MAE 280A – Homework #2

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1. Write state-space equations for the block diagrams in Fig. 1. Compute the corresponding linear ordinary differential equation in u and y.

2. Draw a block diagram involving only intergrators and construct the associated state-space realization associated with the linear ordinary differential equations:

   a) $\ddot{y} + ay = \ddot{u}$
   b) $m\ddot{y} - y = u$
   c) $\ddot{y} + \dot{y} = \dot{u}$

3. The equations of motion of a rigid-body with principal moments of inertia J_1, J_2 and J_3 is given by Euler's equations:

$$J_1\dot{\omega}_1 + \omega_2\,\omega_3(J_3 - J_2) = 0,$$
$$J_2\dot{\omega}_2 + \omega_1\,\omega_3(J_1 - J_3) = 0,$$
$$J_3\dot{\omega}_3 + \omega_1\,\omega_2(J_2 - J_1) = 0,$$

   where the angular velocity vector

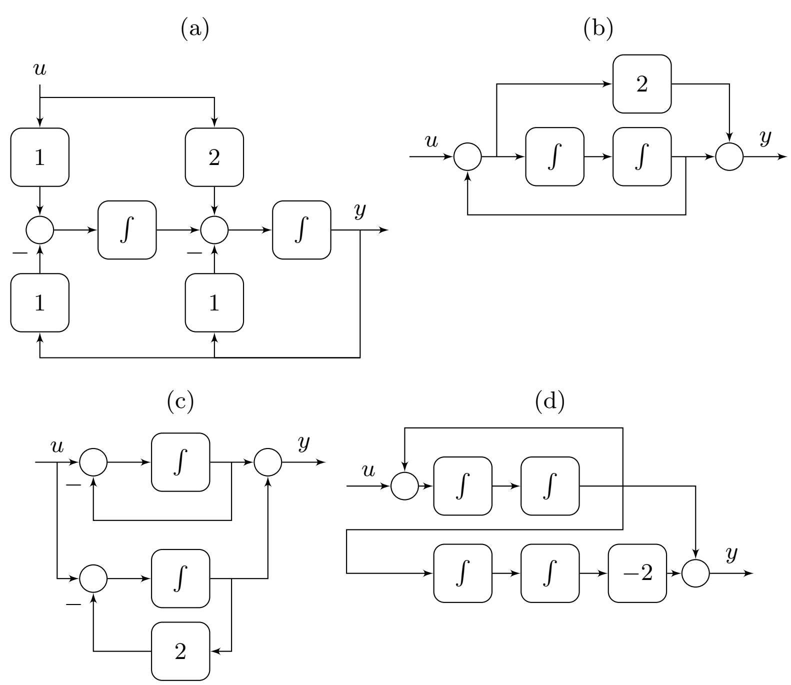
$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

   is the state vector. Verify that

$$\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3) = (\Omega, 0, 0)$$

   is an equilibrium point. Linearize the equations about this equilibrium point.

Figure 1: Block diagrams for problem 1



4. The water level, h, in a rectangular water tank of cross-section area A can be modeled as the integrator:

$$\dot{h} = \frac{1}{A} w_{\text{in}}$$

where w_{in} is the in-flow rate. If water is allowed to flow out from the bottom of the tank through an orifice then

$$\dot{h} = \frac{1}{A} \left(w_{\text{in}} - w_{\text{out}} \right).$$

The out-flow rate can be approximated by

$$w_{\text{out}} = \frac{1}{R} \left(p_{\text{t}} - p_{\text{a}} \right)^{\frac{1}{\alpha}}$$

where the *resistance* $R > 0$ and the exponent α depend on the shape of the out-flow orifice, p_{a} is the ambient pressure outside the tank, and

$$p_{\text{t}} = p_{\text{a}} + \rho\, g\, h,$$

is the pressure at the water level, ρ is the water density and g is the gravitational acceleration. Combine these equations to write a nonlinear ordinary differential equation in state-space relating the water in-flow rate w_{in} with the water tank level, h. Determine a water in-flow rate w_{in} such that the tank system is in equilibrium with a water level $h = \bar{h} > 0$. Linearize the state-space equations about this equilibrium point for $\alpha = 2$.

5. The exponential model:

$$\dot{x} = r\, x$$

has been used by Malthus to study population growth. In this context, x is the current population and the growth rate $r = b - m$, where b is the birth rate and m is the mortality rate. Explain the behavior of this model when $b > m$, $b < m$ or $b = m$.

6. Verhulst suggested that the growth rate in the model of problem 5 often depends on the size of the current population:

$$r = r_0 \left(1 - \frac{x}{k} \right),$$

which leads to the *logistic model*:

$$\dot{x} = r_0 \left(1 - \frac{x}{k} \right) x.$$

Calculate the equilibrium points for this model. Linearize about the equilibrium points. Represent the equation in a block diagram using integrators and use MATLAB to simulate a population with $r_0 = k = 1$ starting at $x(0) = x_0$ for x_0 equal to 0, 1/2, 1 and 2.

7. The Lotka-Volterra model:

$$\dot{x} = r\, x - a\, x\, y$$
$$\dot{y} = e\, a\, x\, y - m\, y$$

where r, a, e and m are positive constants is used to model two populations where one of the species is a prey and the other is a predator, e.g. foxes and rabbits. The variable x is the prey population, y is the predator population, r is the intrinsic rate of prey population increase, a is the death rate of prey per predator encounter, e is the efficiency rate of turning preys into predators, and m is the intrinsic predator mortality rate. Calculate the equilibrium points for this model. Linearize about the equilibrium points. Represent the equations in a block diagram using integrators and use MATLAB to simulate the predator and prey populations for $r = 0.05$, $a = 0.05$, $m = 0.2$, $e = 0.2$, and with $x(0) = 9$, $y(0) = 1$. Try other initial conditions and comment on your findings.