MAE143 B - Linear Control - Winter 2021
Midterm #1, January 26th

Instructions:

1. You have from 9:30AM to 10:50AM to finish this exam.
2. You have until 11:00AM to submit your answers on gradescope.
3. The exam is to be done individually.
4. All answers should be manuscript.
5. No typed answers of any form will be accepted.
6. All graphs and diagrams have to be sketched by hand. Include enough detail.
7. If you use a tablet to do the work make sure it is handwritten.
8. This exam has 4 questions, for a total of 21 points.

Questions:

1. (3 points) **Stabilization**
   A system with transfer-function
   $$G(s) = \frac{s}{s-1}$$
   is to be stabilized using a standard feedback controller. Of the following controllers, which one would be the best choice:
   a) $K(s) = 1$
   b) $K(s) = -2$
   c) $K(s) = 2/s$

   Explain your reasoning.

   Solution: Controller $c)$ performs a pole-zero cancellation of an unstable zero, so the closed-loop system is not internally stable. (+1 point)

   The other two controllers are proportional controllers with which
   $$S(s) = \frac{1}{1 + \frac{Ks}{s-1}} = \frac{s-1}{(1+K)s-1}$$
   The choice $K = 1$ leads to a closed-loop pole at $s = 1/2$, which is not stable. (+1 point)

   The choice $K = -2$ leads to a closed-loop pole at $s = -1$, which is stable. Hence the best choice is controller $b)$. (+1 point)

For the remaining questions consider the following system:
The temperature of a substance in a container left at constant ambient temperature, T_a, with a heat source, q, can be modeled as

$$mc\dot{T}(t) = \frac{1}{R}(T_a - T(t)) + q(t), \tag{1}$$

in which m and c are the substance's mass and specific heat, and R is the overall system's thermal resistance.

2. **Modeling and Open-Loop Response**

(a) (2 points) Calculate the transfer-function $G(s)$ from q to T and draw a block diagram showing the relationship between G and the signals $q(t)$, $T(t)$, and $w(t) = T_a/R$.

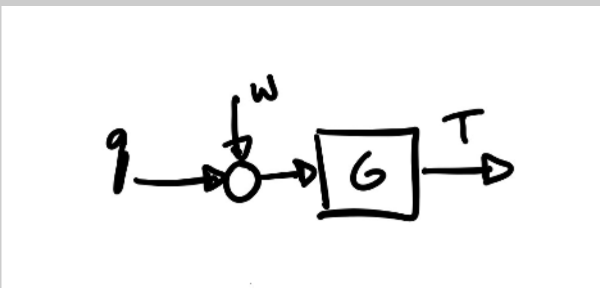
Solution:
Rearrange the equation as

$$\dot{T}(t) + \frac{1}{mcR}T(t) = \frac{1}{mcR}T_a + \frac{1}{mc}q(t) = \frac{1}{mc}\left(w(t) + q(t)\right), \quad w(t) = \frac{T_a}{R}. \tag{2}$$

The transfer-function from q to T is

$$G(s) = \frac{\frac{1}{mc}}{s + \frac{1}{mcR}} \qquad \text{(+1 point)}$$

This is the same transfer-function from w to T so a block-diagram should look like:



(+1 point)

(b) (2 points) Show that when $q(t) = \bar{q}$ is constant and the initial temperature is T_0 then

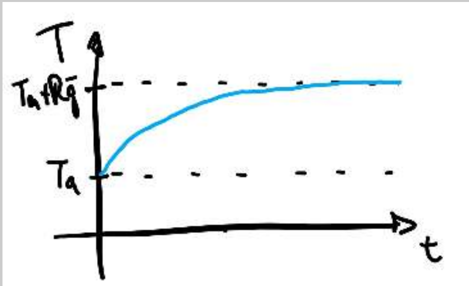
$$T(t) = T_a + (T_0 - T_a)e^{-t/mcR} + R\bar{q}\left(1 - e^{-t/mcR}\right). \tag{3}$$

Sketch the solution assuming that $T_0 = T_a$ and $\bar{q} > 0$.
HINT: You do not have to "derive" the solution if you can quote the book or the lecture notes.

Solution:
If $q(t) + w(t) = \bar{q} + T_a/R$ is a constant then the solution to (2) is

$$\begin{aligned} T(t) &= T_0 e^{-(1/mcR)t} + \frac{\frac{1}{mc}(\bar{q} + T_a/R)}{\frac{1}{mcR}}\left(1 - e^{-(1/mcR)t}\right) \\ &= T_0 e^{-(1/mcR)t} + (R\bar{q} + T_a)\left(1 - e^{-(1/mcR)t}\right) \\ &= T_a + (T_0 - T_a)e^{-(1/mcR)t} + R\bar{q}\left(1 - e^{-(1/mcR)t}\right). \qquad \text{(+1 point)} \end{aligned}$$

A sketch of the solution should look like



(+1 point)

(c) (2 points) A student performed the following experiment. The heat source was turned on until 5 kg of a substance with specific heat $c = 5000$ J/kg K reached a temperature of $75°$ C, at which point the heat source was turned off. The time that it took for the substance's temperature to reach $50°$ C was 900 s. Use this information and the ambient temperature $T_a = 25°$ C to estimate the parameter R.

Solution:
Starting from $T_0 = 75°$ C, the temperature follows (3), that is

$$T(t) = T_a + (T_0 - T_a)e^{-t/mcR}$$

Rearranging

$$e^{-t/mcR} = \frac{T(t) - T_a}{T_0 - T_a}$$

or

$$\frac{t}{mcR} = \log \frac{T_0 - T_a}{T(t) - T_a} \implies R = \frac{t}{mc \log \frac{T_0 - T_a}{T(t) - T_a}} \qquad \text{(+1 point)}$$

At $t = 900$ s

$$R = \frac{900 \text{ s}}{5 \text{ kg} \times 5000 \text{ J/kg K} \log((75-25)/(50-25))} =$$

$$\frac{900}{5 \times 5000 \times \log(2)} K/W \approx 5 \times 10^{-3} K/W \qquad \text{(+1 point)}$$

3. **Proportional Control**

$\boxed{\text{In this question, assume that } mc = R = 1.}$

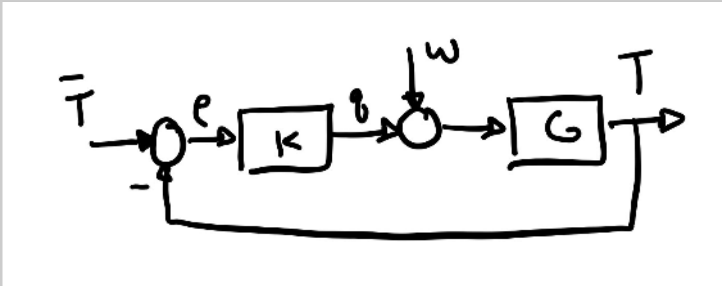
Consider the proportional control law

$$q(t) = Ke(t), \quad e(t) = \bar{T}(t) - T(t) \tag{4}$$

where $\bar{T}(t)$ is a reference temperature.

(a) (1 point) Draw a block diagram of the corresponding closed-loop system showing the relationship between G, K and the signals $\bar{T}$, T, e, q, and w.

Solution: A block-diagram should look like:



(+1 point)

(b) (2 points) Calculate the closed-loop transfer-functions from $\bar{T}$ to e and from w to e.

Solution:
With $mc = R = 1$

$$G(s) = \frac{\frac{1}{mc}}{s + \frac{1}{mcR}} = \frac{1}{s+1}$$

From $\bar{T}$ to e:

$$S(s) = \frac{1}{1+GK} = \frac{1}{1+K\frac{1}{s+1}} = \frac{s+1}{s+1+K} \qquad \text{(+1 point)}$$

From w to e:

$$D(s) = \frac{G}{1+GK} = GS = \frac{1}{s+1} \times \frac{s+1}{s+1+K} = \frac{1}{s+1+K} \qquad \text{(+1 point)}$$

(c) (1 point) What are the values of K for which the closed-loop system is asymptotically stable?

Solution:
For asymptotic stability

$$-(1+K) < 0$$

hence

$$K > -1 \qquad \text{(+1 point)}$$

(d) (2 points) Does the signal $T(t)$ converge to $\bar{T}$ when $\bar{T}(t) = \bar{T}$ is a constant? Explain.

Solution:
No because $\bar{T}(t)$ and $w(t)$ are non-zero constants and $S(0)$ and $D(0)$ are not both zero.

(+2 points)

Alternative explanation: Because $\bar{T}(t)$ and $w(t)$ are non-zero constants and $K(s) = K$ does not have a pole at the origin.

4. **Integral Control**

In this question, assume that $mc = R = 1$.

Consider the integral control law

$$q(t) = K_i \int_0^t e(\tau)\, d\tau, \quad e(t) = \bar{T}(t) - T(t) \tag{5}$$

where $\bar{T}(t)$ is a reference temperature.

(a) (2 points) Calculate the closed-loop transfer-functions from $\bar{T}$ to e and from w to e.

Solution:
The transfer-function of the controller is

$$K(s) = \frac{K_i}{s} \qquad \text{(+1 point)}$$

From $\bar{T}$ to e:

$$S(s) = \frac{1}{1+GK} = \frac{1}{1+\frac{K_i}{s(s+1)}} = \frac{s(s+1)}{s^2+s+K_i} \qquad \text{(+1 point)}$$

From w to e:

$$D(s) = \frac{G}{1+GK} = GS = \frac{1}{s+1} \times \frac{s(s+1)}{s^2+s+K_i} = \frac{s}{s^2+s+K_i} \qquad \text{(+1 point)}$$

(b) (2 points) What are the values of K_i for which the closed-loop system is asymptotically stable?

Solution:
For asymptotic stability the roots of

$$s^2 + s + K_i = 0 \qquad \text{(+1 point)}$$

should have negative real part.
That is the case only if

$$K_i > 0 \qquad \text{(+1 point)}$$

(c) (2 points) Does the signal $T(t)$ converge to $\bar{T}$ when $\bar{T}(t) = \bar{T}$ is a constant? Explain.

Solution:
Yes because $\bar{T}(t)$ and $w(t)$ are non-zero constants and $S(0) = D(0)$ are both zero. (+2 points)
Alternative explanation: Because $\bar{T}(t)$ and $w(t)$ are non-zero constants and $K(s)$ has a pole at the origin.