

3 LQR with Noisy Input

Problem:

Compute a state feedback controller

$$u(t) = Kx(t)$$

that stabilizes the closed loop system and minimizes

$$J := \lim_{t \rightarrow \infty} E [x(t)^T Q x(t) + u(t)^T R u(t)]$$

for the LTI system

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_w w(t), \quad x(0) = 0$$

Assumptions:

- a) $Q \succeq 0$, $R \succ 0$;
- b) (A, B_u) stabilizable;
- c) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

A first step toward a solution:

The closed loop cost is

$$J = \lim_{t \rightarrow \infty} E [x(t)^T (Q + K^T R K) x(t)]$$

and the closed system is

$$\dot{x} = (A + B_u K)x + B_w w, \quad x(0) = 0$$

As before, we seek to relate J to a Gramian.

3.1 System Response to a Noisy Input

Problem: Evaluate

$$J := \lim_{t \rightarrow \infty} E [x(t)^T Q x(t)]$$

for the LTI system

$$\dot{x}(t) = Ax(t) + B_w w(t), \quad x(0) = 0$$

where w is a Gaussian white noise vector with zero mean and covariance matrix $W \succ 0$, i.e.,

$$E [w(t)] = 0, \quad E [w(t)w(\tau)^T] = W\delta(t - \tau).$$

Solution:

Recall that trace is linear and that $\text{trace}(AB) = \text{trace}(BA)$ so that

$$x^T x = \text{trace}(x^T x) = \text{trace}(xx^T)$$

and

$$\begin{aligned} J &= \lim_{t \rightarrow \infty} E [x(t)^T Q x(t)], \\ &= \lim_{t \rightarrow \infty} E [\text{trace} (Q x(t) x(t)^T)]. \end{aligned}$$

Furthermore

$$x(t) = \int_0^t e^{A(t-\tau)} B_w w(\tau) d\tau$$

so that

$$\begin{aligned} J &= \lim_{t \rightarrow \infty} E \left[\text{trace} \left(Q \int_0^t e^{A(t-\tau)} B_w w(\tau) d\tau \int_0^t w^T(\sigma) B_w^T e^{A^T(t-\sigma)} d\sigma \right) \right], \\ &= \lim_{t \rightarrow \infty} \text{trace} \left(Q \int_0^t e^{A(t-\tau)} B_w \int_0^t E [w(\tau)w^T(\sigma)] B_w^T e^{A^T(t-\sigma)} d\sigma d\tau \right), \\ &= \lim_{t \rightarrow \infty} \text{trace} \left(Q \int_0^t e^{A(t-\tau)} B_w \int_0^t W\delta(\tau - \sigma) B_w^T e^{A^T(t-\sigma)} d\sigma d\tau \right), \end{aligned}$$

because $w(t)$ is white.

Integrating on σ

$$\begin{aligned} J &= \lim_{t \rightarrow \infty} \text{trace} \left(Q \int_0^t e^{A(t-\tau)} B_w W B_w^T e^{A^T(t-\tau)} d\tau \right), \\ &= \lim_{t \rightarrow \infty} \text{trace} \left(Q \int_0^t e^{A\eta} B_w W B_w^T e^{A^T\eta} d\eta \right), \quad \eta := t - \tau \\ &= \lim_{t \rightarrow \infty} \text{trace} \left(\left[\int_0^t e^{A^T\eta} Q e^{A\eta} d\eta \right] B_w W B_w^T \right), \\ &= \text{trace} \left(\left[\int_0^\infty e^{A^T\eta} Q e^{A\eta} d\eta \right] B_w W B_w^T \right), \\ &= \text{trace} (X B_w W B_w^T), \end{aligned}$$

where X is the solution to the Lyapunov equation

$$A^T X + X A + Q = 0.$$

Back to our problem, the closed loop cost is

$$J = \lim_{t \rightarrow \infty} E [x(t)^T (Q + K^T R K) x(t)]$$

and the closed system is

$$\dot{x} = (A + B_u K)x + B_w w, \quad x(0) = 0$$

Therefore J can be computed as

$$J = \text{trace} (X B_w W B_w^T)$$

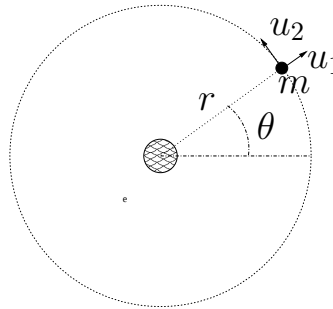
where

$$(A + B_u K)^T X + X(A + B_u K) + Q + K^T R K = 0.$$

As before, the optimal control is again computed as

$$A^T X + X A - X B_u R^{-1} B_u^T X + Q = 0, \quad K = -R^{-1} B_u^T X.$$

3.2 Example: controlling a satellite in circular orbit



Satellite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2 . Continuing...

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= u_1 - \frac{km}{r^2} + w_1, \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= u_2 + w_2, \end{aligned}$$

where w_1 and w_2 are independent white noise disturbances with variances δ_1 and δ_2 .

As before, putting in state space and linearizing around

$$\bar{x}(t) = \begin{pmatrix} \bar{r} \\ \bar{\omega}t \\ 0 \\ \bar{\omega} \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = 0, \quad \bar{w} = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = 0, \quad \bar{\omega} = \sqrt{k/\bar{r}^3}$$

one obtains the linearized system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Problem: Given

$$m = 100 \text{ kg}, \quad \bar{r} = R + 300 \text{ km}, \quad \bar{k} = GM$$

where $G \approx 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2$ is the universal gravitational constant, and $M \approx 5.98 \times 10^{24} \text{ kg}$ and $R \approx 6.37 \times 10^3 \text{ km}$ are the mass and radius of the earth. If the variances δ_1 and δ_2 are $0.1N$ find solutions to the LQR control problem where

$$Q = I, \quad R = \rho I,$$

using u_2 only first, then using u_1 and u_2 .

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% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
% LQR Control - Part I
m = 100          % 100 kg
m =
    100
r = 300E3        % 300 km
r =
    300000
R = 6.37E6       % 6.37 10^3 km
R =
    6370000
G = 6.673E-11 % 6.673 N m^2/kg^2
G =
    6.6730e-11
M = 5.98E24      % 5.98 10^24 kg
M =
    5.9800e+24

% gravitational force constant
k = G * M
k =
    3.9905e+14

% angular velocity (rad/s)
w = sqrt(k/((R+r)^3))
w =
    1.1596e-03

% "ground" velocity (m/s)
v = w * (R + r)
v =
    7.7348e+03

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0]
A =
         0         0    1.0000e+00         0
         0         0         0    1.0000e+00
    4.0343e-06         0         0    1.5470e+04
         0         0   -3.4772e-10         0
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*(r+R))]
Bu =
         0         0
         0         0
    1.0000e-02         0
         0    1.4993e-09
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*(r+R))]
Bw =
         0         0
         0         0
    1.0000e-02         0
         0    1.4993e-09

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% noise variances
W = 0.1 * eye(2)
W =
    1.0000e-01    0
         0    1.0000e-01

% open loop system eigenvalues
eig(A)
ans =
         0
   -8.5533e-58
   2.7105e-20 + 1.1596e-03i
   2.7105e-20 - 1.1596e-03i

% LQR weights
Q = diag([1 1 1 1])
Q =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
R = eye(1)
R =
     1

% Cheapest control (using u2 only)
rho = 1e6
rho =
    1000000
[K,X,S] = lqr(A, Bu(:,2), Q, rho * R);
K = - K
K =
   -1.9375e-03   1.0000e-03   -6.2621e-01   -3.5948e+06
trace(X * Bw * W * Bw')
ans =
    1.6977e+03

% closed loop eigenvalues
Acl = A + Bu(:,2)*K;
eig(Acl)
ans =
   -1.3474e-03 + 2.6060e-03i
   -1.3474e-03 - 2.6060e-03i
   -2.6948e-03
   -2.6079e-10

% Cheapest control (using u1 only)
% WARNING: Recall this can't be done because
% (A,Bu(:,1)) is not controllable
% (in this case also not stabilizable!)
rho = 1
rho =
     1

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[K,X,S] = lqr(A, Bu(:,1), Q, rho * R);
K = - K
K =
    -8.2167e-01    5.7055e-01   -1.4176e+01    5.1248e+08
trace(X * Bw * W * Bw')
ans =
    6.6391e+08

% closed loop eigenvalues
Acl = A + Bu(:,1)*K;
eig(Acl)
ans =
    -7.0882e-02 + 7.0538e-02i
    -7.0882e-02 - 7.0538e-02i
    -1.9839e-10
     3.9580e-18

diary off

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3.3 More on System Response to a Noisy Input

Problem: Evaluate the state mean and asymptotic covariance matrix

$$Y := \lim_{t \rightarrow \infty} E [x(t)x(t)^T]$$

for the LTI system

$$\dot{x}(t) = Ax(t) + B_w w(t), \quad x(0) = 0$$

where w is a Gaussian white noise vector with zero mean and covariance matrix $W \succ 0$, i.e.,

$$E[w(t)] = 0, \quad E[w(t)w(\tau)^T] = W\delta(t - \tau).$$

Solution:

As before

$$x(t) = \int_0^t e^{A(t-\tau)} B_w w(\tau) d\tau$$

For the mean

$$E[x(t)] = E \left[\int_0^t e^{A(t-\tau)} B_w w(\tau) d\tau \right] = \int_0^t e^{A(t-\tau)} B_w E[w(\tau)] d\tau = 0$$

For the variance

$$\begin{aligned} Y &= \lim_{t \rightarrow \infty} E \left[\int_0^t e^{A(t-\tau)} B_w w(\tau) d\tau \int_0^t w^T(\sigma) B_w^T e^{A^T(t-\sigma)} d\sigma \right], \\ &= \lim_{t \rightarrow \infty} \int_0^t e^{A(t-\tau)} B_w \int_0^t E[w(\tau)w^T(\sigma)] B_w^T e^{A^T(t-\sigma)} d\sigma d\tau, \\ &= \lim_{t \rightarrow \infty} \int_0^t e^{A(t-\tau)} B_w \int_0^t W\delta(\tau - \sigma) B_w^T e^{A^T(t-\sigma)} d\sigma d\tau, \\ &= \lim_{t \rightarrow \infty} \int_0^t e^{A(t-\tau)} B_w W B_w^T e^{A^T(t-\tau)} d\tau, \\ &= \lim_{t \rightarrow \infty} \int_0^t e^{A\eta} B_w W B_w^T e^{A^T\eta} d\eta, \quad \eta := t - \tau \end{aligned}$$

where Y is the solution to the Lyapunov equation

$$AY + YA^T + B_w W B_w^T = 0.$$

```

% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% LQR Control - Part II
%
m = 100;                % 100 kg
r = 300E3;              % 300 km
R = 6.37E6;             % 6.37 10^3 km
G = 6.673E-11;          % 6.673 N m^2/kg^2
M = 5.98E24;            % 5.98 10^24 kg
k = G * M;              % gravitational force constant
w = sqrt(k/((R+r)^3)); % angular velocity (rad/s)
v = w * (R + r);        % "ground" velocity (m/s)

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0];
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*(r+R))];
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*(r+R))];

% noise variances
W = 0.1 * eye(2)
W =
    1.0000e-01    0
         0    1.0000e-01

% scale
T = diag([1 r+R 1 r+R])
T =
         1         0         0         0
         0    6670000         0         0
         0         0         1         0
         0         0         0    6670000

% similarity transformation
At = T * A / T
At =
         0         0    1.0000e+00         0
         0         0         0    1.0000e+00
    4.0343e-06         0         0    2.3193e-03
         0         0   -2.3193e-03         0
But = T * Bu
But =
         0         0
         0         0
    1.0000e-02         0
         0    1.0000e-02
Bwt = T * Bw
Bwt =
         0         0
         0         0
    1.0000e-02         0
         0    1.0000e-02

```

```

% LQR weights
Q = diag([1 1 1 1])
Q =
    1    0    0    0
    0    1    0    0
    0    0    1    0
    0    0    0    1
R = eye(1)
R =
    1

% Cheapest control (using u2 only)
rho = 1e6
rho =
    1000000
[Kt,X,S] = lqr(At, But(:,2), Q, rho * R);
Kt = - Kt
Kt =
   -4.0349e-03    1.0000e-03   -1.7280e+00   -7.7560e-01
trace(X * Bwt * W * Bwt')
ans =
    9.9026e+03

% closed loop eigenvalues
Acl = At + But(:,2)*Kt;
eig(Acl)
ans =
   -2.0020e-03 + 3.0623e-03i
   -2.0020e-03 - 3.0623e-03i
   -2.5870e-03
   -1.1650e-03

% gain in original coordinates
K = Kt * T
K =
   -4.0349e-03    6.6700e+03   -1.7280e+00   -5.1733e+06

% state covariance
Y = lyap(Acl, Bwt * W * Bwt')
Y =
    6.6227e+02    5.3438e+02    9.4369e-16   -3.0343e+00
    5.3438e+02    5.9154e+03    3.0343e+00   -2.6201e-14
    9.4369e-16    3.0343e+00    4.3655e-03   -2.1558e-03
   -3.0343e+00   -2.6201e-14   -2.1558e-03    2.1878e-02
sqrt(trace(Y))
ans =
    8.1103e+01
sqrt(diag(Y))
ans =
    2.5735e+01
    7.6912e+01
    6.6072e-02
    1.4791e-01

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```

% optimal control (using u2 only)
rho = 1
rho =
    1
[Kt,X,S] = lqr(At, But(:,2), Q, rho * R);
Kt = - Kt
Kt =
   -2.0501e+00    1.0000e+00   -8.9080e+02   -1.4636e+01
trace(X * Bwt * W * Bwt')
ans =
    2.2464e+03

% closed loop eigenvalues
Acl = At + But(:,2)*Kt;
eig(Acl)
ans =
   -7.0859e-02 + 7.0562e-02i
   -7.0859e-02 - 7.0562e-02i
   -3.4789e-03
   -1.1596e-03

% gain in original coordinates
K = Kt * T
K =
   -2.0501e+00    6.6700e+06   -8.9080e+02   -9.7620e+07

% state covariance
Y = lyap(Acl, Bwt * W * Bwt')
Y =
    2.6762e+02    5.3438e+02    4.1633e-16   -9.8990e-01
    5.3438e+02    1.9506e+03    9.8990e-01    9.9476e-14
    4.1633e-16    9.8990e-01    1.2162e-03   -2.1558e-03
   -9.8990e-01    9.9476e-14   -2.1558e-03    2.6995e-01
sqrt(trace(Y))
ans =
    4.7101e+01
sqrt(diag(Y))
ans =
    1.6359e+01
    4.4165e+01
    3.4874e-02
    5.1956e-01

% optimal control (using u1 and u2)
R = eye(2);
rho = 1
rho =
    1
[Kt,X,S] = lqr(At, But, Q, rho * R);
Kt = - Kt
Kt =
   -1.0003e+00    1.6359e-02   -1.4179e+01   -2.3277e-05
   -1.6359e-02   -9.9987e-01   -2.3277e-05   -1.4177e+01
trace(X * Bwt * W * Bwt')

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```

ans =
    2.8356e-02

% closed loop eigenvalues
Acl = At + But*Kt;
eig(Acl)
ans =
    -7.0895e-02 + 7.1691e-02i
    -7.0895e-02 - 7.1691e-02i
    -7.0884e-02 + 6.9372e-02i
    -7.0884e-02 - 6.9372e-02i

% gain in original coordinates
K = Kt * T
K =
    -1.0003e+00    1.0912e+05   -1.4179e+01   -1.5526e+02
    -1.6359e-02   -6.6691e+06   -2.3277e-05   -9.4557e+07

% state covariance
Y = lyap(Acl, Bwt * W * Bwt')
Y =
    3.5263e-03   -4.3368e-19    6.7763e-21   -4.0692e-06
   -4.3368e-19    3.5270e-03    4.0692e-06    5.7598e-20
    6.7763e-21    4.0692e-06    3.5267e-05   -1.3976e-20
   -4.0692e-06    5.7598e-20   -1.3976e-20    3.5274e-05
sqrt(trace(Y))
ans =
    8.4402e-02
sqrt(diag(Y))
ans =
    5.9382e-02
    5.9388e-02
    5.9386e-03
    5.9392e-03

diary off

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