + b_r/J_r)\,V(s) = r/J_r(T(s) + W(s))$$

   from which

   $$V(s) = G(s)(U(s) + W(s)), \qquad G(s) = \frac{\frac{r}{J_r}}{s + \frac{b_r}{J_r}} = \frac{0.1}{s+1}.$$

   A block-diagram should look like:



   (+1 point)

2. **Proportional-Integral Speed Control**

   (a) (2 points) Consider the proportional plus integral control law

   $$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau, \quad e(t) = \bar{v}(t) - v(t) \tag{3}$$

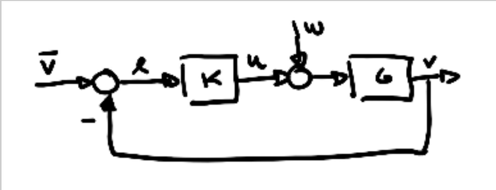
   where $\bar{v}(t)$ is a reference velocity. Calculate the open-loop transfer-function $K(s)$ from e to u and draw a closed-loop block-diagram showing the relationship between G, K and the signals u, w, e, v, and $\bar{v}$.

   Solution:
   Apply the Laplace transform

   $$U(s) = K_p E(s) + \frac{K_i}{s} E(s) = K(s)E(s), \qquad K(s) = \frac{K_p s + K_i}{s}.$$

   (+1 point)

A block-diagram should look like



(+1 point)

(b) (2 points) Calculate the closed-loop transfer-functions from $\bar{v}$ to e and from w to e.

Solution:
Because

$$K(s) = \frac{K_p s + K_i}{s}, \qquad G(s)K(s) = \frac{K_p s + K_i}{s} \times \frac{0.1}{s+1} = \frac{0.1(K_p s + K_i)}{s(s+1)}$$

From $\bar{v}$ to e:

$$S(s) = \frac{1}{1+GK} = \frac{1}{1 + \frac{0.1(K_p s + K_i)}{s(s+1)}} = \frac{s(s+1)}{s^2 + s(1 + 0.1K_p) + 0.1K_i} \qquad \text{(+1 point)}$$

From w to e:

$$D(s) = \frac{G}{1+GK} = GS = \frac{0.1}{s+1}\frac{s(s+1)}{s^2 + s(1+0.1K_p) + 0.1K_i} \qquad \text{(+1 point)}$$
$$= \frac{0.1s}{s^2 + s(1+0.1K_p) + 0.1K_i}$$

(c) (4 points) Calculate K_p and K_i so that the closed-loop system has $\omega_n = 1\text{rad/s}$ and $\zeta = 1$. What is the transfer-function of the corresponding controller? Where are the closed-loop poles located?

Solution:
From the closed-loop transfer-functions

$$\omega_n^2 = 1 = 0.1K_i, \qquad 2\zeta\omega_n = 2 = 1 + 0.1K_p$$

from which

$$K_i = 10, \qquad K_p = 10(2-1) = 10. \qquad \text{(+2 points)}$$

The controller is

$$K(s) = \frac{10(s+1)}{s} \qquad \text{(+1 point)}$$

In that case

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 1 = (s+1)^2$$

so the two poles are located at $s = -1$. (+1 point)
NOTE TO GRADERS: In the next questions that ask for K_p and K_i from part (c), they should be graded and awarded full marks if their answer is correct for whatever compatible K_p and K_i they have obtained here, even if the answer to part (c) is correct.

(d) (2 points) For the K_p and K_i you calculated in part (c) is the closed-loop internally stable? Explain.

Solution:
Yes because

$$S(s) = \frac{s}{s+1}$$

is asymptotically stable... (+1 point)
and the pole-zero cancellation that happens in the product

$$K(s)G(s) = \frac{10(s+1)}{s}\frac{0.1}{s+1} = \frac{1}{s}$$

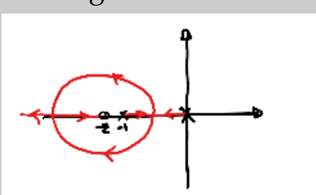
is that of a stable pole $s = -1$. (+1 point)

(e) (4 points) Explain how can you use the root-locus method to calculate values of K_p and K_i for which the closed-loop system is asymptotically stable. Locate the points in your root-locus diagram that correspond to the K_p and K_i you calculated in part (c).

Solution:
In order to study the stability of the closed loop using the root-locus there are several options. One is to have $K_i/K_p = z$ be a zero and look for $\alpha = K_p$... (+1 point)
that stabilizes the loop transfer-function

$$L(s) = K(s)G(s) = \frac{(s+z)}{s}\frac{0.1}{s+1} = \frac{0.1(s+z)}{s(s+1)} \qquad \text{(+1 point)}$$

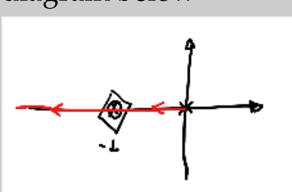
which gives rise to a root-locus diagram such as



from which positive values of $\alpha = K_p$ will lead to asymptotic stability. (+1 point)

NOTE TO GRADERS: Variations will happen depending on where z is located. In fact, this diagram is not even necessary for points, if the next one is correct.

The choice of $K_p/K_i = 10/10 = 1$ in part (c) means that $z = 1$ producing a pole zero-cancelation. It also means that all closed-loop poles are located in $s = -1$, indicated by the diamond in the diagram below



(+1 point)

(f) (1 point) Does the signal $v(t)$ converge to a given constant $\bar{v}$ when $w(t) = -mgr$ and K_p and K_i are as in part (c)? Explain.

Solution:
Yes because that K is stabilizing and

$$e = S\bar{v} - Dw \qquad \text{(+1/2 point)}$$

and

$$S(0) = D(0) = 0 \qquad \text{(+1/2 point)}$$

Another explanation: Yes because $K(s)$ is stabilizing and has a pole at the origin.

(g) (3 points) Use the frequency response method to calculate the steady-state value of $u(t)$ when $v(t) = \bar{v}$ and $w(t) = -mgr$. What does the signal $u(t)$ converge to when $\bar{v} = 0$? Explain.

Solution:
From Chapter 4

$$e = S\bar{v} - Dw, \qquad u = Ke$$

so that

$$u = KS\bar{v} - KGSw = Q\bar{v} - Hw \qquad \text{(+1/2 point)}$$

In this case

$$\begin{aligned} Q &= KS = \frac{10(s+1)}{s}\frac{s(s+1)}{(s+1)^2} = 10, \\ H &= 1 - S = 1 - \frac{s(s+1)}{(s+1)^2} = 1 - \frac{s}{s+1} = \frac{1}{s+1} \end{aligned} \qquad \text{(+1/2 point)}$$

Using the frequency response, and because both inputs are constant,

$$\bar{u} = \lim_{t\to\infty} u(t) = Q(0)\bar{v} - H(0)(-mgr) = 10\,\bar{v} + 10 \qquad \text{(+1 point)}$$

When $\bar{v} = 0$ then $u(t)$ converges to $\bar{u} = mgr = 10$Nm, which is the torque required to balance the mass weight. (+1 point)

(h) (1 point (bonus)) What happens in part (g) if $\bar{v} < -1$m/s? Explain.

Solution:
From part (g), the input torque should converge to

$$\bar{u} = Q(0)\bar{v} - H(0)(-mgr) = 10\,\bar{v} + 10 < 0$$

A negative torque at the winch means that the cable will have a negative tension. Instead of reaching the desired speed, the mass will simply free fall at -1m/s. (+1 point)

(i) (1 point (bonus)) Show that when $\bar{v}(t) = 0$ and K_p and K_i are as in (c), the closed-loop transfer-function from $w(t)$ to

$$x(t) = x(0) + \int_0^t v(\tau)\, d\tau$$

is equal to

$$F(s) = \frac{0.1}{(s+1)^2}$$

Sketch the complete response $x(t)$ when $\bar{v} = 0$, $x(0) = v(0) = 0$, and $w(t) = -mgr$. Explain.

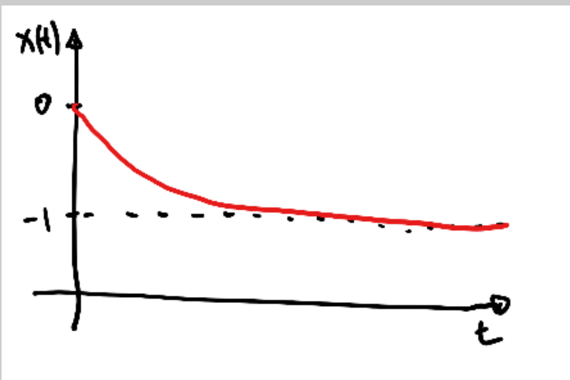
Solution:
From Chapter 4 and $\bar{v} = 0$

$$v = H\bar{v} + Dw = Dw$$

so that

$$x = Fw, \qquad F = s^{-1}D = \frac{1}{s}\frac{0.1s}{(s+1)^2} = \frac{0.1}{(s+1)^2} \qquad \text{(+1/2 bonus point)}$$

A sketch of the solution is as in



because both poles are real and the steady state value is $-F(0)mgr = -0.1 \times 10 = -1$m.

(+1/2 bonus point)