

State-Space and Linearization

In this chapter we introduce ideas that can be used to implement controllers on physical hardware. The resulting block diagrams and equations also serve as the basis for simulation of dynamic systems in computers, a topic that we use to motivate the introduction of *state-space models*. The state-space formalism provides a framework for computing linearized models from nonlinear differential equations, and sometimes relate the stability of the linearized model to the stability of a more complex nonlinear model. We finish with a discussion about possible issues that can arise when a linear controller is used in feedback with a nonlinear system.

5.1 Realization of Dynamic Systems

The simplest dynamic system for which we can envision a construction is the *integrator*. Any device that is capable of storing mass, charge or energy in some form is basically an integrator. Indeed, we have already met one such device in § 2.8: the *toilet water tank*. In Fig. 2.16, the water level, y , is the results of integration of the water input flow, u ,

$$y(t) = \frac{1}{A} \int_0^t u(\tau) d\tau,$$

where A is the constant cross-section area of the tank. The voltage across the terminals of a *capacitor*, v , is the integral of the current, i ,

$$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau,$$

where C is the capacitor's capacitance. A *fly-wheel* with negligible friction integrates the input torque, f , to produce an angular velocity, ω ,

$$\omega(t) = \frac{1}{J} \int_0^t f(\tau) d\tau,$$

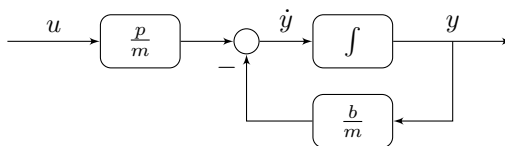


Fig. 5.1: First-order differential equation, $m \dot{y} + b y = p u$, $m > 0$

where J is the wheel's moment of inertia. Of significance in modern control and signal processing algorithms, an integrator can be implemented via an approximate integration rule, such as the *trapezoidal integration rule*:

$$y(kT) - y(kT - T) = \int_{kT-T}^{kT} u(t) dt \approx \frac{T}{2} (u(kT) - u(kT - T))$$

which can be implemented in the form of the recursive algorithm:

$$y(kT) = y(kT - T) + \frac{T}{2} (u(kT) - u(kT - T))$$

indexed by the integer k and where T is a small enough *sampling period* at which periodic samples of the continuous input $u(t)$ are obtained. We will discuss some aspects of digital control in § 8.

Two difficulties are common to all physical realizations of integrator: a) providing enough storage capacity, be it in the form of mass, volume, charge, energy, or numbers in a computer; b) controlling the *losses* or *leaks*. The storage capacity should be sized to fit the application in hand, and losses must be kept under control with proper materials and engineering. In the following discussion we assume that these issues have been worked out and proceed to utilize integrators to realize more complex dynamic systems.

We have already used integrators in § 2 to represent simple differential equations. Here we expand on the basic idea to cover differential equations of higher-order that might also involve the derivative of the input. As seen in § 3.2, transfer-functions and linear ordinary differential equations are closely related, and the techniques and diagrams obtained from differential equations can be readily used to implement the corresponding transfer-function.

Let us start by revisiting the diagram in Fig. 2.3 which represents the linear ordinary differential equation (2.3). This diagram is reproduced in Fig. 5.1. The trick used to represent the differential equation (2.3) into a block diagram with integrators was to isolate the highest derivative:

$$\dot{y} = \frac{p}{m} u - \frac{b}{m} y.$$

The highest derivative is then integrated in a series of cascade integrators from which all lower-order derivatives become available. For example, in order to

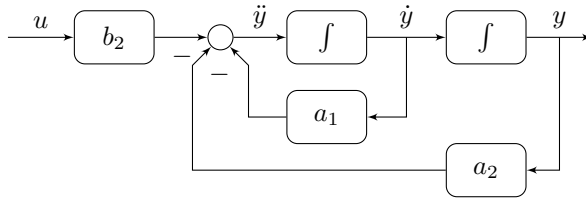


Fig. 5.2: Second-order differential equation, $\ddot{y} + a_1\dot{y} + a_2y = b_2u$

represent the second-order differential equation

$$\ddot{y} + a_1\dot{y} + a_2y = b_2u$$

into a block diagram with integrators we first isolate the second derivative

$$\ddot{y} = b_2u - a_1\dot{y} - a_2y$$

which is then integrated twice. The input signal u and the signals y and $\dot{y}$ are run through amplifiers and a summer is used to reconstruct $\ddot{y}$ as shown in Fig. 5.2. Of course, one could use the exact same scheme to implement the associated transfer-function:

$$\frac{Y}{U} = \frac{b_2}{s^2 + a_1s + a_2}.$$

In this case, isolate the term with the highest power of s :

$$s^2Y = b_2U - s a_1Y - a_2Y$$

and replace integrators by ' s^{-1} '. The result is Fig. 5.3, which is virtually the same as Fig 5.2. Initial conditions for the differential equation, $y(0)$ and $\dot{y}(0)$ in this case, are implicitly incorporated in the block diagrams as initial conditions for the integrators. For example, the second integrator in Fig. 5.2 implements the definite integral:

$$y(t) = y(0) + \int_0^t \dot{y}(\tau) d\tau.$$

Physically, $y(0)$ is the amount of the quantity being integrated, water, current, torque, etc, which is present in the integrator at $t = 0$. More about that in § 5.2.

A slightly more challenging task is to represent the differential equation

$$\ddot{y} + a_1\dot{y} + a_2y = b_0\ddot{u} + b_1\dot{u} + b_2u$$

using only integrators. The difficulty is how to handle the derivatives of the input signal u . One idea is to use linearity. First solve

$$\ddot{z} + a_1\dot{z} + a_2z = u.$$

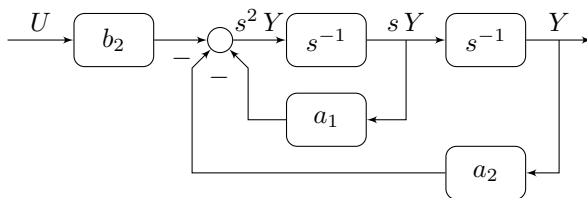


Fig. 5.3: Second-order transfer-function, $G(s) = b_2/(s^2 + a_1 s + a_2)$

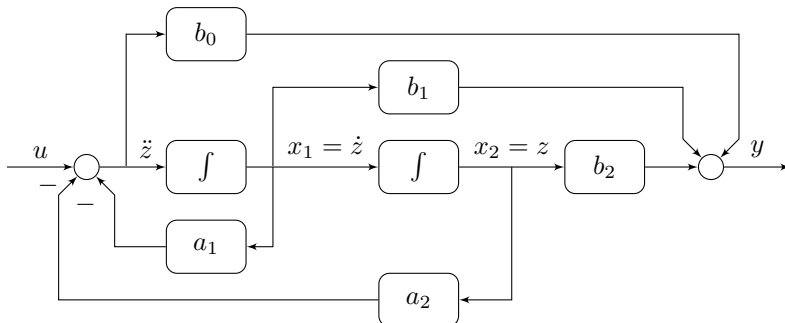


Fig. 5.4: Second-order differential equation, $\ddot{y} + a_1 \dot{y} + a_2 y = b_0 \ddot{u} + b_1 \dot{u} + b_2 u$

One can use the diagram in Fig. 5.2 with $b_2 = 1$ for that. A solution to

$$\ddot{y} + a_1 \dot{y} + a_2 y = u_0 + u_1 + u_2, \quad u_0 = b_0 \ddot{u}, \quad u_1 = b_1 \dot{u}, \quad u_2 = b_2 u$$

can be calculated from z , $\dot{z}$ and $\ddot{z}$ using superposition

$$y = b_0 \ddot{z} + b_1 \dot{z} + b_2 z.$$

This idea is implemented in the diagram of Fig. 5.4. See also **P2.31**.

Here is an alternative: isolate the highest derivative and collect the right-hand terms with same degree, that is

$$\ddot{y} = b_0 \ddot{u} + (b_1 \dot{u} - a_1 \dot{y}) + (b_2 u - a_2 y).$$

Now integrate¹ twice to obtain

$$y(t) = b_0 u(t) + \int_0^t \left[b_1 u(\tau) - a_1 y(\tau) + \int_0^\tau (b_2 u(\sigma) - a_2 y(\sigma)) d\sigma \right] d\tau.$$

These operations are represented in the diagram in Fig. 5.5. Because Fig. 5.4 and 5.5 represent the exact same system, the realization of a differential equation, or its corresponding transfer-function, is not unique. Other forms are possible which we do not have room to discuss here. They can be found, for instance, in the excellent [Kai80].

¹ Assuming zero initial conditions $y(0) = \dot{y}(0) = 0$.

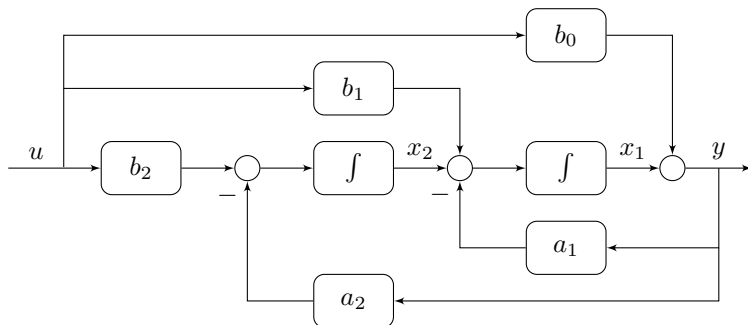


Fig. 5.5: Second-order differential equation, $\ddot{y} + a_1\dot{y} + a_2y = b_0\ddot{u} + b_1\dot{u} + b_2u$

Besides the issue of uniqueness, a natural question is: can we apply these ideas to *any* differential equation and obtain a block diagram using only integrators? The answer is *no*. Consider for example:

$$y = \dot{u}$$

where u is an input and y is the output. It is not possible to represent this equation in a diagram where y is obtained as a function of u without introducing a derivative block. In general, by a straightforward generalization of the techniques discussed above, it should be possible to represent a general linear ordinary differential equation of the form (3.11) using only integrators if the highest derivative of the input signal u appearing in (3.11) is not higher than the highest derivative of the output signal y . That is, if $m \leq n$. In terms of the associated rational transfer-function, G in (3.12), it means that the degree of the numerator, m , is no higher than the degree of the denominator, n , that is G is proper. As we will soon argue, it is not possible to physically implement a differentiator, hence one should not ordinarily find transfer-function models of physical systems which are not proper. Nor one should expect to be able to implement a controller obtained from a transfer-function which is not proper.

One obstruction for implementing differentiators is the required amount of amplification: differentiators need to deliver an ever increasing gain to high-frequency signals. Indeed, the transfer-function of a differentiator, $G(s) = s$, produces in the presence of the sinusoidal input, $u(t) = \cos(\omega t)$, with frequency ω and unit amplitude, a steady-state response

$$y_{ss}(t) = |j\omega| \cos(\omega t + \angle j\omega) = -|\omega| \sin(\omega t).$$

See § 3.8. As ω grows, the amount of amplification required to produce $y_{ss}(t)$ grows as well, becoming unbounded as $\omega \rightarrow \infty$. Clearly no physical system can have this capability. Even if one is willing to construct one such system in the hope that the input signals have low enough frequency content, the system will likely face problems if it has to deal with discontinuities² of the input signal.

²Or high enough derivative.

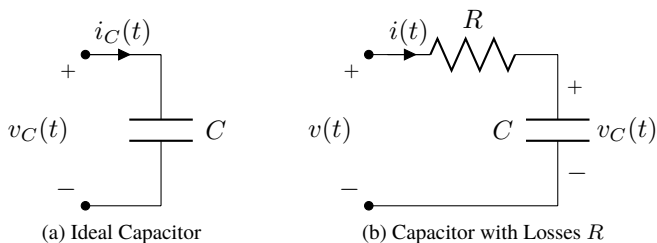


Fig. 5.6: Capacitor circuit models

Using the Laplace transform formalism, one might find useful to think of the generalized derivative of a signal at a discontinuity producing an impulse, e.g. $d/dt \, 1(t) = \delta(t)$. A physical system that is modeled as a differentiator will not be able to produce an impulse, as discussed in § 3.

That is however not to say that *components* of a system can not be modeled as differentiators or have improper transfer functions. Take for example the electrical circuit in Fig. 5.6. The relationship between the voltage and current of an *ideal* capacitor is the differentiator:

$$i_C(t) = C \dot{v}_C(t).$$

The transfer-function of the ideal capacitor component is therefore

$$\frac{I_C(s)}{V_C(s)} = sC,$$

which is not proper. A *real* capacitor will however have losses, which can be represented by some small nonzero resistance R appearing in series with the capacitor in the circuit of Fig. 5.6. The complete circuit relations are

$$Ri(t) + v_C(t) = v(t), \quad i(t) = C \dot{v}_C(t),$$

from which we obtain the transfer-function:

$$G(s) = \frac{I(s)}{V(s)} = \frac{sC}{1 + sRC},$$

after eliminating v_C . See **P2.30** and **P3.28**. The overall circuit has a proper transfer-function, $G(s)$. The smaller the losses, i.e. R , the closer the circuit behaves as an ideal capacitor, hence a differentiator. Indeed

$$\lim_{R \rightarrow 0} G(s) = sC$$

which is not proper. It was nevertheless very useful to work with the *ideal capacitor* model and its improper component transfer-function to understand

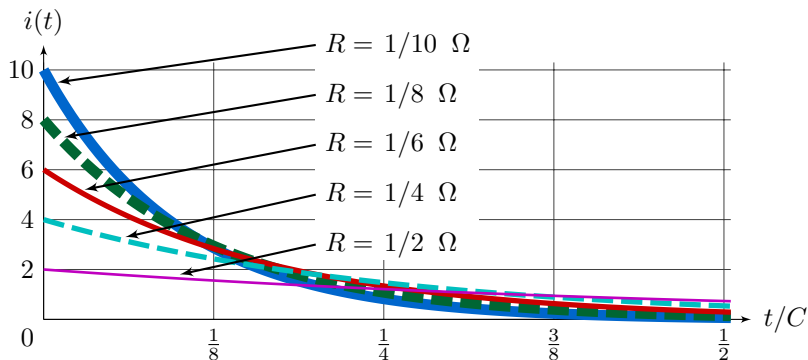


Fig. 5.7: Normalized response of a capacitor with losses to a 1V step input

the overall circuit behavior and its limitations. It is in this spirit that one should approach improper transfer-functions.

Even if the issue of gain could be addressed, there is still potential problems with losses in practical differentiators. For example, the unit step response of the ideal capacitor with model sC is the impulse $C\delta(t)$. The unit step response of the capacitor with losses is

$$\mathcal{L}^{-1} \left\{ \frac{sC}{1 + sRC} \times \frac{1}{s} \right\} = \frac{1}{R} e^{-\frac{1}{RC}t}, \quad t \geq 0.$$

which approaches an impulse of amplitude C as $R \rightarrow 0$. A capacitor with losses would therefore produce large *spikes* of current for small amounts of time every time a 1V source is connected to its terminals, as plotted³ in Fig. 5.7. Such intense but short lived currents and voltages will cause other problems to the circuit and the physical materials it is made of, one of them is that the materials will stop responding linearly.

On the issue of high-frequency gains, the steady-state response of the capacitor with losses to a sinusoidal input of frequency $\omega > 0$ and unit amplitude, $u(t) = \cos(\omega t)$, is:

$$y_{ss}(t) = |G(j\omega)| \cos(\omega t + \angle G(j\omega)).$$

At high frequencies the gain is approximately

$$|G(j\omega)| \approx \frac{1}{R}$$

which shows that the amount of amplification in this circuit is limited by the losses represented by the resistor R . Note also that

$$\lim_{R \rightarrow 0} |G(j\omega)| = \omega C$$

³The response plotted in Fig. 5.7 is still idealized as it assumes a voltage source that has a step discontinuity. This model could be improved to account for that as well if needed.

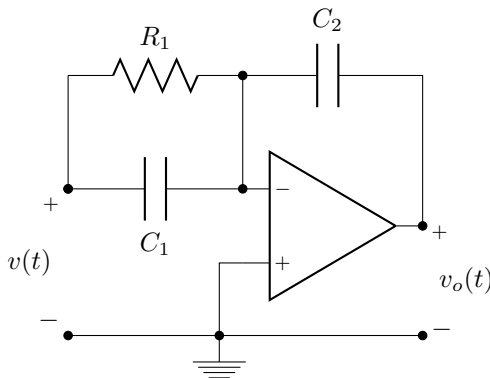


Fig. 5.8: Circuit for PI control

revealing the high-frequency amplification of a differentiator when R is small.

We conclude our discussion with another electric circuit. The circuit in Fig. 5.8 contains a resistor, R_1 , two capacitors, C_1 and C_2 , and an (operational) amplifier with very high gain, the triangular circuit element. You have shown in **P3.31** that the transfer-function from the voltage, v_i , to the voltage, v_o , is:

$$\frac{V_o(s)}{V_i(s)} = -K \frac{s - z}{s}, \quad K = \frac{C_1}{C_2}, \quad z = -\frac{1}{R_1 C_1}.$$

Therefore, by adjusting the ratio between the two capacitors C_1 and C_2 and the resistor R_1 it is possible to set the gain K and the zero z to be exactly the ones needed to implement the PI controller (4.14). The particular case when the capacitor C_1 is removed from the circuit, i.e. $C_1 = 0$, is important. In this case the circuit transfer-function reduces to

$$\frac{V_o(s)}{V_i(s)} = -\frac{K}{s}, \quad K = \frac{1}{R_1 C_2},$$

which is a pure integrator with gain K (**P2.32** and **P3.34**). This circuit can be combined with amplifiers to build a physical realization for the diagrams in Fig. 5.1-5.5. Electronic hardware with specialized circuitry implementing blocks such as the one in Fig. 5.8 were manufactured, sold and used for analysis and simulation of dynamic systems in the second half of the twentieth century under the name *analog computers*, an example of one used by NASA is shown in Fig. 5.9. These computers have all but being replaced by *digital computers* where dynamic systems are analyzed and simulated via numerical integration.

5.2 State-Space Models

After learning how to convert differential equations and transfer-functions into block diagrams using integrators we will now introduce a formalism to trans-

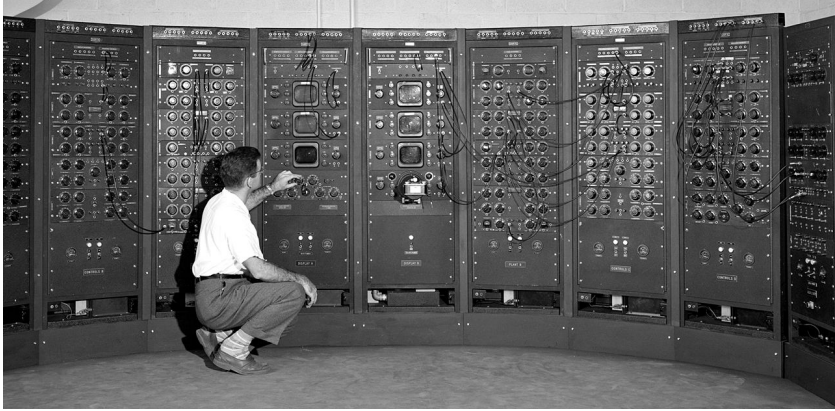


Fig. 5.9: Analog Computer used by NASA in the mid twentieth century. The device was located in the Engine Research Building at the Lewis Flight Propulsion Laboratory, now John H. Glenn Research Center, Cleveland Ohio.

form differential equations of arbitrary order into a set of *vector* first-order differential equations. The key is to look at the integrators in our block diagrams. Start by defining a new variable for each variable being integrated. For example, in the diagram in Fig. 5.4, define two variables

$$x_1 = \dot{z}, \quad x_2 = z,$$

one per output of each integrator. Next write two first-order differential equations at the input of each integrator:

$$\begin{aligned} \dot{x}_1 &= u - a_1 x_1 - a_2 x_2, \\ \dot{x}_2 &= x_1. \end{aligned}$$

Then rewrite the output, y , in terms of x_1 and x_2 after removing any derivatives:

$$\begin{aligned} y &= b_0 \dot{x}_1 + b_1 x_1 + b_2 x_2, \\ &= b_0(u - a_1 x_1 - a_2 x_2) + b_1 x_1 + b_2 x_2, \\ &= (b_1 - a_1 b_0) x_1 + (b_2 - a_2 b_0) x_2 + b_0 u. \end{aligned}$$

With the help of vectors and matrices we rearrange:

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ y &= [b_1 - a_1 b_0 \quad b_2 - a_2 b_0] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + [b_0] u. \end{aligned} \tag{5.1}$$

A similar procedure works for the diagram in Fig. 5.5. With x_1 and x_2

representing the output of each integrator, we write the differential equations:

$$\begin{aligned}\dot{x}_1 &= b_1 u - a_1 y + x_2, \\ \dot{x}_2 &= b_2 u - a_2 y.\end{aligned}$$

Note that these equations depend on the output, y . Because

$$y = b_0 u + x_1,$$

is a function of x_1 only, we substitute to obtain:

$$\begin{aligned}\dot{x}_1 &= -a_1 x_1 + x_2 + (b_1 - a_1 b_0)u, \\ \dot{x}_2 &= -a_2 x_1 + (b_2 - a_2 b_0)u.\end{aligned}$$

Rearranging using vectors and matrices:

$$\begin{aligned}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + [b_0] u.\end{aligned}\tag{5.2}$$

Equations (5.1) and (5.2) are in a special form we call *state-space* equations. Differential equations for linear time-invariant systems are in state-space form when they match the template:

$$\begin{aligned}\dot{x} &= Ax + Bu, \\ y &= Cx + Du,\end{aligned}\tag{5.3}$$

for some appropriate quadruple of constant matrices (A, B, C, D) . As before, u and y denote the input and output signals. The vector x is known as the *state vector*. The terminology *state* alludes to the fact that knowledge of the state (of the system), x , and the input, u , implies that all signals in the diagram, including the output, y , can be determined. Furthermore, knowledge of $x(t)$ at a given instant of time, say at $t = 0$, and knowledge of the inputs $u(t)$, $t \geq 0$, is enough to predict or reconstruct any signal at time $t \geq 0$.

In (5.3), not only the state, x , but also the input, u , and output, y , can be vectors. This means that state-space is capable of providing a uniform representation for single-input-single-output (SISO) as well as multiple-input-multiple-output (MIMO) systems. Furthermore, the matrix and linear algebra formalism enables the use of compact and powerful notation. For instance, applying the Laplace transform⁴ to the state-space equations (5.3) we obtain:

$$\begin{aligned}sX(s) - x(0^-) &= AX(s) + BU(s), \\ Y(s) &= CX(s) + Du(s).\end{aligned}$$

⁴The Laplace transform of a matrix or vector is to be interpreted entry-wise

We can *easily* solve for $X(s)$ in the first equation

$$\begin{aligned} sX(s) - AX(s) &= (sI - A)X(s) = BU(s) + x(0^-) \\ \implies X(s) &= (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0^-) \end{aligned}$$

and compute

$$Y(s) = G(s)U(s) + F(s, x(0^-))$$

where $G(s)$ is the transfer-function

$$G(s) = C(sI - A)^{-1}B + D, \quad (5.4)$$

and the function

$$F(s, x(0^-)) = C(sI - A)^{-1}x(0^-) \quad (5.5)$$

parametrizes the response to the *initial state*, $x(0^-)$, which plays the role of initial conditions. The simplicity of the formulas hides the complexity of the underlying calculations. For example, with

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [b_1 - a_1b_0 \quad b_2 - a_2b_0], \quad D = [b_0],$$

we compute

$$sI - A = \begin{bmatrix} s + a_1 & a_2 \\ -1 & s \end{bmatrix}$$

from which

$$(sI - A)^{-1} = \frac{\text{Adj}(sI - A)}{|sI - A|} = \frac{1}{s^2 + a_1s + a_2} \begin{bmatrix} s & -a_2 \\ 1 & s + a_1 \end{bmatrix}, \quad (5.6)$$

and

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D, \\ &= \frac{1}{s^2 + a_1s + a_2} [b_1 - a_1b_0 \quad b_2 - a_2b_0] \begin{bmatrix} s & -a_2 \\ 1 & s + a_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + [b_0], \\ &= \frac{b_1s - a_1b_0s + b_2 - a_2b_0}{s^2 + a_1s + a_2} + b_0, \\ &= \frac{b_0s^2 + b_1s + b_2}{s^2 + a_1s + a_2}. \end{aligned}$$

Because $\text{Adj}(sI - A)$ is a polynomial matrix and B , C and D are real matrices, the associated transfer-function $G(s)$ is always *rational*. The denominator of

$G(s)$ is equal to the determinant $|sI - A|$, a polynomial of degree n , where n is the dimension of the state vector. The determinant equation

$$|sI - A| = 0$$

is the *characteristic equation*. If s_0 is a root of the characteristic equation then it is an *eigenvalue* of the square matrix A and satisfy:

$$A x_0 = s_0 x_0, \quad x_0 \neq 0.$$

The vector x_0 corresponding to an eigenvalue s_0 is an *eigenvector* of A . Eigenvalues of A are poles of the transfer function $G(s)$. A matrix that has all eigenvalues with negative real part is called a *Hurwitz matrix* and internal stability⁵ of the the state-space system (A, B, C, D) is equivalent to A being a Hurwitz matrix. Another consequence of formula (5.4) is that

$$\lim_{|s| \rightarrow \infty} (sI - A)^{-1} = 0 \quad \implies \quad \lim_{|s| \rightarrow \infty} G(s) = D.$$

Consequently, the transfer-function of a linear time-invariant system that can be put in state-space form always satisfy (3.17). In other words, $G(s)$ is always rational and proper. Furthermore, if $D = 0$ then (3.21) holds and $G(s)$ is strictly proper.

Another deceptively simple formula is the impulse response:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{C(sI - A)^{-1}B + D\} = Ce^{At}B + D\delta(t), \quad t \geq 0$$

Of course the hat-trick is to be able to pull out the *rabbit*:

$$\mathcal{L}^{-1}\{(sI - A)^{-1}\} = e^{At}, \quad t \geq 0$$

from one's empty hat. The exponential⁶ function of a matrix hides⁷ the com-

⁵See § 4.6.

⁶There are many ways to make peace with the notion of a square matrix exponential. One is through the power series expansion

$$e^A = \sum_{i=0}^{\infty} \frac{1}{i!} A^i = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots$$

which is a direct extension of the standard power series of the scalar exponential function. For the most part, the exponential function of a matrix operates and has properties similar to the ones of the regular exponential, e.g. $d/dt e^{At} = Ae^{At} = e^{At}A$. Beware that some properties only hold if the matrices commute, e.g. $e^{A+B} \neq e^A e^B$ unless $AB = BA$, or are nonsingular, e.g. $\int e^{A\tau} d\tau = A^{-1}e^{At}$. Note that A , A^{-1} and e^A all commute, that is $Ae^A = e^A A$, and $AA^{-1} = A^{-1}A = I$.

⁷For example, for $A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}$ the matrix exponential e^A is given by

$$e^A = \frac{e^{-\frac{a_1}{2}}}{\Delta} \begin{bmatrix} \Delta \cosh\left(\frac{1}{2}\Delta\right) - a_1 \sinh\left(\frac{1}{2}\Delta\right) & -2a_2 \sinh\left(\frac{1}{2}\Delta\right) \\ 2 \sinh\left(\frac{1}{2}\Delta\right) & \Delta \cosh\left(\frac{1}{2}\Delta\right) + a_1 \sinh\left(\frac{1}{2}\Delta\right) \end{bmatrix}$$

where $\Delta = \sqrt{a_1^2 - 4a_2}$.

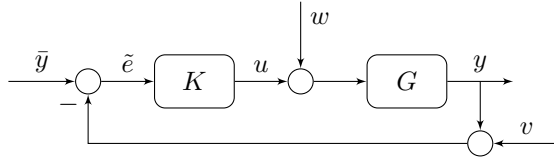


Fig. 5.10: Closed-loop feedback configuration with reference, $\bar{y}$, input disturbance, w , and measurement noise, v

plexities which will ultimately be able to correctly compute the response of linear systems even in the most complicated cases, e.g. for systems with multiple roots or complex-conjugate roots (see § 3). Likewise, in response to a non-zero initial-condition we have that

$$\begin{aligned} f(t, x(0^-)) &= \mathcal{L}^{-1}\{F(s, x(0^-))\} = \\ &\mathcal{L}^{-1}\{C(sI - A)^{-1}x(0^-)\} = Ce^{At}x(0^-), \quad t \geq 0, \end{aligned}$$

and⁸

$$\begin{aligned} \|f(t, x(0^-))\|_2^2 &= \int_{0^-}^{\infty} |f(t, x(0^-))|^2 dt = \\ &\int_{0^-}^{\infty} f(t, x(0^-))^T f(t, x(0^-)) dt = \\ &x(0^-)^T P x(0^-) \leq \max_j \lambda_j(P) \|x(0^-)\|_2^2 \end{aligned} \quad (5.7)$$

where $\lambda_j(P)$ denotes the j th (real) eigenvalue of the $n \times n$ symmetric matrix

$$P = \int_{0^-}^{\infty} e^{A^T t} C^T C e^{At} dt,$$

which is known as the *Observability Gramian*. It is possible to show that when A is Hurwitz then the matrix is finite and positive-semidefinite, $P \succeq 0$, which implies that all $\lambda_j(P)$ are real, finite and bounded, and therefore $\|f(t, x(0^-))\|_2$ is bounded by the 2-norm of the initial state $x(0^-)$ [Kai80, Theorem 2.6-1 and Corollary 2.6-2]. A natural consequence of boundedness of $\|f(t, x(0^-))\|_2$ is that

$$\lim_{t \rightarrow \infty} f(t, x(0^-)) = 0.$$

As mentioned before, state-space provides a unified treatment of SISO and MIMO systems. To see how this can work, consider the block diagram in

⁸The next lines may be a bit too advanced for this course. The important idea is that $\|f(t, x(0^-))\|_2 \leq \max_j \lambda_j(P) \|x(0^-)\|_2^2$, that is $\|f(t, x(0^-))\|$ is bounded by the norm of the initial state.

Fig. 4.8 reproduced for convenience in Fig. 5.10. Collect all inputs and outputs into the vectors

$$\mathbf{u} = \begin{pmatrix} \bar{y} \\ w \\ v \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y \\ u \end{pmatrix}.$$

Assume that G is strictly proper⁹ and let the system, G , and the controller, K , have the state-space representations:

$$\begin{aligned} \dot{x}_g &= A_g x_g + B_g u_g, & \dot{x}_k &= A_k x_k + B_k u_k, \\ y_g &= C_g x_g, & y_k &= C_k x_k + D_k u_k. \end{aligned}$$

The closed-loop diagram in Fig. 5.10 implies the following connections:

$$u_g = w + u, \quad u = y_k, \quad u_k = \bar{y} - y - v, \quad y = y_g.$$

After some algebra, we obtain first-order vector equations for the system state:

$$\begin{aligned} \dot{x}_g &= A_g x_g + B_g(w + u), \\ &= A_g x_g + B_g C_k x_k + B_g w + B_g D_k(\bar{y} - y - v), \\ &= (A_g - B_g D_k C_g) x_g + B_g C_k x_k + B_g w + B_g D_k \bar{y} - B_g D_k v, \end{aligned}$$

the controller state:

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k(\bar{y} - y - v), \\ &= -B_k C_g x_g + A_k x_k + B_k \bar{y} - B_k v, \end{aligned}$$

and the *outputs*:

$$\begin{aligned} y &= y_g = C_g x_g, \\ u &= y_k = C_k x_k. \end{aligned}$$

These equations are put in form (5.3) after defining the closed-loop state vector and rearranging:

$$\mathbf{x} = \begin{pmatrix} x_g \\ x_k \end{pmatrix}, \quad \begin{aligned} \dot{\mathbf{x}} &= \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u}, \\ \mathbf{y} &= \mathbf{C} \mathbf{x} + \mathbf{D} \mathbf{u}. \end{aligned}$$

where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} A_g - B_g D_k C_g & B_g C_k \\ -B_k C_g & A_k \end{bmatrix}, & \mathbf{C} &= \begin{bmatrix} C_g & 0 \\ 0 & C_k \end{bmatrix}, \\ \mathbf{B} &= \begin{bmatrix} B_g D_k & B_g & -B_g D_k \\ B_k & 0 & -B_k \end{bmatrix}, & \mathbf{D} &= 0. \end{aligned}$$

Closed-loop internal stability (Lemma 4.3) can be shown to be equivalent to matrix $\mathbf{A}$ being Hurwitz.

⁹Slightly messier formulas are available in the case of a proper system.

5.3 Nonlinear Systems and Linearization

Nonlinear dynamic systems can also be represented in state-space form:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), \\ y(t) &= g(x(t), u(t)).\end{aligned}\tag{5.8}$$

When f and g are continuous and differentiable functions and the state vector, x , and the input, u , are within a *small* neighborhood of a point $(\bar{x}, \bar{u})$ or trajectory $(\bar{x}(t), \bar{u}(t))$, it is natural to expect that the behavior of the nonlinear system can be approximated by that of a properly defined linear system. The procedure used to compute such linear system is known as *linearization* and the resulting system as a *linearized* approximation.

Consider the simplest case where $(\bar{x}, \bar{u})$ is a single point at which both f and g are continuous and differentiable. Expanding f and g in series around $(\bar{x}, \bar{u})$ we obtain

$$\begin{aligned}f(x, u) &\approx f(\bar{x}, \bar{u}) + A(x - \bar{x}) + B(u - \bar{u}), \\ g(x, u) &\approx g(\bar{x}, \bar{u}) + C(x - \bar{x}) + D(u - \bar{u}),\end{aligned}$$

where

$$A = \left. \frac{\partial f}{\partial x} \right|_{\substack{x=\bar{x} \\ u=\bar{u}}}, \quad B = \left. \frac{\partial f}{\partial u} \right|_{\substack{x=\bar{x} \\ u=\bar{u}}}, \quad C = \left. \frac{\partial g}{\partial x} \right|_{\substack{x=\bar{x} \\ u=\bar{u}}}, \quad D = \left. \frac{\partial g}{\partial u} \right|_{\substack{x=\bar{x} \\ u=\bar{u}}}.\tag{5.9}$$

For reasons which will become clear soon, one is often interested in special points $(\bar{x}, \bar{u})$ satisfying the nonlinear equation

$$f(\bar{x}, \bar{u}) = 0.$$

Such points are called *equilibrium points*. Indeed, at $(\bar{x}, \bar{u})$ we have $\dot{x} = f(\bar{x}, \bar{u}) = 0$, hence, *in the absence of perturbations*, trajectories of a dynamic system starting at an equilibrium point $(\bar{x}, \bar{u})$ will simply stay at $(\bar{x}, \bar{u})$. Around an equilibrium point we define deviations

$$\tilde{x}(t) = x(t) - \bar{x}, \quad \tilde{u}(t) = u(t) - \bar{u}, \quad \tilde{y}(t) = y(t) - g(\bar{x}, \bar{u}),\tag{5.10}$$

to obtain the *linearized system*

$$\begin{aligned}\dot{\tilde{x}}(t) &= A \tilde{x}(t) + B \tilde{u}(t), \\ \tilde{y}(t) &= C \tilde{x}(t) + D \tilde{u}(t)\end{aligned}\tag{5.11}$$

in standard state-space form (5.3). The next lemma, due to Lyapunov and presented without a proof¹⁰, links asymptotic stability of the linearized system with asymptotic stability of the nonlinear system around the equilibrium point.

¹⁰See for instance [Kha96, Theorem 3.7].

Lemma 5.1 (Lyapunov) *Consider the nonlinear dynamic system in state-space form defined in (5.8). Let $(\bar{x}, \bar{u})$ be an equilibrium point and consider the linearized system (5.11) for which the quadruple (A, B, C, D) is given in (5.9).*

If A is Hurwitz then there is $\epsilon > 0$ for which any trajectory with initial condition in $\|x(0) - \bar{x}\| < \epsilon$ and input $u(t) = \bar{u}$, $t \geq 0$, converges asymptotically to the equilibrium point $(\bar{x}, \bar{u})$, that is $\lim_{t \rightarrow \infty} x(t) \rightarrow \bar{x}$.

On the other hand, if A has at least one eigenvalue with positive real part then for any $\epsilon > 0$ there exists at least one trajectory with initial condition in $\|x(0) - \bar{x}\| < \epsilon$ and input $u(t) = \bar{u}$, $t \geq 0$, that diverges from the equilibrium point $(\bar{x}, \bar{u})$, that is $\lim_{t \rightarrow \infty} \|x(t) - \bar{x}\| > 0$.

This lemma is of major significance for control systems. The first statement means that it is enough to check if a linearized version of a nonlinear system around an equilibrium point is asymptotically stable in order to ensure convergence to that equilibrium point. The lemma's main weakness is that it does not tell anything about the size of the neighborhood of the equilibrium point in which convergence to equilibrium happens, that is, the size of ϵ . The second statement says that instability of the linearized system implies instability of the original nonlinear system. Note that Lemma 5.1 is inconclusive when A has eigenvalues on the imaginary axis.

For some systems, linearizing around an equilibrium point might be too restrictive. Take for example an airplane or aircraft in orbit which cannot be in equilibrium with zero velocities. Another example is a bicycle. In such cases it is useful to linearize around a time-dependent *equilibrium trajectory* $(\bar{x}(t), \bar{u}(t))$ satisfying

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t)).$$

As before, we can define deviations from the equilibrium trajectory

$$\tilde{x}(t) = x(t) - \bar{x}(t), \quad \tilde{u}(t) = u(t) - \bar{u}(t), \quad \tilde{y}(t) = y(t) - g(\bar{x}(t), \bar{u}(t))$$

and the *time-varying* linearized system:

$$\begin{aligned} \dot{\tilde{x}}(t) &= A(t) \tilde{x}(t) + B(t) \tilde{u}(t), \\ \tilde{y}(t) &= C(t) \tilde{x}(t) + D(t) \tilde{u}(t), \end{aligned} \tag{5.12}$$

where $(A(t), B(t), C(t), D(t))$ are computed as in (5.9). Note the potential dependence on time due to the evaluation at $(\bar{x}(t), \bar{u}(t))$. Stability of time-varying linear systems is a much more complicated subject which is out of the scope of this book. See also the discussion in § 3.6.

A familiar example in which equilibrium trajectories, as opposed to equilibrium points, arise naturally is a simple mass subject to Newton's *second* law, $m\ddot{y} = u$, where u is an external force, which we write in state-space form:

$$\dot{x} = f(x, u), \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dot{y} \\ y \end{pmatrix}, \quad f(x, u) = \begin{pmatrix} (1/m)u \\ x_1 \end{pmatrix}.$$

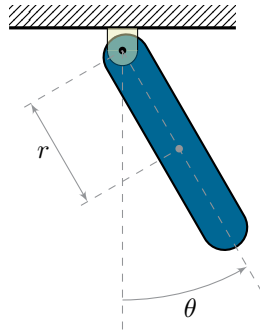


Fig. 5.11: Simple pendulum

Equilibrium trajectories in which the external force is zero, $\bar{u} = 0$, satisfy:

$$\begin{aligned} \dot{\bar{x}}(t) = f(\bar{x}(t), 0) &\implies \\ \dot{\bar{x}}_1 = 0, \quad \dot{\bar{x}}_2 = \bar{x}_1(t) &\implies \\ \bar{x}_1(t) = v, \quad \bar{x}_2(t) = \bar{x}(0) + vt, &\quad (5.13) \end{aligned}$$

which is the familiar statement that a mass in equilibrium will be at rest or at constant velocity, that is Newton's *first* law. See § 5.6 for another example.

In the next sections we will illustrate how to obtain linearized systems from nonlinear models through a series of simple examples.

5.4 Simple Pendulum

Consider the simple planar pendulum depicted in Fig. 5.11. The equations of motion of the simple pendulum obtained from Newton's law in terms of the pendulum's angle θ is the second-order nonlinear differential equation

$$J_r \ddot{\theta} + b \dot{\theta} + m g r \sin \theta = u$$

where

$$J_r = J + m r^2 > 0, \quad (5.14)$$

and J is the pendulum's moment of inertia about its center of mass, m is the pendulum's mass, b is the (viscous) friction coefficient, and r is the distance to the pendulum's center of mass. For example, if the pendulum is a uniform cylindrical rod of length ℓ then $r = \ell/2$ and the moment of inertia about its center of mass¹¹ is $J = m \ell^2/12$. The input u is a torque, which is applied by a motor mounted on the axis of the pendulum. We assume that the motor is

¹¹In this case the quantity $J_r = J + m r^2 = m \ell^2/12 + m \ell^2/4 = m \ell^2/3$ which is the rod's moment of inertia about one of its ends.

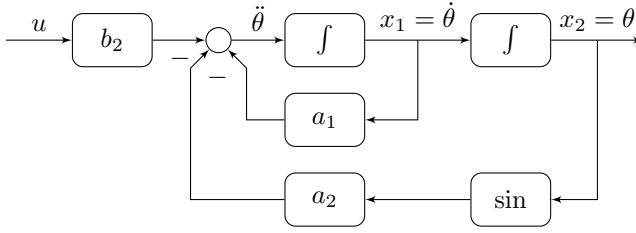


Fig. 5.12: Block diagram for a simple pendulum

attached in such a way that the pendulum can rotate freely and possibly complete multiple turns around its axis. The motor will not be modeled¹².

We isolate the highest derivative to write:

$$\ddot{\theta} = b_2 u - a_1 \dot{\theta} - a_2 \sin \theta,$$

$$a_1 = \frac{b}{J_r} \geq 0, \quad a_2 = \frac{m g r}{J_r} > 0, \quad b_2 = \frac{1}{J_r} > 0.$$

This differential equation is represented as a block diagram in Fig. 5.12, which is a version of Fig. 5.2 modified to accommodate for the nonlinear block ‘sin’. Following § 5.2, we represent the pendulum’s nonlinear equation of motion in state-space form (5.8) after defining:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \theta \end{pmatrix}, \quad f(x, u) = \begin{pmatrix} b_2 u - a_1 x_1 - a_2 \sin x_2 \\ x_1 \end{pmatrix}, \quad g(x, u) = x_2.$$

We set $\bar{u} = 0$ and look for equilibrium points by solving the system of equations:

$$f(\bar{x}, \bar{u}) = \begin{pmatrix} -a_1 \bar{x}_1 - a_2 \sin \bar{x}_2 \\ \bar{x}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solutions must satisfy $\bar{x}_1 = \sin \bar{x}_2 = 0$ or, in other words:

$$\bar{x}_1 = 0 \quad \text{and} \quad \bar{x}_2 = k\pi, \quad k \in \mathbb{Z}.$$

We compute derivatives (5.9)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -a_1 & -a_2 \cos x_2 \\ 1 & 0 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} b_2 \\ 0 \end{bmatrix}, \quad \frac{\partial g}{\partial x} = [0 \quad 1], \quad \frac{\partial g}{\partial u} = [0].$$

which we first evaluate at

$$(\bar{x}, \bar{u}) = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0 \right)$$

¹²See **P2.16** and **P3.25** for a model of a DC motor.

to obtain the linearized system (5.11) with matrices:

$$A^0 = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, \quad B^0 = \begin{bmatrix} b_2 \\ 0 \end{bmatrix}, \quad C^0 = [0 \quad 1], \quad D^0 = [0].$$

When $a_1 > 0$ and $a_2 > 0$ the matrix A is Hurwitz¹³. The transfer-function associated with the linearized system is

$$\begin{aligned} G^0(s) &= C^0(sI - A^0)^{-1}B^0 + D^0 \\ &= \frac{1}{s^2 + a_1s + a_2} [0 \quad 1] \begin{bmatrix} s & -a_2 \\ 1 & s + a_1 \end{bmatrix} \begin{bmatrix} b_2 \\ 0 \end{bmatrix} = \frac{b_2}{s^2 + a_1s + a_2} \end{aligned}$$

where (5.6) was used in place of $(sI - A^0)^{-1}$. Because A^0 is Hurwitz, $G^0(s)$ is asymptotically stable and Lemma 5.1 guarantees that trajectories starting close enough to $\theta = 0$ and with small enough velocity will converge to 0, as we would expect from a physical pendulum. We expect that a physical pendulum will eventually converge to $\theta = 0$ no matter the initial conditions, since the energy available due to the initial conditions will eventually dissipate due to friction. This is also true in the simple pendulum model when $b > 0$. The idea to use energy or a related positive measure of the state of a system in order to assess stability is the main idea behind Lyapunov functions, which are widely used with linear and nonlinear systems. Lyapunov stability is discussed in detail in standard nonlinear systems and control references, e.g. [Kha96].

Linearization around the next equilibrium point

$$(\bar{x}, \bar{u}) = \left(\begin{pmatrix} 0 \\ \pi \end{pmatrix}, 0 \right)$$

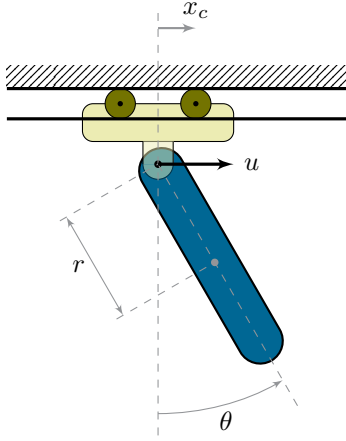
produces the linearized system (5.11) with matrices:

$$A^\pi = \begin{bmatrix} -a_1 & a_2 \\ 1 & 0 \end{bmatrix}, \quad B^\pi = \begin{bmatrix} b_2 \\ 0 \end{bmatrix}, \quad C^\pi = [0 \quad 1], \quad D^\pi = [0].$$

This time however, matrix A is never Hurwitz¹⁴. The transfer-function associated with the linearized system is

$$G^\pi(s) = C^\pi(sI - A^\pi)^{-1}B^\pi + D^\pi = \frac{b_2}{s^2 + a_1s - a_2}$$

which is not asymptotically stable. According to Lemma 5.1, trajectories starting close enough to $\theta = \pi$ will diverge from π , again as we would expect from a physical pendulum. All other equilibria in which $\bar{x}_2$ is a higher integer multiple of π will lead to one of the above linearized systems.



(a) Pendulum suspended in a rail



(b) Inverted pendulum in a cart

Fig. 5.13: Pendulum in carts

5.5 Pendulum in a Cart

We now complicate the pendulum by attaching it to a cart that can move only in the x_c direction, as shown in Fig. 5.13a. In Fig 5.13a the cart is on a rail, which would be the case, for example, in a crane. The same model can be used to describe the *inverted* pendulum in a cart shown in Fig. 5.13b. Without delving into the details of the derivation, the equations of motion for the pendulum are the pair of coupled nonlinear second-order differential equations:

$$\begin{aligned} (J_p + m_p r^2) \ddot{\theta} + m_p r \ddot{x}_c \cos \theta + b_p \dot{\theta} + m_p g r \sin \theta &= 0 \\ m_p r \ddot{\theta} \cos \theta + (m_p + m_c) \ddot{x}_c + b_c \dot{x}_c - m_p r \dot{\theta}^2 \sin \theta &= u \end{aligned} \quad (5.15)$$

where θ is the pendulum's angle and x_c is the cart's position, shown in the diagram in Fig. 5.13a. The positive constants J_p , m_p and b_p are the pendulum's moment of inertia, mass, and viscous friction coefficient, r is the distance to the pendulum's center of mass, m_c and b_c are the cart's mass and viscous friction coefficient. An important difference is that the input, u , is a force applied to the cart, as opposed to a torque applied to the pendulum. The goal being to equilibrate the pendulum by moving the cart, as done in the Segway[®] Personal Transporter shown in Fig. 5.13b.

When $\ddot{x}_c = 0$, the first equation in (5.15) reduces to the equation of motion of the simple pendulum developed in § 5.4 but with a zero input torque. Note

¹³The eigenvalues of A are the roots of the characteristic equation $0 = |sI - A^0| = s^2 + a_1 s + a_2$ which is similar to equation (4.11) studied in § 4.2. See also § 6.1.

¹⁴The characteristic equation $0 = |sI - A^\pi| = s^2 + a_1 s - a_2$ will always have a root with positive real part. See § 6.1.

that the term ‘ $-m_p r \ddot{x}_c \cos \theta$ ’ is effectively a torque, applied to the pendulum by virtue of accelerating the cart in the x_c direction. As expected, a positive acceleration produces a negative torque. Check that in Fig. 5.13a!

As an intermediate step toward a state-space model, these equations can be cast into the second-order *vector* differential equation:

$$M(q) \ddot{q} + F(q, \dot{q}) = Gu, \quad q = \begin{pmatrix} \theta \\ x_c \end{pmatrix}$$

where q is the *configuration* vector and

$$M(q) = \begin{bmatrix} J_p + m_p r^2 & m_p r \cos q_1 \\ m_p r \cos q_1 & m_p + m_c \end{bmatrix}, \quad F(q, \dot{q}) = \begin{pmatrix} b_p \dot{q}_1 + m_p g r \sin q_1 \\ b_c \dot{q}_2 - m_p r \dot{q}_1^2 \sin q_1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix $M(q)$ is known as the *mass matrix*. A vector second-order system is the standard form for models of mechanical and electrical systems derived from physical principles, such as the present model of the pendulum in a cart. For instance, a finite-element modeler of mechanical systems will produce models that conform to the above vector second-order form.

One can go from vector second-order to state-space by defining:

$$x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}, \quad f(x, u) = f(q, \dot{q}, u) = \begin{pmatrix} \dot{q} \\ M^{-1}(q) [Gu - F(q, \dot{q})] \end{pmatrix}.$$

When moving from vector second-order to state-space one needs to *invert* the mass matrix, which may not always be a trivial task. Setting $\bar{u} = 0$ we calculate the equilibrium points:

$$\bar{x}_1 = \bar{q}_1 = k\pi, \quad k \in \mathbb{Z}, \quad \bar{x}_3 = \bar{x}_4 = \bar{\dot{q}}_1 = \bar{\dot{q}}_2 = 0.$$

Note that the constant $x_2 = q_2 = x_c$ is arbitrary, which means that equilibrium of the pendulum does not depend on a particular value of x_c . In the case of the inverted pendulum, this is due to the fact that the coordinate $q_2 = x_c$ does not appear directly in the equations of motion. This is analogous to what happens in the car model (2.1), and indicates the existence of equilibrium trajectories such as (5.13), in which the velocity $\dot{q}_2 = \dot{x}_c$ is constant as opposed to zero. Indeed, for the pendulum in a cart, $M(q) = M(q_1) = M(\theta)$ and $F(q, \dot{q}) = F(q_1, \dot{q}) = F(\theta, \dot{\theta}, \dot{x}_c)$ so that a reduced state-space model is possible with:

$$x = \begin{pmatrix} \theta \\ \dot{\theta} \\ \dot{x}_c \end{pmatrix}, \quad f(x, u) = \begin{pmatrix} \dot{x}_1 \\ M^{-1}(x_1) [Gu - F(x)] \end{pmatrix}. \quad (5.16)$$

Equilibrium points for (5.16) are:

$$\bar{x}_1 = \bar{q}_1 = k\pi, \quad k \in \mathbb{Z}, \quad \bar{x}_2 = \bar{x}_3 = \bar{\dot{q}}_1 = \bar{\dot{q}}_2 = 0.$$

Linearized matrices (5.9) can be computed from (5.16) when $\bar{x}_1 = 0$:

$$A^0 = \frac{1}{J} \begin{bmatrix} 0 & J & 0 \\ -g m_r r & -b_p m_r / m_p & b_c r \\ g m_p r^2 & b_p r & -b_c J_r / m_p \end{bmatrix}, \quad B^0 = \frac{1}{J} \begin{bmatrix} 0 \\ -r \\ J_r / m_p \end{bmatrix},$$

or when $\bar{x}_1 = \pi$:

$$A^\pi = \frac{1}{J} \begin{bmatrix} 0 & J & 0 \\ g m_r r & -b_p m_r / m_p & -b_c r \\ g m_p r^2 & -b_p r & -b_c J_r / m_p \end{bmatrix}, \quad B^\pi = \frac{1}{J} \begin{bmatrix} 0 \\ r \\ J_r / m_p \end{bmatrix},$$

where we defined and used the positive quantities

$$J_r = J_p + m_p r^2, \quad m_r = m_p + m_t, \quad J = J_r \frac{m_r}{m_p} - m_p r^2,$$

to simplify the entries in the matrices. Both sets of linearized matrices are very similar except for a couple of sign changes. However, these small changes are fundamental for understanding the behavior of the system around each equilibrium.

For example, consider that damping is zero, $b_c = b_t = 0$, which is simpler to analyze. In this case the eigenvalues of A^0 are

$$0, \quad j\sqrt{g(m_t + m_t)r}, \quad -j\sqrt{g(m_t + m_t)r},$$

where the imaginary eigenvalues are indicative of an oscillatory system. Indeed when the damping coefficients b_c and b_p are positive, all eigenvalues of A^0 have negative real part (verify) and from Lemma 5.1, the equilibrium point $\theta = 0$ is asymptotically stable.

If $b_c = b_t = 0$ then the eigenvalues of A^π are all real

$$0, \quad \sqrt{g(m_t + m_t)r}, \quad -\sqrt{g(m_t + m_t)r},$$

and one of them always have positive real part. When the damping coefficients b_c and b_p are positive, two of the eigenvalues of A^π have negative real part but one remains on the right-hand side of the complex plane. From Lemma 5.1, the equilibrium point $\theta = \pi$ is unstable.

5.6 Car Steering

Our third example is that of a simplified 4-wheel vehicle traveling as depicted in Fig. 5.14. Without any slip, the wheels of the car remain tangent to circles centered at the virtual point c , as shown in the figure. A real car uses a more complicated steering mechanism to traverse the same geometry shown in Fig. 5.14: only the front wheels move, as opposed to the front axle, and real

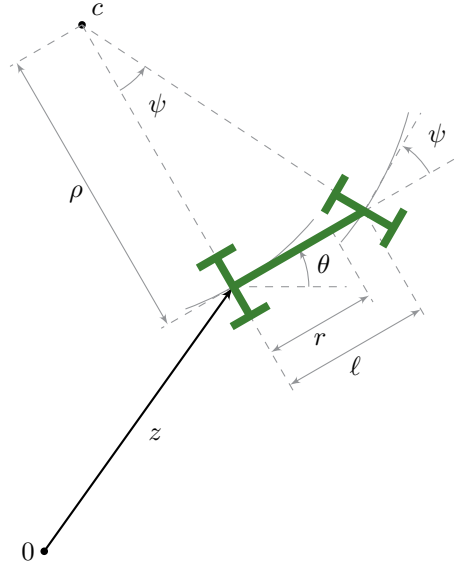


Fig. 5.14: Schematic of a simplified car steering in the plane; the car is turning around a virtual circle of radius $\rho = \ell / \tan \psi$ centered at c

tires allow some *slip* to occur. The front axle steering angle, ψ , is related to the radius of the circle that goes by the mid-point of the rear axle by the formula

$$\rho = \frac{\ell}{\tan \psi}.$$

We assume that the steering angle ψ is in the interval $(-\pi/2, \pi/2)$. If v is the rear axle's mid-point tangential velocity, then v is related to the car's angular velocity, $\dot{\theta}$, by

$$v = \rho \dot{\theta}.$$

If $z = (z_x, z_y)$ is the position of the mid-point of the rear axle then the velocity vector $\dot{r}$ is

$$\dot{z}_x = v \cos \theta, \quad \dot{z}_y = v \sin \theta.$$

We can put these equations together in state-space form (5.8) with:

$$x = \begin{pmatrix} z_x \\ z_y \\ \theta \end{pmatrix}, \quad u = \tan \psi, \quad f(x, u) = \begin{pmatrix} v \cos \theta \\ v \sin \theta \\ u v / \ell \end{pmatrix}.$$

Note that we chose as input $u = \tan \psi$ rather than $u = \psi$. Therefore, by inverting u , we obtain $\psi = \tan^{-1} u \in (-\pi/2, \pi/2)$, which automatically enforces that ψ is always within its permissible range.

When $v \neq 0$, the car does not have any equilibrium points because there exists no $\bar{\theta}$ such that $\cos \bar{\theta} = \sin \bar{\theta} = 0$. For this reason we linearize around a moving trajectory. For example, a straight horizontal line

$$\bar{x}(t) = \begin{pmatrix} z_x(t) \\ z_y(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} vt \\ 0 \\ 0 \end{pmatrix}, \quad \bar{u}(t) = 0 \quad (5.17)$$

satisfies $\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t))$ and hence is an equilibrium trajectory. We compute derivatives to obtain

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & -v \sin \theta \\ 0 & 0 & v \cos \theta \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ v/\ell \end{bmatrix}$$

which are evaluated to compute the matrices of the linearized system (5.12):

$$A(t) = A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix}, \quad B(t) = B = \begin{bmatrix} 0 \\ 0 \\ v/\ell \end{bmatrix}.$$

In this case, the matrices $A(t)$ and $B(t)$ happen to not depend on the time t , and hence the linearized system is in fact time-invariant. See the footnote¹⁵ for an example of a trajectory that leads to a *time-varying* linearized system. The model discussed in this section does not take into account the inertial properties of the car and, for this reason, is known as a *kinematic* model. A full dynamic nonlinear model is obtained with the addition of the equation:

$$\dot{v} = \frac{\ell^2 \dot{f} - b \ell^2 v - J_r v u \dot{u}}{m \ell^2 + J_r u^2}, \quad J_r = J + m r^2$$

where m and J are the car's mass and moment of inertia, r is the distance measured along the car main axis from the mid-point of the rear axle to the

¹⁵ Consider a trajectory in which $\bar{u}(t) = \bar{u} \neq 0$. An equilibrium trajectory satisfies:

$$\dot{\bar{x}}(t) = \begin{pmatrix} \dot{z}_x(t) \\ \dot{z}_y(t) \\ \dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} v \cos \bar{\theta}(t) \\ v \sin \bar{\theta}(t) \\ \bar{u} v/\ell \end{pmatrix} = f(\bar{x}(t), \bar{u}) \implies \bar{\theta}(t) = \bar{\theta}(0) + \omega t, \quad \omega = \bar{u} v/\ell.$$

With $\bar{\theta}(0) = 0$,

$$\bar{x}(t) = \begin{pmatrix} z_x(t) \\ z_y(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \bar{z}_x(0) + (v/\omega) \sin(\omega t) \\ \bar{z}_y(0) - (v/\omega) \cos(\omega t) \\ \omega t \end{pmatrix}, \quad \bar{u}(t) = \bar{u}.$$

which are circles centered at $(\bar{z}_x(0), \bar{z}_y(0))$ with radius $\rho = v/\omega = \ell/\bar{u}$. This time however

$$A(t) = \begin{bmatrix} 0 & 0 & -v \sin(\omega t) \\ 0 & 0 & v \cos(\omega t) \\ 0 & 0 & 0 \end{bmatrix}, \quad B(t) = B = \begin{bmatrix} 0 \\ 0 \\ v/\ell \end{bmatrix},$$

and the linearized system is therefore time-varying.

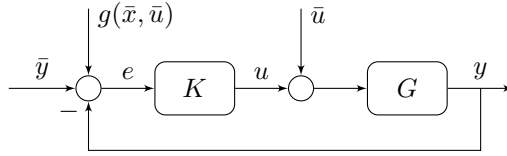


Fig. 5.15: Closed-loop feedback configuration with reference, $\bar{y}$, reference offset, $g(\bar{x}, \bar{u})$, input offset, $\bar{u}$

car's center of mass, f is a tangential force applied at the rear axle, and b is a (viscous) damping coefficient. Note that when u is small or J_r is small then

$$\dot{v} \approx \frac{1}{m}f - \frac{b}{m}v$$

which is the same equation as (2.1) used before to model a car moving in a straight line.

5.7 Linear Control of Nonlinear Systems

Suppose that one designs a linear controller based on a model, $\tilde{G}$, linearized around the equilibrium point $(\bar{x}, \bar{u})$ of a certain nonlinear system, G . Two questions need to be answered: a) what is the exact form of the controller? b) under what conditions will this controller stabilize the original nonlinear system?

To answer the first question, observe that the linearized system (5.11) is developed in terms of deviations (5.10) and the signals on a feedback controller designed based on the standard diagram in Fig. 1.8 are:

$$\tilde{u} = K(\bar{y} - \tilde{y}), \quad \tilde{u} = u - \bar{u}, \quad \tilde{y} = y - g(\bar{x}, \bar{u}),$$

The actual control that needs to be applied to the original system, G , is therefore:

$$u = \bar{u} + K[\bar{y} + g(\bar{x}, \bar{u}) - y]. \quad (5.18)$$

In practice, whenever possible, most control systems are linearized about a zero input $\bar{u}$ (can you tell why?), which simplifies the diagram. Moreover, if integral control is used, one can often dispense with the input $\bar{u}$ if it is constant. Compare the block diagram in Fig. 5.15 with the one in Fig. 4.8 and recall the discussion in § 4.4 on how integral control can reject the constant *disturbance*, $\bar{u}$. We will discuss integral control for nonlinear systems at the end of this section.

For a concrete example, consider a linear proportional controller for the nonlinear simple pendulum discussed in § 5.4. We will show in § 6.5 that a static gain K can be selected to stabilize the pendulum around the equilibrium point $(\bar{u}, \bar{\theta}) = (0, 0)$, which is asymptotically stable, or around $(\bar{u}, \bar{\theta}) = (0, \pi)$, which is unstable. Following Fig. 5.15, such controller must be implemented as in Fig. 5.16 after setting $g(\bar{x}, \bar{u}) = \bar{\theta}$ and $\bar{u} = 0$.

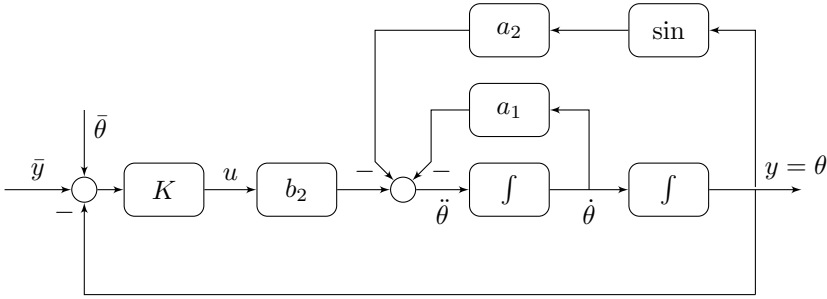


Fig. 5.16: Linear control of the simple pendulum; $\bar{\theta} = 0$ around stable equilibrium, and $\bar{\theta} = \pi$ around unstable equilibrium

One must be specially careful when a linear controller is used to move a nonlinear system away from its natural equilibrium point. If the reference $\bar{y}$ is constant, $K = K(s)$ is a dynamic linear controller, and the nonlinear system, G , is described in state-space by equations (5.8), then the closed-loop equilibrium point must satisfy:

$$f(\hat{x}, \hat{u}) = 0, \quad \hat{u} = \bar{u} + K(0)(\bar{y} + g(\bar{x}, \bar{u}) - \hat{y}), \quad \hat{y} = g(\hat{x}, \hat{u}) \quad (5.19)$$

where $K(0)$ is the constant steady-state gain of the controller¹⁶. This closed-loop equilibrium point, $\hat{x}$, is not necessarily equal to the open-loop equilibrium, $\bar{x}$. When the reference input is zero, $\bar{y} = 0$, the controller is a regulator (see § 4.4). From Lemma 5.1, we know that a regulator designed to stabilize a model linearized at $(\bar{x}, \bar{u})$ also stabilizes the original nonlinear system in a neighborhood of the equilibrium point $(\bar{x}, \bar{u})$. If the initial conditions of the closed-loop system are close enough to $\bar{x}$, then we should expect that x converges to $\bar{x}$, u converges to $\bar{u}$, and y converges to $y = g(\bar{x}, \bar{u})$ so that $\hat{x} = \bar{x}$ and $\hat{u} = \bar{u}$. This is a partial answer to question b).

An interesting case is that of a controller which is designed to satisfy

$$K(0) = 0, \quad (5.20)$$

which means that the controller has a zero at the origin. This effect is often obtained with the addition of a high-pass or *washout filter* of the form

$$\frac{s}{s + a}, \quad a > 0$$

to the controller. Washout filters are used in applications where the controller should have no authority on the system's steady-state response but need to act during transients. A typical application is in the control of electrical power systems, in which the steady-state response is dictated by the loads of the circuit

¹⁶This is the case even if $K(s)$ is not asymptotically stable, as long as the closed-loop is internally stable § 4.6. A formal analysis is in § 8.2.

and should not be affected by the controller [PSGL96]. Another application is in flight control. See [FPEN09, § 10.3] for a complete design of a yaw damper for a Boeing 747 aircraft which uses a washout filter to preserve pilot authority in closed-loop.

When the controller is not a regulator, that is $\bar{y} \neq 0$, the closed-loop controller will typically modify the closed-loop equilibrium point. Tracking is all about modifying the natural (open-loop) equilibrium of systems. In light of Lemma 5.1, a necessary condition for stability of the closed-loop system when $\bar{y}$ is a constant is that the controller K stabilizes the closed-loop system linearized at the new equilibrium point $(\hat{x}, \hat{u})$, given in (5.19). If $\bar{y}$ is close to $g(\bar{x}, \bar{u})$ and the nonlinearities in f and g are *mild*, then one can *hope* that stabilization of the system linearized at $(\bar{x}, \bar{u})$ might also imply stabilization of the system linearized at $(\hat{x}, \hat{u})$, but there are no guarantees that can be offered in all cases. As with linear systems, there will likely be a non-zero steady-state error, i.e. $\bar{y} \neq \hat{y}$. In many cases, one can enforce a zero closed-loop steady-state error using integral control. With an integrator in the loop, one often dispenses with the inputs $g(\bar{x}, \bar{u})$ and $\bar{u}$ in the closed-loop diagram of Fig. 5.15, which reduces the controller to the standard block diagram in Fig. 1.8:

$$u = K(\bar{y} - y).$$

An informal¹⁷¹⁸ argument to support integral control in nonlinear systems is

¹⁷One problem with this argument is that there might not exist $(\hat{x}, \hat{u})$ such that

$$f(\hat{x}, \hat{u}) = 0, \quad \bar{y} = g(\hat{x}, \hat{u}).$$

For example, consider the open-loop stable first-order linear system with an input saturation

$$\dot{x} = f(x, u) = -x + \tan^{-1}(u), \quad y = g(x, u) = x.$$

In equilibrium $g(\bar{x}, \bar{u}) = \bar{x} = \tan^{-1}(\bar{u}) \in (-\pi/2, \pi/2)$, hence no controller can drive the system to produce a constant output that is equal to a reference $\bar{y}$ such that $|\bar{y}| > \pi/2$.

¹⁸A more subtle obstacle is that even when there exists an equilibrium, it might not be reachable. Consider for example an integral controller in feedback with the nonlinear system:

$$\dot{y} = \frac{1}{y-1} + u, \quad \dot{u} = \bar{y} - y, \quad \bar{y} = 2.$$

In state-space:

$$\dot{x} = f(x), \quad x = \begin{pmatrix} y \\ u \end{pmatrix}, \quad f(x) = \begin{pmatrix} 1/(x_1 - 1) + x_2 \\ 2 - x_1 \end{pmatrix}, \quad g(x) = x_1,$$

which has an equilibrium point at $(\bar{x}_1, \bar{x}_2) = (2, -1)$. Because

$$A = \left. \frac{\partial f}{\partial x} \right|_{\substack{x_1=2 \\ x_2=-1}} = \begin{bmatrix} -\frac{1}{(x_1-1)^2} & 1 \\ -1 & 0 \end{bmatrix}_{\substack{x_1=2 \\ x_2=-1}} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

is Hurwitz this equilibrium point is asymptotically stable. For instance, with initial conditions $(x_1(0), x_2(0)) = (3, 0)$ the closed-loop system converges to $\bar{y} = \bar{x}_1 = 2$ at t grows. However, with initial condition $(x_1(0), x_2(0)) = (0, 0)$, the system never reaches this equilibrium. As $x_1(t)$ tries to grow continuously toward 2, the value of $x_2(t)$ grows linearly but the value of $1/(x_1 - 1)$ becomes more and more negative as $x_1(t)$ approaches 1. Indeed, $x_1(t)$ can never exceed one, so it never reaches equilibrium.

as follows: when $\bar{y}$ is constant and K contains an integrator, $K(0) \rightarrow \infty$ and therefore one must have $\bar{y} \rightarrow y$ if the closed-loop system is to reach equilibrium. If the integrator is the first element in the controller, that is

$$x_{k1}(t) = x_{k1}(0) + \int_0^t \bar{y} - y(\tau) d\tau,$$

where x_{k1} is the first state of the controller, we have:

$$\dot{x}_{k1}(t) = \bar{y} - y(t).$$

If the closed loop system is in equilibrium with a constant $\bar{y}$ and constant $\bar{x}_{k1}$, then $\hat{y} = \bar{y}$. If the corresponding equilibrium point, $(\hat{x}, \hat{y})$, is asymptotically stable, then

$$\lim_{t \rightarrow \infty} x(t) = \hat{x}, \quad \lim_{t \rightarrow \infty} u(t) = \hat{u}, \quad \lim_{t \rightarrow \infty} y(t) = \hat{y} = \bar{y},$$

given that the initial conditions are close enough to $(\hat{x}, \hat{u})$. As with linear systems, the price to be paid is a more complex dynamic system to contend with. On the other hand, there is solace in the fact that $g(\bar{x}, \bar{u})$ nor $\bar{u}$ need to be accurately estimated. In the case of the simple pendulum, a formal analysis of the effectiveness of integral control will be provided in § 8.2.

Problems

P5.1. Write state-space equations for the block diagrams in Fig. 5.17 and in Fig. 5.18 and compute the transfer-function from u to y .

P5.2. You have shown in **P2.1** that the ordinary differential equation

$$m \dot{v} + b v = m g$$

is a simplified description of the motion of an object of mass m dropping vertically under constant gravitational acceleration, g , and linear air resistance, $-b v$. Let the gravitational force, mg , be the input and the vertical velocity, v , be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.3. Repeat **P5.2** considering the vertical position, $x = \int_0^t v(\tau) d\tau$, as the output.

P5.4. Repeat **P5.2** considering the vertical acceleration, $\dot{v}$, as the output. *Hint: Use the original equation to obtain $\dot{v}$ as a function of v .*

P5.5. The ordinary differential equation

$$m \dot{v} + b v|v| = m g$$

is a simplified description of the motion of an object of mass m dropping vertically under constant gravitational acceleration, g , and quadratic

air resistance, $-b v|v|$. Let the gravitational force, mg , be the input and the vertical velocity, v , be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.6. Use the block-diagram obtained in **P5.5** to simulate the velocity of an object with $m = 70\text{kg}$, $b = 0.0175\text{kg/m}$, and $g = 10\text{m/s}^2$ falling with zero initial velocity. Compare your solution with the one given in **P2.5**.

P5.7. Use **P5.5** to redo **P2.4**.

P5.8. Calculate the equilibrium points of the state-space representation obtained in **P5.5**. Linearize the state-space equations about the equilibrium points and compute the corresponding transfer-functions. Are the equilibrium points asymptotically stable?

P5.9. You have shown in **P2.7** that the ordinary differential equation

$$(J_1 r_2^2 + J_2 r_1^2) \dot{\omega}_1 = r_2^2 \tau, \\ \omega_2 = (r_1/r_2) \omega_1$$

is a simplified description of the motion of a rotating machine driven by a belt without slip as in Fig. 2.18, where ω_1 is the angular velocity of the driving shaft and ω_2 is the machine's angular velocity. Let the torque, τ , be the input and the machine's angular velocity, ω_2 , be the output

Fig. 5.17: Block diagrams for **P5.1**

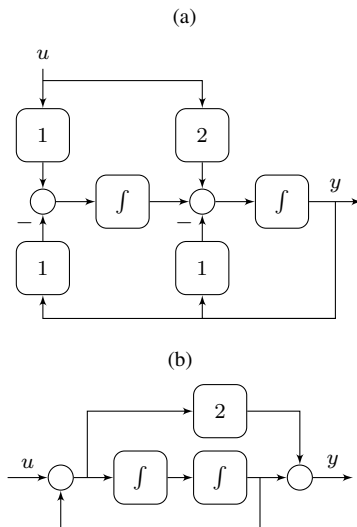
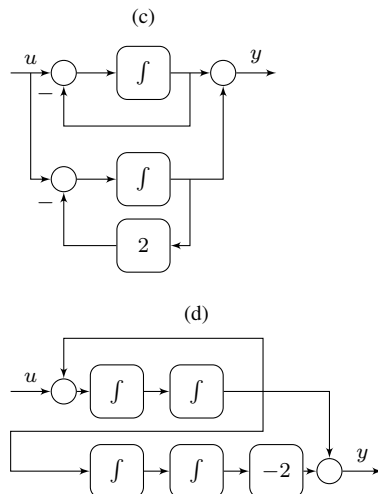


Fig. 5.18: Block diagrams for **P5.1**



and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.10. Repeat **P5.9** considering the machine's angle, $\theta_2 = \int_0^t \omega_2(\tau) d\tau$, as the output.

P5.11. Repeat **P5.9** considering the machine's angular acceleration, $\dot{\omega}_2$, as the output. *Hint: Use the original equation to obtain $\dot{\omega}_2$ as a function of ω_1 .*

P5.12. You have shown in **P2.10** that the ordinary differential equation

$$\begin{aligned} J \dot{\omega} + (b_1 + b_2) \omega &= \tau + g r (m_1 - m_2), \\ J &= J_1 + J_2 + r^2 (m_1 + m_2), \\ v_1 &= r \omega, \end{aligned}$$

is a simplified description of the motion of the elevator in Fig. 2.19, where ω is the angular velocity of the driving shaft and v_1 is the elevator's load linear velocity. Let the torque, τ , and the gravitational torque, $g r (m_1 - m_2)$, be inputs and the elevator's linear velocity, v_1 , be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.13. Compute the transfer-function from the two inputs to the single output in **P5.12**. The transfer-function will be a 1×2 matrix. *Hint: Use the state-space equations and equation (5.6).*

P5.14. You have shown in **P2.16** that the ordinary differential equation

$$J \dot{\omega} + \left(b + \frac{K_t K_e}{R_a} \right) \omega = \frac{K_t}{R_a} v_a$$

is a simplified description of the motion of the rotor of the DC motor in Fig. 2.20, where ω is the rotor angular velocity. Let the armature voltage, v_a , be the input and the angular velocity, ω , be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.15. Repeat **P5.14** considering the rotor's angle, $\theta_2 = \int_0^t \omega_2(\tau) d\tau$, as the output.

P5.16. Recall from **P2.16** that the rotor torque:

$$\tau = K_t i_a.$$

The armature current, i_a , is related to the armature voltage, v_a , and the rotor angular velocity, ω , through

$$v_a = R_a i_a + K_e \omega.$$

Repeat **P5.14** considering the rotor torque as the output. At what (constant) angular velocity the motor attains its highest torque?

P5.17. You have shown in **P2.31** that the ordinary differential equation

$$R C_2 \dot{v}_o + R C_1 \dot{v}_i + v_i = 0.$$

is an approximate model for the electric circuit in Fig. 2.27. Let the input voltage, v_i , be the input and the output voltage, v_o , be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.18. You have shown in **P2.28** that the ordinary differential equations:

$$\begin{aligned} m_1 \ddot{x}_1 + (b_1 + b_2) \dot{x}_1 + \\ (k_1 + k_2) x_1 - b_2 \dot{x}_2 - k_2 x_2 &= 0, \\ m_2 \ddot{x}_2 + b_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) &= f_2 \end{aligned}$$

constitute a simplified description of the motion of the mass-spring-damper system in Fig. 2.25 where x_1 and x_2 are displacements and f_2 is a force applied on the mass m_2 . Let the force, f_2 , be the input and the displacement, x_2 , be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equations in state-space form.

P5.19. Let $m_1 = m_2 = 1\text{kg}$, $b_1 = b_2 = 0.1\text{kg/s}$, $k_1 = 1\text{N/m}$, $k_2 = 2\text{N/m}$. Use MATLAB to compute the transfer-function from the force f_2 to the displacement x_2 . Is this system asymptotically stable?

P5.20. The equations of motion of a rigid-body with principal moments of inertia J_1 , J_2 and J_3 is given by Euler's equations:

$$\begin{aligned} J_1 \dot{\omega}_1 + \omega_2 \omega_3 (J_3 - J_2) &= 0, \\ J_2 \dot{\omega}_2 + \omega_1 \omega_3 (J_1 - J_3) &= 0, \\ J_3 \dot{\omega}_3 + \omega_1 \omega_2 (J_2 - J_1) &= 0, \end{aligned}$$

where the angular velocity vector

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

is the state vector. Verify that

$$\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3) = (\Omega, 0, 0)$$

is an equilibrium point. Linearize the equations about this equilibrium point. Show that if $J_2 < J_1 < J_3$ or $J_3 < J_1 < J_2$ then this is an unstable equilibrium point. Interpret this result.

P5.21. Compute the linearized equations for the pendulum in a cart model developed in § 5.5. Let $m_p = 2\text{kg}$, $m_c = 10\text{kg}$, $\ell = 1\text{m}$, $b_p = 0.01\text{kg m}^2/\text{s}$, $b_c = 0.1\text{km/s}$, $r = \ell/2$, $J_p = m\ell^2/12$, and use MATLAB to compute the transfer-function from u to θ and from u to $\dot{x}_c$ around the equilibrium points calculated with $\theta = 0$ and $\theta = \pi$ and $\ddot{u} = r$. Are the equilibria asymptotically stable?

P5.22. We have shown in § 2.8 that the water level, h , in a rectangular water tank of cross-section area A can be modeled as the integrator:

$$\dot{h} = \frac{1}{A} w_{\text{in}}$$

where w_{in} is the in-flow rate. If water is allowed to flow out from the bottom of the tank through an orifice then

$$\dot{h} = \frac{1}{A} (w_{\text{in}} - w_{\text{out}}).$$

The out-flow rate can be approximated by

$$w_{\text{out}} = \frac{1}{R} (p_t - p_a)^{1/\alpha}$$

where the *resistance* $R > 0$ and the exponent α depend on the shape of the out-flow orifice, p_a is the ambient pressure outside the tank, and

$$p_t = p_a + \rho g h,$$

is the pressure at the water level, ρ is the water density and g is the gravitational acceleration. Combine these equations to write a nonlinear differential equation in state-space relating the water in-flow rate w_{in} with the water tank level, h . Determine a water in-flow rate w_{in} such that the tank system is in equilibrium with a water level $h = \bar{h} > 0$. Linearize the state-space equations about this equilibrium point for $\alpha = 2$ and compute the corresponding transfer-function. Is this equilibrium point asymptotically stable?

P5.23. The exponential model:

$$\dot{x} = r x$$

has been used by Malthus to study population growth. In this context, x is the current population and the growth rate $r = b - m$, where b is the birth rate and m is the mortality rate. Explain the behavior of this model when $b > m$, $b < m$ or $b = m$.

P5.24. Verhulst suggested that the growth rate in the model of **P5.23** often depends on the size of the current population:

$$r = r_0(1 - x/k),$$

which leads to the *logistic model*:

$$\dot{x} = r_0(1 - x/k)x.$$

Calculate the equilibrium points for this model. Linearize about the equilibrium points and classify the equilibria as asymptotically stable or unstable. Represent the equation in a block diagram using integrators and use MATLAB to simulate a population with $r_0 = k = 1$ starting at $x(0) = x_0$ for x_0 equal to 0, 1/2, 1 and 2.

P5.25. The Lotka-Volterra model:

$$\begin{aligned}\dot{x} &= r x - a x y \\ \dot{y} &= e a x y - m y\end{aligned}$$

where r , a , e and m are positive constants is used to model two populations where one of the species is a prey and the other is a predator, e.g. foxes and rabbits. The variable x is the prey population, y is the predator population, r is the intrinsic rate of prey population increase, a is the death rate of prey per predator encounter, e is the efficiency rate of turning preys into predators, and m is the intrinsic predator mortality rate. Calculate the equilibrium points for this model. Linearize about the equilibrium points and classify the equilibria as asymptotically stable or unstable. Represent the equations in a block diagram using integrators and use MATLAB to simulate the predator and prey populations for $r = 0.05$, $a = 0.05$, $m = 0.2$, $e = 0.2$, and with $x(0) = 9$, $y(0) = 1$. Try other initial conditions and comment on your findings.