MAE143 B - Linear Control - Winter 2022
Final, March 17th

Instructions:

1. This exam is open book. You may use whatever written materials of your choice, including your notes and the book, and a calculator with no communication capabilities.
2. You have 120 minutes. Don't get stuck! Come back later if you have time.
3. Write the answers on your "Blue Book."
4. Do not forget to write your name and student ID.
5. The exam has 6 questions for a total of 36 points.

Questions:

1. **Root Locus I.**

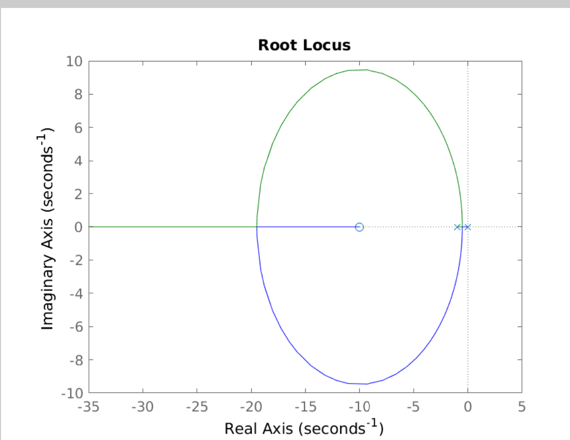
   Consider the loop transfer-function

$$L(s) = \frac{s+10}{s(s+1)}$$

   and answer the following questions.

   (a) (4 points) Sketch the root-locus diagram for $L(s)$.

   Solution:
   Sketch should look like:



   (+4 points)

   NOTE TO GRADERS: pay attention to correct poles and zeros, number of branches, location of the asymptotes, etc, when grading.

   (b) (1 point) Based on your root-locus diagram, what is the range of values of $\alpha > 0$ such that the zeros of $1 + \alpha L(s)$ are asymptotically stable? Explain.

   Solution: The zeros of $1 + \alpha L(s)$ are asymptotically stable for all $\alpha > 0$ because all the branches of the root locus remain on the left-hand side of the complex plane.

   (+1 point)

(c) (1 point) Based on your root-locus diagram, is there a value of $\alpha > 0$ for which all zeros of $1 + \alpha L(s)$ are real and have negative real part less than -10? Explain.

Solution: As the roots move from the poles to the zeros, there exists a value of $\bar{\alpha} > 0$ that places both zeros exactly at the real axis (near -20 in my plot) and for any value of $\alpha > \bar{\alpha}$ the roots remain real and with real part less that -10. (+1 point)

2. **Root Locus II.**

Consider the loop transfer-function

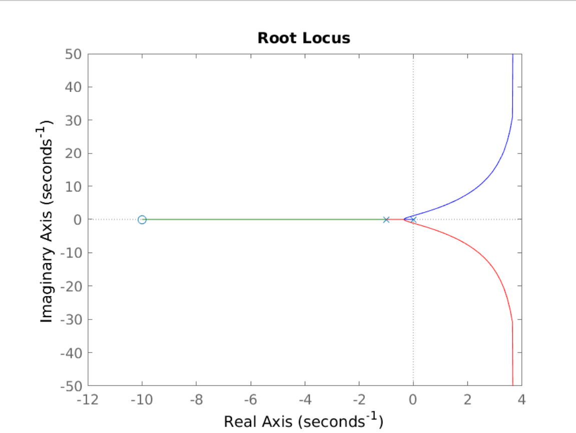
$$L(s) = \frac{s + 10}{s(s + 1)^2}$$

and answer the following questions.

(a) (4 points) Sketch the root-locus diagram for $L(s)$.

Solution:

Sketch should look like:



(+4 points)

NOTE TO GRADERS: pay attention to correct poles and zeros, number of branches, location of the asymptotes, etc, when grading.

(b) (1 point) Based on your root-locus diagram, are there values of $\alpha > 0$ for which at least one zeros of $1 + \alpha L(s)$ is *not* asymptotically stable? Explain.

Solution: Because the asymptotes are on the right-hand side of the complex plane, there is a positive value $\bar{\alpha} > 0$ from which two of the zeros for any $\alpha > \bar{\alpha}$ are on the right-hand side of the complex plane. (+1 point)

(c) (1 point) Based on your root-locus diagram, is there a value of $\alpha > 0$ for which all zeros of $1 + \alpha L(s)$ are real and asymptotically stable? Explain.

Solution: Yes, any $0 < \alpha < \bar{\alpha}$ from part (b) is such that all three roots are asymptotically stable. (+1 point)

3. **Bode and Nyquist Plots.**

Consider the loop transfer-function

$$L(s) = \frac{s+10}{s^2-1}$$

and answer the following questions.

(a) (4 points) Sketch the magnitude and phase of the Bode plots for $L(s)$.

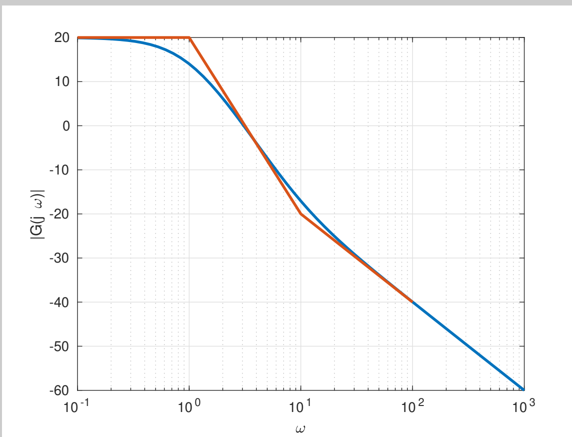
Solution:

Start by normalizing the transfer-function

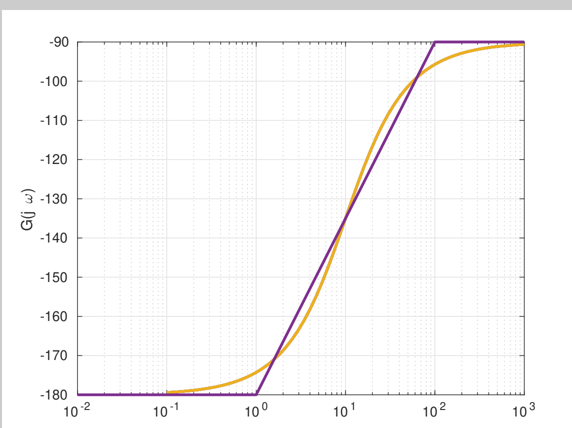
$$L(s) = -10\frac{1+s/10}{(1-s)(1+s)}.$$

Note the negative gain! (+1 point)

Sketches should look like:



(+1.5 point)



(+1.5 point)

NOTE TO GRADERS: pay attention to correct slopes at poles and zeros, correct phase offset because of the negative gain.

(b) (4 points) Sketch the Nyquist plot for $L(s)$.

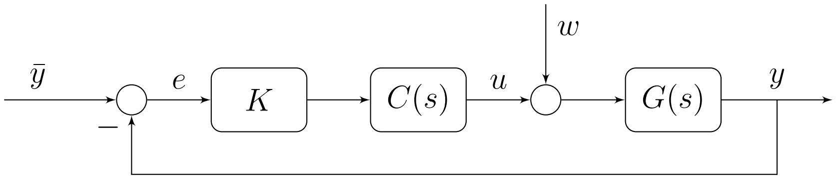
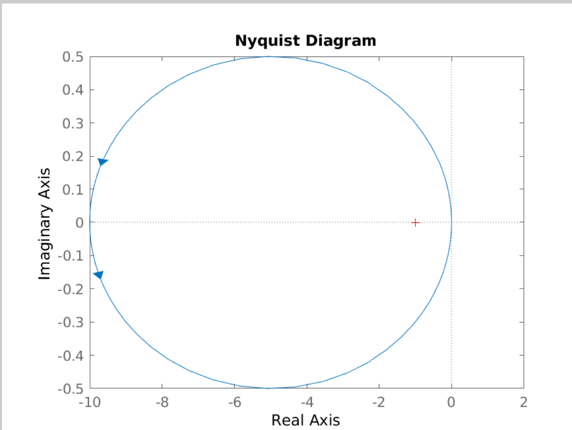


Figure 1

Solution:
Sketches should look like:



(+4 points)

NOTE TO GRADERS: pay attention to correct crossing of quadrants, direction of plot, etc, when grading.

(c) (2 points) Based on the Nyquist diagram, determine the range of values of $\alpha > 0$ for which the zeros of $1 + \alpha L(s)$ are asymptotically stable. Explain.

Solution:
The number of poles of $L(s)$ in the right-hand side of the complex plane if $P_\Gamma = 1$.
(+1 point)

If $\alpha > 0.1$ then $-10 < -1/\alpha < 0$ and the number of encirclements is 1 and $Z_\Gamma = P_\Gamma - \#\text{encirclements} = 1 - 1 = 0$, which indicates that there are no closed-loop unstable poles, that is the zeros of $1 + \alpha L(s)$ are asymptotically stable. (+1 point)

One may also note that if $\alpha < 0.1$ then $-1/\alpha < -10$ and the number of encirclements is 0 and $Z_\Gamma = P_\Gamma - \#\text{encirclements} = 1$, which indicates that there is one unstable closed-loop pole.

4. **Feedback Analysis I.**

Consider the feedback diagram in Fig. 1 in which

$$G(s) = \frac{s+10}{s+1}, \qquad C(s) = \frac{1}{s}, \qquad L(s) = C(s)\,G(s) = \frac{s+10}{s(s+1)},$$

and answer the following questions. *HINT: use your answers from Question 1!*

(a) (2 points) Assume $w(t) = 0$. Is the closed-loop capable of asymptotically tracking a constant non-zero reference input $\bar{y}$? Explain.

Solution: Yes, because the loop transfer-function $L(s) = G(s)C(s)$ has a pole at zero (+1 point)

... and according to Question 1 the closed-loop system is asymptotically stable for any $\alpha > 0$. (+1 point)

(b) (2 points) Assume $w(t) = \bar{w} \neq 0$. Is the closed-loop capable of asymptotically tracking a constant non-zero reference input $\bar{y}$? Explain.

Solution: Yes, because the $C(s)$ has a pole at zero (+1 point)

... and according to Question 1 the closed-loop system is asymptotically stable for any $\alpha > 0$. (+1 point)

(c) (2 points) Assume $\bar{y}(t) = 0$ and $w(t) = \bar{w} \neq 0$. Calculate the steady-state value of $u(t)$.

Solution:

The transfer-function from w to u is

$$u = -Hw, \qquad H = \frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{\frac{s+10}{s(s+1)}}{1 + \frac{s+10}{s(s+1)}} = \frac{s+10}{s^2 + s + s + 10} \qquad \text{(+1 point)}$$

and the steady state value of $u(t)$ is given by the frequency response method

$$u_{\text{ss}}(t) = -H(0)\bar{w} = -1 \times \bar{w} = -\bar{w}. \qquad \text{(+1 point)}$$

But you knew that already because that is the only way in which you would achieve asymptotic tracking!

5. **Feedback Analysis II.**

Now let

$$G(s) = \frac{1}{s}, \qquad C(s) = \frac{s+10}{s+1}, \qquad L(s) = C(s)\,G(s) = \frac{s+10}{s(s+1)},$$

and answer the following questions. *HINT: use your answers from Question 1!*

(a) (2 points) Assume $w(t) = 0$. Is the closed-loop capable of asymptotically tracking a constant non-zero reference input $\bar{y}$? Explain.

Solution: Yes, because the loop transfer-function $L(s) = G(s)C(s)$ has a pole at zero (+1 point)

... and according to Question 1 the closed-loop system is asymptotically stable for any $\alpha > 0$. (+1 point)

(b) (2 points) Assume $w(t) = \bar{w} \neq 0$. Is the closed-loop capable of asymptotically tracking a constant non-zero reference input $\bar{y}$? Explain.

Solution: No, because the $C(s)$ does not have a pole at zero. (+2 point)

6. **Feedback Design.**

Consider the feedback diagram in Fig. 1 in which

$$G(s) = \frac{1}{s}.$$

(a) (4 points) Design a controller transfer-function $C(s)$ so that the closed-loop system is capable of asymptotically tracking a constant non-zero reference input $\bar{y}$ while rejecting a constant non-zero disturbance w. Use a graphical method (root-locus or Nyquist) to justify the existence of a stabilizing gain. You do not need to explicitly calculate a value of gain, but you need to show that a suitable choice exists.

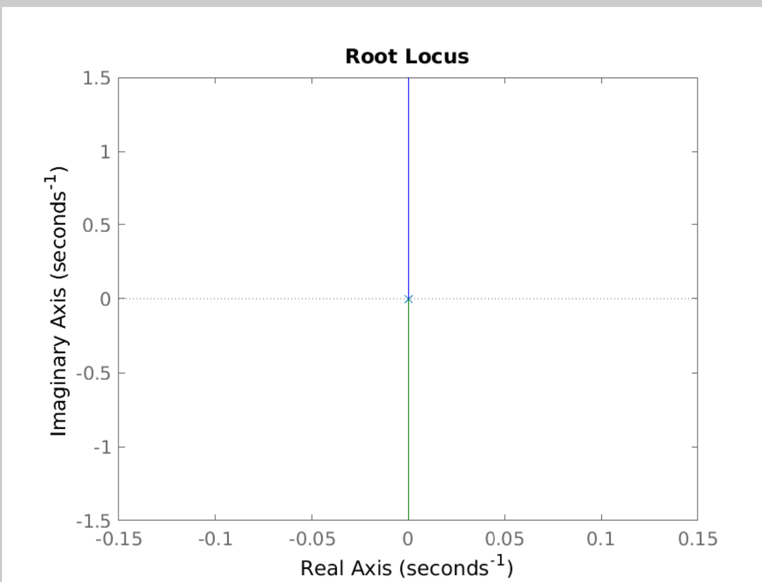
Solution:

In order to reject the constant disturbance it is necessary for the controller to have a pole at $s = 0$. (+1 point)

Unfortunately the integral controller in which

$$C(s) = \frac{1}{s}$$

is not stabilizing. Its root locus looks like

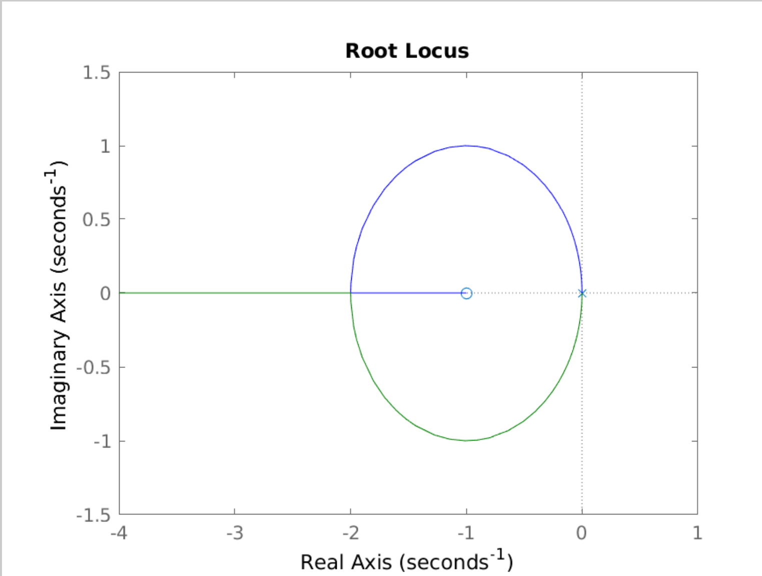


which only has imaginary roots. (+1 point)

The addition of a zero

$$C(s) = \frac{s+z}{s}, \quad z > 0 \qquad \text{(+1 point)}$$

should fix the problem allowing the root locus to veer toward the negative side of the complex plane, as in



shown for $z = 1$. In this case, any $K = \alpha > 0$ leads to a asymptotically stable closed-loop system. (+1 point)

NOTE TO GRADERS: an analysis using Bode plots and the Nyquist criterion is also possible and should be considered correct.

NOTE TO GRADERS: there are other possible solutions.