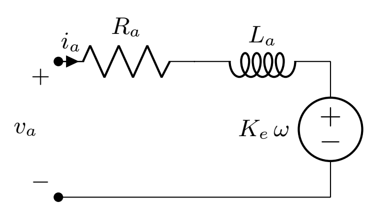
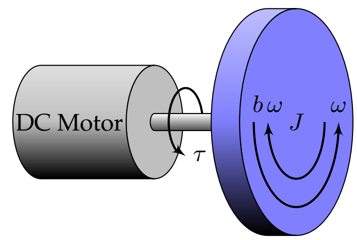
MAE143 B - Linear Control - Spring 2020
Midterm 1, January 28th

Instructions:

1. This exam is **open book**. You can consult any printed or written material of your liking.
2. You have 70 minutes.
3. Most questions show you the answer, and you're asked to prove it. Move on. Do not get stuck!
4. Write your answers on your "Blue Book".
5. Do not forget to write your name and student number.
6. This exam has 4 questions, 28 total points and 2 bonus points.
7. Good luck!

Consider the DC motor electrical and mechanical schematics:





You have shown in the homework that the motion of the rotor of a DC motor shown schematically above can be described by the differential equation:

$$J\,\dot{\omega} + \left(b + \frac{K_e K_t}{R_a}\right)\omega = \frac{K_t}{R_a} v_a$$

where v_a is the motor's input voltage and all other constants are positive.

Answer the following questions:

Questions:

1. **Modeling and Open-Loop Response I**

   (a) (2 points) Calculate the transfer-function, $G_\omega(s)$, from $u(t) = v_a(t)$ to $y(t) = \omega(t)$.

   Solution: Applying the Laplace transform to the differential equation:

   $$J\,(sY(s) - y_0) + \left(b + \frac{K_e K_t}{R_a}\right) Y(s) = \frac{K_t}{R_a} V_a(s) \qquad \text{(+1 point)}$$

   from which

   $$\left(s + \left(\frac{b}{J} + \frac{K_e K_t}{J R_a}\right)\right) Y(s) = \frac{K_t}{J R_a} V_a(s) + y_0$$

   or

   $$Y(s) = G_\omega(s) V_a(s) + \frac{1}{s - \lambda} y_0$$

   where

   $$G_\omega(s) = \frac{\frac{K_t}{J R_a}}{s - \lambda} \qquad \text{(+1 point)}$$

   NOTE TO GRADERS: It is fine to assume zero initial conditions.

   (b) (2 points) Calculates the poles and zeros of $G_\omega(s)$. Is $G_\omega(s)$ asymptotically stable?

   Solution: $G_\omega(s)$ has no zeros and one pole at $s = \lambda$. (+1 point)
   Because all constanst are non-negative $\lambda < 0$ and G_ω is asymptotically stable. (+1 point)

   (c) (2 points) Show that if $y(0) = y_0$ then

   $$y(t) = y_0\, e^{-\lambda t} + \tilde{y}\,(1 - e^{\lambda t}),$$

   $$\tilde{y} = \frac{V_a K_t / R_a}{b + K_e K_t / R_a}, \qquad \lambda = -\frac{b}{J} - \frac{K_e K_t}{J R_a},$$

   is the response to a voltage step input $u(t) = v_a(t) = V_a 1(t)$.

   Solution: NOTE TO GRADERS: Here one can recognize a cannonical first order system and copy the response from the book, for example, or calculate it directly.
   For example, using Laplace transforms from part (a) with $V_a(s) = V_a/s$:

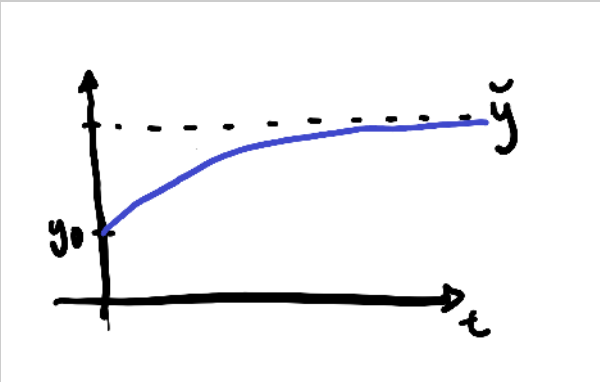
   $$Y(s) = \frac{\frac{V_a K_t}{J R_a}}{s(s - \lambda)} + \frac{1}{s - \lambda} y_0 = \frac{V_a K_t / R_a}{-J\lambda}\left(\frac{1}{s} - \frac{1}{s - \lambda}\right) + \frac{y_0}{s - \lambda} \qquad \text{(+1 point)}$$

   from which

   $$y(t) = \tilde{y}(1 - e^{\lambda t}) + y_0 e^{\lambda t}, \qquad \tilde{y} = \frac{V_a K_t / R_a}{-J\lambda}. \qquad \text{(+1 point)}$$

   (d) (1 point) Sketch the response from part (c).

   Solution: A sketch should look like:



(+1 point)

2. **Proportional Velocity Control**

In this question, consider the DC motor from Question 1 and assume that

$$J = b = K_t = K_e = R_a = 1.$$

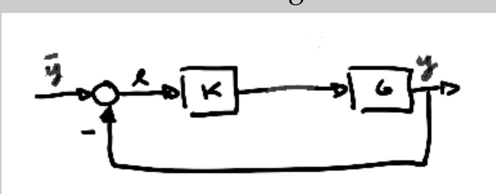
Consider the proportional control law

$$u(t) = Ke(t), \quad e(t) = \bar{y}(t) - y(t)$$

where $\bar{y}(t)$ is a reference angular velocity, $\bar{\omega}(t)$.

(a) (1 point) Draw a block diagram of the corresponding closed-loop system showing the relationship between G, K and the signals $\bar{y}$, e, u, and y.

Solution: A block-diagram should look like:



(+1 point)

(b) (2 points) Calculate the closed-loop transfer-functions from $\bar{y}$ to e and $\bar{y}$ to y.

Solution: Substituting numerical values

$$G_\omega(s) = \frac{1}{s+1+1} = \frac{1}{s+2}$$

From $\bar{y}$ to e:

$$S(s) = \frac{1}{1+GK} = \frac{1}{1+K\frac{1}{s+2}} = \frac{s+2}{s+2+K}$$ (+1 point)

From $\bar{y}$ to y:

$$H(s) = \frac{GK}{1+GK} = \frac{K\frac{1}{s+2}}{1+K\frac{1}{s+2}} = \frac{K}{s+2+K}$$ (+1 point)

(c) (2 points) What are the values of K for which the closed-loop system is asymptotically stable?

Solution:
For asymptotic stability

$$-2 - K < 0 \qquad \text{(+1 point)}$$

hence

$$K > -2 \qquad \text{(+1 point)}$$

(d) (2 points) Does the signal $y(t)$ converge to $\bar{y}$ when $\bar{y}(t) = \bar{y}$ is a non-zero constant? Explain.

Solution:
No because

$$e = S\bar{y} \qquad \text{(+1 point)}$$

and

$$S(0) = \frac{2}{2+K} \neq 0 \qquad \text{(+1 point)}$$

Another explanation: G does not have a pole at the origin gets full points.

3. **Tracking Design**

Consider the proportional control law from Question 2 in which K is chosen so that the closed-loop is asymptotically stable.

(a) (2 points) Calculate the steady-state error in response to a reference angular velocity $\bar{y}(t) = \bar{y}\cos(t)$.

Solution:
Use the frequency response method

$$e^{ss}(t) = |S(j)|\cos(t + \angle S(j)) \qquad \text{(+1 point)}$$

Because

$$S(j) = \frac{j+2}{j+2+K} = \frac{\sqrt{5}}{\sqrt{1+(2+K)^2}}\angle\phi, \quad \phi = \tan^{-1} 1/2 - \tan^{-1} 1/(2+K) \qquad \text{(+1/2 point)}$$

then

$$e^{ss}(t) = \frac{\sqrt{5}}{\sqrt{1+(2+K)^2}}\cos(t + \tan^{-1} 1/2 - \tan^{-1} 1/(2+K)) \qquad \text{(+1/2 point)}$$

(b) (3 points) Show how the parameters a, b, and K of the controller with transfer-function

$$K(s) = \frac{K(s+a)^2}{s^2+b^2}$$

can be selected so that the closed-loop is asymptotically stable and the steady-state error in response to the reference angular velocity from part (a) is zero.

Solution: For the steady-state error to be zero K must have a pole at $s = j$, hence $b = 1$. (+1 point)

To make life easier, we can choose $a = 2$ so that

$$K(s)G(s) = \frac{1}{s+2}\frac{K(s+2)^2}{s^2+1} = \frac{K(s+2)}{s^2+1} \qquad \text{(+1 point)}$$

NOTE TO GRADERS: other choices that lead to closed-loop asymptotic stability are fine.
For asymptotic stability

$$S = \frac{1}{1+KG} = \frac{1}{1+\frac{K(s+2)}{s^2+1}} = \frac{s^2+1}{s^2+Ks+1+2K} \qquad \text{(+1 point)}$$

In this form, any $K > 0$ will be such that the closed-loop is asymptotically stable. (+1 point)

(c) (2 points) Select suitable a, b, and K and calculate the corresponding sensitivity transfer-function $S(s) = (1 + K(s)G(s))^{-1}$. Calculate also its poles and zeros.

Solution: From part (b)

$$S = \frac{1}{1+KG} = \frac{1}{1+\frac{K(s+2)}{s^2+1}} = \frac{s^2+1}{s^2+Ks+1+2K} \qquad \text{(+1 point)}$$

which has zeros at $s = \pm j$ (+1/2 point)
and poles at

$$s^2 + Ks + 1 + 2K = 0, \qquad s = \frac{-K \pm \sqrt{K^2 - 4(1+2K)}}{2} \qquad \text{(+1/2 point)}$$

which have negative real part for any $K > 0$.

4. **Proportional Position Control**

(a) (2 points) Calculate the transfer-function, $G_\theta(s)$, from $u(t) = v_a(t)$ to

$$y(t) = \theta(t) = \int_0^t \omega(\tau)\, d\tau.$$

Solution: Because θ is the integral of ω

$$G_\theta(s) = \frac{\Theta(s)}{U(s)} = \frac{1}{s}\frac{\Omega(s)}{U(s)} = \frac{1}{s}G_\omega(s) = \frac{\frac{K_t}{JR_a}}{s(s-\lambda)} \qquad \text{(+2 points)}$$

(b) (2 points) Calculates the poles and zeros of $G_\theta(s)$. Is $G_\theta(s)$ asymptotically stable?

Solution: $G_\theta(s)$ has no zeros, one pole at $s = \lambda$, and one pole at $s = 0$. (+1 point)
Because of the pole at $s = 0$, G_θ is NOT asymptotically stable. (+1 point)

Assume that

$$J = b = K_t = K_e = R_a = 1,$$

and consider the proportional control law

$$u(t) = Ke(t), \quad e(t) = \bar{y}(t) - y(t)$$

where $\bar{y}(t)$ is a reference angular position, $\bar{\theta}(t)$.

(c) (2 points) What are the values of K for which the closed-loop system is asymptotically stable?

Solution: Substituting numerical values

$$G(s) = G_\theta(s) = \frac{1}{s(s+2)}$$

In closed-loop

$$S(s) = \frac{1}{1+KG(s)} = \frac{1}{1+K\frac{1}{s(s+2)}} = \frac{s(s+2)}{s^2+2s+K} \qquad \text{(+1 point)}$$

which is asymptotically stable for all $K > 0$. (+1 point)

(d) (1 point) Pick a stabilizing value of K and calculate the corresponding closed-loop poles.

Solution: For example, for $K = 1$

$$s^2 + 2s + K = s^2 + 2s + 1 = (s+1)^2$$

which has two poles at $s = -1$. (+1 point)

(e) (2 points (bonus)) Does the signal $y(t)$ converge to $\bar{y}$, when $\bar{y}(t) = \bar{y}$ is a non-zero constant? Explain.

Solution:
Yes because

$$e = S\bar{y} \qquad \text{(+1 point)}$$

and

$$S(0) = 0 \qquad \text{(+1 point)}$$

Another explanation: G has a pole at the origin gets full points.