MAE143 B - Linear Control - Winter 2020
Midterm 2, February 18th

Instructions:

1. This exam is **open book**. You can consult any printed or written material of your liking.
2. You have 70 minutes.
3. Most questions show you the answer, and you're asked to prove it. Move on. Do not get stuck!
4. Write your answers on your "Blue Book".
5. Do not forget to write your name and student number.
6. This exam has 4 questions, 24 total points and 3 bonus points.
7. Good luck!

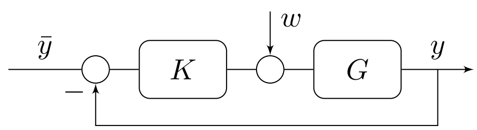


Figure 1: Feedback system diagram

Questions:

1. **Feedback Design I**

   Consider the standard feedback diagram from Fig. 1 in which

$$G(s) = \frac{1}{s(s+1)}.$$

   Consider the proportional controller

$$K(s) = K_p$$

   and answer the following questions:

   (a) (2 points) Calculate the closed-loop transfer functions

$$S = \frac{1}{1+KG}, \qquad Q = \frac{K}{1+KG},$$
$$D = \frac{G}{1+KG}, \qquad H = \frac{KG}{1+KG}.$$

Solution:

$$S = \frac{1}{1+\frac{K_p}{s(s+1)}} = \frac{s(s+1)}{s^2+s+K_p} \qquad Q = K_p S = \frac{K_p s(s+1)}{s^2+s+K_p} \qquad \text{(+1 point)}$$

$$D = \frac{1}{s(s+1)} S = \frac{1}{s^2+s+K_p} \qquad H = \frac{K_p}{s(s+1)} S = \frac{K_p}{s^2+s+K_p}. \qquad \text{(+1 point)}$$

   (b) (2 points) For what values of $K_p > 0$ is the closed-loop system internally stable?

Solution: This closed-loop transfer function is a standard second-order transfer-function in which $\omega_n^2 = K_p$, $\zeta = 1/(2\sqrt{K_p})$ which for any $K_p > 0$ is such that $\zeta > 0$, hence asymptotically stable. (+1 point)

It is internally stable because no pole-zero cancellation occured when forming the product KG. (+1 point)

(c) (1 point) Can this closed-loop system asymptotically track a constant reference input $\bar{y}$? Explain.

Solution: Yes because G has a pole at zero. (+1 point)

(d) (1 point) Can this closed-loop system asymptotically reject a constant input disturbance w? Explain.

Solution: No because K does not have a pole at zero. (+1 point)

2. **Feedback Design II**

Consider the standard feedback diagram from Fig. 1 in which

$$G(s) = \frac{s}{s+1}.$$

Consider the proportional-integral controller

$$K(s) = K_p + K_i s^{-1}$$

and answer the following questions:

(a) (2 points) Calculate the closed-loop transfer functions

$$S = \frac{1}{1+KG}, \qquad Q = \frac{K}{1+KG},$$
$$D = \frac{G}{1+KG}, \qquad H = \frac{KG}{1+KG}.$$

Solution:

$$S = \frac{1}{1 + \frac{K_p s + K_i}{s}\frac{s}{s+1}} = \frac{s+1}{(1+K_p)s + 1 + K_i} \qquad Q = \frac{K_p s + K_i}{s} S = \frac{K_p s + K_i}{s[(1+K_p)s + 1 + K_i]}$$

(+1 point)

$$D = \frac{s}{s+1} S = \frac{s}{(1+K_p)s + 1 + K_i} \qquad H = \frac{K_p s + K_i}{s}\frac{s}{s+1} S = \frac{K_p s + K_i}{s^2 + s + K_p}.$$ (+1 point)

(b) (2 points) For what values of $K_p > 0$ and $K_i > 0$ is the closed-loop system internally stable?

Solution: None because of the pole-zero cancellation at the origin. (+2 points)

(c) (1 point) Can this closed-loop system asymptotically track a constant reference input $\bar{y}$? Explain.

Solution: No, because it is not internally stable. (+1 point)

(d) (1 point) Can this closed-loop system asymptotically reject a constant input disturbance w? Explain.

Solution: No, because it is not internally stable. (+1 point)

3. **System Realization and State Space**

   In the Midterm I you solved a question involving a controller of the form

$$K(s) = \frac{K(s+a)^2}{s^2+b^2}.$$

   (a) (3 points) Draw a block-diagram using only integrators, amplifiers and summers that can realize $K(s)$.

Solution:

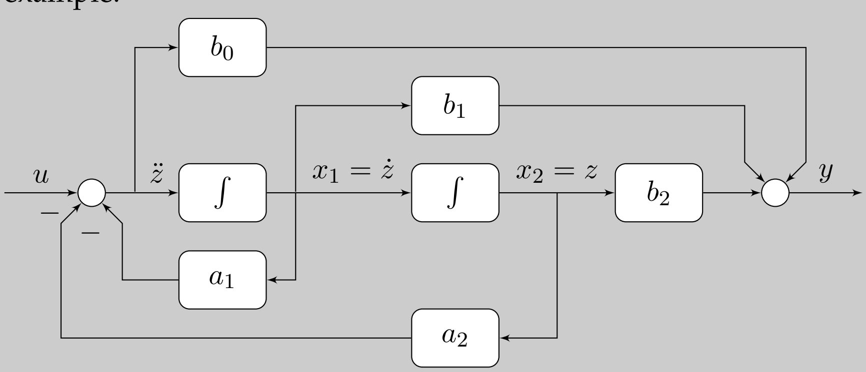
The transfer-function is a standard second-order system

$$K(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} = \frac{K s^2 + 2aKs + Ka^2}{s^2+b^2}$$

for the choices of

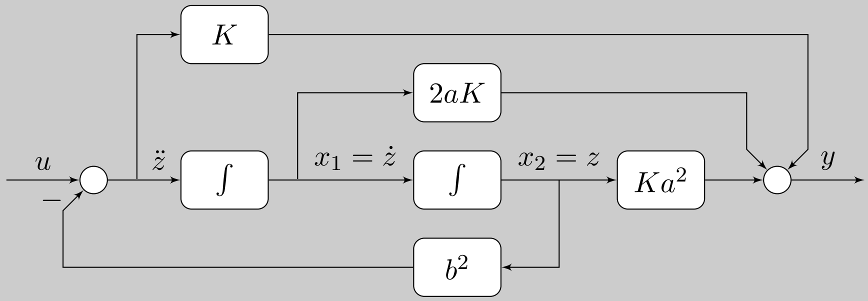
$$b_0 = K, \qquad b_1 = 2aK, \qquad b_2 = Ka^2, \qquad a_1 = 0, \qquad a_2 = b^2.$$ (+1 point)

for which a standard block-diagram such as the ones in Chapter 5 will be a realization. For example:



(+1 point)

which can be specialized to the problem in hand as



(+1 point)

NOTE TO GRADERS: There are other correct solutions to this questions.

(b) (3 points) (+1 bonus point) Write state-space equations associated with the block-diagram you obtained in part (a).

Solution: One possible state-space realization follows from the choice of x_1 and x_2 as the outputs of the integrators in part (a):

$$\dot{x}_1 = -b^2 x_2 + u \qquad \text{(+1 point)}$$
$$\dot{x}_2 = x_1 \qquad \text{(+1 point)}$$

and the output

$$\begin{aligned} y &= 2aKx_1 + Ka^2 x_2 + K(u - b^2 x_2) \\ &= 2aKx_1 + K(a^2 - b^2)x_2 + Ku \end{aligned} \qquad \text{(+1 point)}$$

Bonus point for putting it in matrix form:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & -b^2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 2aK & K(a^2 - b^2) \end{bmatrix} x + \begin{bmatrix} K \end{bmatrix} u \end{aligned} \qquad \text{(+1 bonus point)}$$

NOTE TO GRADERS: There are other correct solutions to this questions.

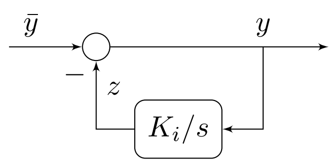


Figure 2: System for Question 4

4. **Filter**

Answer the following questions to see how the feedback system in Fig. 2 can be used as a filter.

(a) (2 points) Calculate the transfer-functions from the input $\bar{y}$ to the outputs y and z.

Solution: Let

$$K(s) = \frac{K_i}{s}$$

so that

$$y = \bar{y} - z = \bar{y} - Ky \qquad \Longrightarrow \qquad y = \frac{1}{1+K}\bar{y}, \quad z = Ky = \frac{K}{1+K}\bar{y}$$

or

$$\frac{Y(s)}{\bar{Y}(s)} = S(s) = \frac{1}{1+\frac{K_i}{s}} = \frac{s}{s+K_i}, \qquad \frac{Z(s)}{\bar{Y}(s)} = H(s) = \frac{K_i}{s}\frac{s}{s+K_i} = \frac{K_i}{s+K_i}$$

(b) (1 point) For what values of $K_i > 0$ is the system asymptotically stable?

Solution: All values of $K_i > 0$, for which the pole $s = -K_i$ has negative real part. (+1 point)

(c) (3 points) Use the frequency-response method to explain why when $\omega \gg K_i$ and

$$\bar{y}(t) = a + b\cos(\omega t)$$

that

$$y_{ss}(t) \approx b\cos(\omega t), \qquad z_{ss}(t) \approx a.$$

Solution: The steady state solution to the above input is given by

$$\begin{aligned} y_{ss}(t) &= aS(0) + b|S(j\omega)|\cos(\omega t + \angle S(j\omega)) \\ &= b|S(j\omega)|\cos(\omega t + \angle S(j\omega)) \end{aligned}$$ (+1 point)

and

$$\begin{aligned} z_{ss}(t) &= aH(0) + b|H(j\omega)|\cos(\omega t + \angle S(j\omega)) \\ &= a + b|H(j\omega)|\cos(\omega t + \angle H(j\omega)) \end{aligned}$$ (+1 point)

from which the approximations provided follow after noticing that for $\omega \gg K_i$

$$|S(j\omega)| = \frac{|\omega|}{\sqrt{\omega^2 + K_i^2}} \approx 1, \quad \angle S(j\omega) = \pi/2 - \tan^{-1}\omega/K_i \approx 0, \quad |H(j\omega)| = \frac{|K_i|}{\sqrt{\omega^2 + K_i^2}} \approx 0$$

(+1 point)

(d) (2 points (bonus)) Explain how one could use the above system to remove the DC component of a signal and, at the same time, produce an estimate of that DC component.

Solution: After choosing K_i small enough, for all relevant $\omega \gg K_i$, z will be approximately equal to the DC value of the signal (the a above) and the signal y will be approximately equal to the input signal (the $b\cos(\omega t)$ above). (+2 bonus points)