MAE143 B - Linear Control - Winter 2022
Midterm, February 8th

Instructions:

1. This exam is **open book**. You can use any **written** or **printed** material you choose.
2. This exam has 3 questions, for a total of 19 points and 3 bonus points.
3. You have 70 minutes.
4. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
5. Write your answers on your "Blue Book".
6. Do not forget to write your name and student number.

Questions:

1. **Modeling**

   A DC motor is modeled by the ordinary differential equation:

   $$J\,\dot{\omega}(t) + \left(b + K_e \frac{K_t}{R_a}\right)\omega(t) = \frac{K_t}{R_a} v_a(t) \qquad \text{(SYS)}$$

   in which $\omega(t)$ is the angular velocity of the motor and $v_a(t)$ is the voltage applied. The constants J, b, and K_t are associated with the mechanical parts of the motor and K_e and R_a are associated with the electrical parts of the motor. See **P2.41–P2.26** in the textbook for more.

   (a) (1 point) Show that the transfer function from $V_a(s)$ to $\Omega(s)$ is

   $$G(s) = \frac{\Omega(s)}{V_a(s)} = \frac{b}{s+a}, \qquad b = \frac{K_t}{JR_a}, \qquad a = \left(\frac{b}{J} + \frac{K_e K_t}{JR_a}\right).$$

   IMPORTANT: Use b and a as the motor parameters for the rest of this exam.

   Solution:
   Rewrite (SYS) as

   $$\dot{\omega}(t) + \frac{1}{J}\left(b + K_e \frac{K_t}{R_a}\right)\omega(t) = \dot{\omega}(t) + a\omega(t) = bv_a(t) = \frac{K_t}{JR_a} v_a(t)$$

   from which

   $$G(s) = \frac{b}{s+a} \qquad \text{(+1 point)}$$

   by inspection.
   NOTE TO GRADERS: some students might prefer to explicitly apply the Laplace transform.

(b) (2 points) Use any method you would like to show that the response of the DC motor to a step input $v_a(t) = \bar{v}_a\, 1(t)$ and zero initial state is

$$\omega_{\text{step}}(t) = \frac{\bar{v}_a\, b}{a}(1 - e^{-at})1(t).$$

Solution:
The Laplace transform of a step input is $V_a(s) = \bar{v}_a s^{-1}$ and the system output is given by

$$\Omega(s) = \frac{b}{s+a}V_a(s) = \frac{\bar{v}_a b}{s(s+a)} = \frac{\bar{v}_a b/a}{s} - \frac{\bar{v}_a b/a}{s+a} \qquad \text{(+1 point)}$$

after performing an expansion in partial fractions. The final response is then

$$\omega(t) = \mathcal{L}^{-1}\left\{\frac{\bar{v}_a b/a}{s} - \frac{\bar{v}_a b/a}{s+a}\right\} = \frac{\bar{v}_a b}{a}(1 - e^{-at})1(t). \qquad \text{(+1 point)}$$

NOTE TO GRADERS: some students might prefer to solve a differential equation.

(c) (1 point) The time-constant of a system, τ, as measured in a step response, is the time it takes the system to reach $1 - e^{-1} \approx 0.63$ of the steady-state value. Show that for (SYS) the time constant is $\tau = a^{-1}$.

Solution:
From $\omega_{\text{step}}(t)$, the steady state value of $\omega(t)$ is equal to $\bar{v}_a b/a$. (+1 point)
Dividing $\omega_{\text{step}}(t)$ by $\bar{v}_a b/a$ and equating to $1 - e^{-1}$ at $t = \tau$ we obtain

$$\frac{a}{b}\omega_{\text{step}}(\tau) = 1 - e^{-a\tau} \qquad \text{(+0.5 point)}$$

which is equal to $1 - e^{-1}$ if

$$a\tau = 1 \qquad \Longrightarrow \qquad \tau = a^{-1}. \qquad \text{(+0.5 point)}$$

(d) (2 points) Show that a DC motor that takes $\tau = 10$ s to reach 0.63 of its steady state 1200 rpm in response to a 12 V step voltage $v_a(t)$ has parameters

$$b = 10 \text{ rpm / (V s)}, \qquad a = 0.1 \text{ s}^{-1}.$$

Solution:
From part (c), $\tau = 10$ s is the time-constant and therefore

$$a = \tau^{-1} = 0.1 \text{ s}^{-1}. \qquad \text{(+1 point)}$$

The steady state value 1200 rpm is

$$\frac{b}{a} 12 \text{ V} = 1200 \text{ rpm}, \qquad \text{(+0.5 point)}$$

hence

$$b = \frac{1200}{12} a \text{ rpm/V} = 100 \times 0.1 \text{ rpm/(V s)} = 10 \text{ rpm/(V s)}. \qquad \text{(+0.5 point)}$$

2. **Proportional control**

Consider the connection of (SYS) with the proportional feedback controller

$$v_a(t) = K_p\, e(\tau), \qquad e(t) = \bar{\omega}(t) - \omega(t). \qquad \text{(P-CTRL)}$$

(a) (1 point) Calculate the transfer function, $S(s)$, from the reference input $\bar{w}(t)$ to the error signal $e(t)$.

Solution:
The controller transfer-function is $K(s) = K_p$ so that the sought after transfer-function is

$$S(s) = \frac{1}{1 + K(s)G(s)} = \frac{1}{1 + K_p b/(s + a)} = \frac{s + a}{s + a + bK_p} \qquad \text{(+1 point)}$$

(b) (1 point) Calculate a value of K_p such that $S(s)$ is asymptotically stable with a and b from Q1(d).

Solution:
Any K_p such that $s = -(a + bK_p) < 0$ works. For example, if $a = 0.1$, $b = 10$, then $K_p = 1$ is such that $-(0.1 + 10 \times 1) = -10.1 < 0$. (+1 point)

(c) (1 point) Is the closed-loop system capable of asymptotically tracking a constant reference input $\bar{\omega}(t) = \bar{\omega}$, $t \geq 0$, with the gain you calculated in Q2(b)?

Solution:
No because the product $K(s)G(s) = K_p b/(s + a)$ does not have a pole at the origin. (+1 point)
NOTE TO GRADERS: some students might give an more detailed answer or might want to characterize the zeros of $S(s)$.

(d) (1 point) Is the closed-loop system internally stable with the feedback controller (P-CTRL) and the gain you calculated in Q2(b)? Explain.

Solution:
Yes, because $S(s)$ is asymptotically stable and there is no pole-zero cancellation in forming the product $K(s)G(s)$. (+1 point)

3. **Proportional-Integral controller**

Consider the connection of (SYS) with the proportional-integral feedback controller

$$v_a(t) = K_p\, e(t) + K_i \int_0^t e(\tau)\, d\tau, \qquad e(t) = \bar{\omega}(t) - \omega(t). \qquad \text{(PI-CTRL)}$$

(a) (2 points) Calculate the transfer-function, $S(s)$, from the reference input $\bar{w}(t)$ to the error signal $e(t)$.

Solution:
The controller transfer-function is $K(s) = K_p + K_i/s$ so that the sought after transfer-function is

$$S(s) = \frac{1}{1 + K(s)G(s)} = \frac{1}{1 + (K_p s + K_i)b/(s^2 + as)} = \frac{s(s+a)}{s^2 + (a + bK_p)s + bK_i}$$

(+2 points)

(b) (1 point) Show that the poles of the transfer-function $S(s)$ are the roots of the polynomial

$$p(s) = s^2 + 2\xi\omega_n s + \omega_n^2, \qquad \omega_n = \sqrt{bK_i}, \qquad \xi = \frac{a + bK_p}{2\sqrt{bK_i}}.$$

Solution:
The poles are the zeros of the denominator of the transfer-function from part (a)

$$\begin{aligned} p(s) &= s^2 + (a + bK_p)s + bK_i \\ &= s^2 + 2\frac{(a + bK_p)}{2\sqrt{bK_i}}\sqrt{bK_i}s + (\sqrt{bK_i})^2 \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2. \end{aligned}$$

(+1 point)

(c) (3 points) Calculate values of K_p and K_i such that $S(s)$ is asymptotically stable with a and b from Q1(d). Is the resulting closed-loop response oscillatory?

Solution:
Take $K_i = b^{-1} = 0.1$ and $K_p = (2 - a)/b = 0.19$ such that

$$p(s) = s^2 + (a + bK_p)s + bK_i = s^2 + 2s + 1 = (s+1)^2 \qquad \text{(+1 point)}$$

which makes $S(s)$ asymptotically stable because the pair of repeated poles is at $s = -1 < 0$. (+1 point)

The response is not oscillatory because both poles are real. (+1 point)

(d) (1 point) Is the closed-loop system capable of asymptotically tracking a constant reference input $\bar{\omega}(t) = \bar{\omega}$, $t \geq 0$, with the gains you calculated in Q3(c)?

Solution:

Yes because $S(s)$ is asymptotically stable and the product $K(s)G(s) = (K_p s + K_i)b/(s^2 + as)$ does have a pole at the origin. (+1 point)

NOTE TO GRADERS: some students might give an more detailed answer or might want to characterize the zeros of $S(s)$.

(e) (1 point) Is the closed-loop system capable of asymptotically rejecting a constant input disturbance acting on the system input?

Solution:

Yes because $S(s)$ and $D(s) = G(s)S(s)$ are asymptotically stable and the product $K(s) = (K_p s + K_i)/s$ does have a pole at the origin. (+1 point)

NOTE TO GRADERS: some students might give an more detailed answer or might want to characterize the zeros of $D(s)$.

(f) (1 point) Is the closed-loop system internally stable with the feedback controller (PI-CTRL) and the gains you calculated in Q3(c)? Explain.

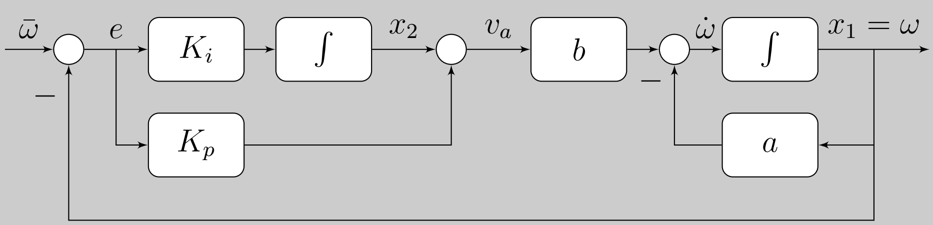
Solution:

Yes, because $S(s)$ is asymptotically stable and there is no pole-zero cancellation in forming the product $K(s)G(s)$. (+1 point)

(g) (3 points (bonus)) Draw a block diagram with integrators that represent the closed-loop connection of the controller (PI-CTRL) with the system (SYS). Clearly identify the signals $\omega(t)$, $\dot{\omega}(t)$, $\bar{\omega}(t)$, $e(t)$, and $v_a(t)$ in your diagram. Use the diagram to write a state-space representation for the closed-loop system in which $\bar{w}(t)$ is the input and $e(t)$ is the output.

Solution:

One possible diagram is



(+1 point)

As for a state-space realization, take x_1 and x_2 as indicated in the diagram so that

$$
\begin{aligned}
e &= \bar{\omega} - x_1 \\
\dot{x}_1 &= -ax_1 + b(x_2 + K_p e) = -(a + bK_p)x_1 + bx_2 + bK_p\bar{\omega}, \\
\dot{x}_2 &= K_i e = -K_i x_1 + K_i\bar{\omega}
\end{aligned}
$$

(+1 point)

In matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -(a + bK_p) & b \\ -K_i & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} bK_p \\ 0 \end{bmatrix} \bar{\omega}$$

$$e = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 1 \end{bmatrix} \bar{\omega}$$

(+1 point)