

4 Optimal State Estimation

Consider the LTI system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_w w(t), \\ y(t) &= C_y x(t) + D_{yw} w(t), \\ z(t) &= C_z x(t)\end{aligned}$$

Problem:

Compute $(\hat{A}, F)$ such that the output of the *state estimator*

$$\begin{aligned}\dot{\hat{x}}(t) &= \hat{A} \hat{x}(t) - F y(t), \\ \hat{z}(t) &= C_z \hat{x}(t)\end{aligned}$$

stabilizes the state estimation error and minimizes the cost function

$$J := \lim_{t \rightarrow \infty} E \left[(z(t) - \hat{z}(t))^T (z(t) - \hat{z}(t)) \right].$$

Assumptions:

- a) (A, C_y) detectable;
- b) $w(t)$ is a Gaussian zero mean white noise with variance $W \succ 0$.

Solution:
Define

$$e(t) := x(t) - \hat{x}(t)$$

so that

$$z(t) - \hat{z}(t) = C_z e(t).$$

Now write the dynamics of the estimation error

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t), \\ &= Ax(t) - \hat{A}\hat{x}(t) + Fy(t) + B_w w(t), \\ &= (A - \hat{A} + FC_y)x(t) + \hat{A}e(t) + (B_w + FD_{yw})w(t)\end{aligned}$$

First observation: if A is unstable $x(t)$ is unbounded and so is $e(t)$, with one exception. If

$$\hat{A} = A + FC_y \quad \implies \quad A - \hat{A} + FC_y = 0$$

and

$$\dot{e}(t) = (A + FC_y)e(t) + (B_w + FD_{yw})w(t).$$

That is, the dynamics of $e(t)$ and $x(t)$ are *decoupled*!
(note that $z(t) - \hat{z}(t) = C_z e(t)$ is also decoupled from $x(t)$).
This choice of $\hat{A}$ produces the *state observer*

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + F(\hat{y} - y), \\ \hat{y} &= C_y \hat{x}, \\ \hat{z} &= C_z \hat{x}.\end{aligned}$$

in which the only unknown is the *observer gain* F . The problem has been reduced to “find the gain F ”.

Simplified problem:

Compute the stabilizing state estimation gain F so as to minimize

$$J := \lim_{t \rightarrow \infty} E [e(t)^T C_z^T C_z e(t)]$$

where $e(t)$ is the state of the LTI system

$$\dot{e}(t) = (A + FC_y)e(t) + (B_w + FD_{yw})w(t).$$

Assumptions:

- a) (A, C_y) is detectable;
- b) $w(t)$ is a Gaussian zero mean white noise with variance $W \succ 0$.

Solution:

As before

$$J = \text{trace} [X(B_w + FD_{yw})W(B_w + FD_{yw})^T]$$

where X is the Gramian

$$(A + FC_y)^T X + X(A + FC_y) + C_z^T C_z = 0.$$

However, in this form, F appears simultaneously at the cost and at the Lyapunov equation, and we can not use completion of squares to determine F .
Use duality!

4.1 Duality

Assume $W \succeq 0$. Consider the cost function

$$J = \text{trace} (X B W B^T) ,$$

where X is the solution to the Lyapunov equation

$$A^T X + X A + C^T C = 0.$$

An alternative (dual) expression for J can be obtained from

$$\begin{aligned} J &= \text{trace} (X B W B^T) , \\ &= \text{trace} \left(\left[\int_0^\infty e^{A^T t} C^T C e^{A t} dt \right] B W B^T \right) , \\ &= \int_0^\infty \text{trace} \left(C e^{A t} B W B^T e^{A^T t} C^T \right) dt , \\ &= \text{trace} \left(C \left[\int_0^\infty e^{A t} B W B^T e^{A^T t} dt \right] C^T \right) . \end{aligned}$$

That is

$$J = \text{trace} (C Y C^T) ,$$

where Y is the solution to the Lyapunov equation

$$A Y + Y A^T + B W B^T = 0.$$

Recall that Y is also the state covariance matrix!

Back to the determination of F , using the dual formulation

$$J = \text{trace} (C_z Y C_z^T)$$

where

$$(A + FC_y)Y + Y(A + FC_y)^T + (B_w + FD_{yw})W(B_w + FD_{yw})^T = 0.$$

The gain F now appears only in the Lyapunov equation!

Assuming that

$$B_w W D_{yw}^T = 0, \quad D_{yw} W D_{yw}^T \succ 0,$$

(for simplicity only, without loss of generality!) the above equation becomes

$$(A + FC_y)Y + Y(A + FC_y)^T + B_w W B_w^T + F D_{yw} W D_{yw}^T F^T = 0.$$

We can now complete the squares to obtain the ARE

$$AY + Y A^T - Y C_y^T (D_{yw} W D_{yw}^T)^{-1} C_y Y + B_w W B_w^T = 0$$

and the associated optimal gain

$$F = -Y C_y^T (D_{yw} W D_{yw}^T)^{-1}.$$

4.2 Summary on Estimation

Problem:

Given the LTI system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_w w(t), \\ y(t) &= C_y x(t) + D_{yw} w(t), \\ z(t) &= C_z x(t)\end{aligned}$$

compute the estimation gain F that stabilizes the state estimation errors and that the output of the *state estimator*

$$\begin{aligned}\dot{\hat{x}}(t) &= A \hat{x}(t) + F(\hat{y}(t) - y(t)), \\ \hat{y}(t) &= C_y \hat{x}(t), \\ \hat{z}(t) &= C_z \hat{x}(t)\end{aligned}$$

minimizes the cost function

$$J := \lim_{t \rightarrow \infty} E \left[(z(t) - \hat{z}(t))^T (z(t) - \hat{z}(t)) \right].$$

Assumptions:

- a) (A, C_y) detectable;
- b) $w(t)$ is a Gaussian zero mean white noise with variance $W \succ 0$;
- c) $B_w W D_{yw}^T = 0$, $D_{yw} W D_{yw}^T \succ 0$.

Solution:

Find the stabilizing solution to the ARE

$$AY + YA^T - YC_y^T (D_{yw} W D_{yw}^T)^{-1} C_y Y + B_w W B_w^T = 0.$$

The optimal gain is

$$F = -YC_y^T (D_{yw} W D_{yw}^T)^{-1}.$$

Things we can infer from the form of the solution:

- 1) $A + FC_y$ is stable even if A is not!

What happens if A is unstable and $\hat{A} = A + FC_y + \delta$?

- 2) The assumption $D_{yw}WD_{yw}^T \succ 0$ reads “no combination of the measurement is error free”.

- 3) If $D_{yw}WD_{yw}^T \succeq 0$, then part of the state could be reconstructed exactly from the measurements.

While this might sounds easier, we do not know how to solve this problem :).

- 4) State estimation is “equivalent” to the state feedback problem

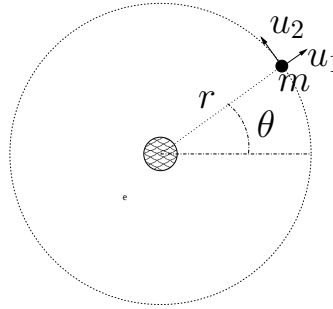
$$\min J = \lim_{t \rightarrow \infty} E \left[x^T B_w W B_w^T x + u^T D_{yw} W D_{yw}^T u \right],$$

for the LTI system

$$\begin{aligned} \dot{x} &= A^T x + C_y^T u + C_z^T w, \\ u &= F^T x \end{aligned}$$

- 5) The optimal gain F does not depend on C_z ! The optimal cost does!

4.3 Example: estimating the state of a satellite in circular orbit



Satellite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2 . Continuing...

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= u_1 - \frac{km}{r^2} + w_1, \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= u_2 + w_2, \end{aligned}$$

where w_1 and w_2 are independent white noise disturbances with variances δ_1 and δ_2 . For the purpose of estimation we set $u_1 = u_2 = 0$ for the moment.

As before, putting in state space and linearizing

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Consider that you have a noisy measurement of θ (x_2)

$$y = [0 \quad 1 \quad 0 \quad 0] x + v$$

where $E\{vv^T\} = \delta_3$.

Problem: Given

$$m = 100 \text{ kg}, \quad \bar{r} = R + 300 \text{ km}, \quad \bar{k} = GM$$

where $G \approx 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2$ is the universal gravitational constant, and $M \approx 5.98 \times 10^{24} \text{ kg}$ and $R \approx 6.37 \times 10^3 \text{ km}$ are the mass and radius of the earth. If the variances $\delta_1 = \delta_2 = \delta_3 = 0.1N$, estimate the state of the satellite.

```

% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% State estimation - Part I
%
m = 100;                % 100 kg
r = 300E3;              % 300 km
R = 6.37E6;             % 6.37 10^3 km
G = 6.673E-11;          % 6.673 N m^2/kg^2
M = 5.98E24;           % 5.98 10^24 kg
k = G * M;              % gravitational force constant
w = sqrt(k/((R+r)^3)); % angular velocity (rad/s)
v = w * (R + r);        % "ground" velocity (m/s)

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0];
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*(r+R))];
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*(r+R))];

% noise variances
W = 0.1 * eye(2)
W =
    1.0000e-01    0
           0    1.0000e-01

% any measurement that does not include x2 (theta) is not observable!
% for instance
Cy = [1 0 0 0; 0 0 1 0; 0 0 0 1];

% system
sys = ss(A, Bw, Cy, 0)

a =
           x1           x2           x3           x4
x1           0           0           1           0
x2           0           0           0           1
x3  4.034e-06           0           0  1.547e+04
x4           0           0 -3.477e-10           0

b =
           u1           u2
x1           0           0
x2           0           0
x3  0.01           0
x4           0  1.499e-09

c =
           x1  x2  x3  x4
y1      1    0    0    0
y2      0    0    1    0
y3      0    0    0    1

d =

```

```

        u1  u2
y1      0   0
y2      0   0
y3      0   0

Continuous-time model.
% take the comment out of next line to see error messages!
%est = kalman(sys, W, eye(size(Cy,1)));
% back to the problem
% measuring x2 (theta)
Cy = [0 r+R 0 0];

% augment noise matrices
Bwa = [Bw zeros(4,1)]
Bwa =
           0           0           0
           0           0           0
    1.0000e-02           0           0
           0    1.4993e-09           0
Dywa = [zeros(1,2) 1]
Dywa =
           0           0           1

% scale
T = diag([1 r+R 1 r+R])
T =
           1           0           0           0
           0    6670000           0           0
           0           0           1           0
           0           0           0    6670000

% similarity transformation
At = T * A / T
At =
           0           0    1.0000e+00           0
           0           0           0    1.0000e+00
    4.0343e-06           0           0    2.3193e-03
           0           0   -2.3193e-03           0
But = T * Bu
But =
           0           0
           0           0
    1.0000e-02           0
           0    1.0000e-02
Bwt = T * Bwa
Bwt =
           0           0           0
           0           0           0
    1.0000e-02           0           0
           0    1.0000e-02           0
Cyt = Cy / T
Cyt =
           0           1           0           0
Dywt = Dywa

```

```

Dywt =
    0    0    1

Ww = W
Ww =
    1.0000e-01    0
           0    1.0000e-01

Wv = 0.1
Wv =
    1.0000e-01

Wt = [Ww zeros(2, 1); zeros(1, 2) Wv]
Wt =
    1.0000e-01    0    0
           0    1.0000e-01    0
           0    0    1.0000e-01

% compute using dual state feedback
[F,X,S] = lqr(At', Cyt', Bwt * Wt * Bwt', Dywt * Wt * Dywt');
F = - F'
F =
    8.9073e+00
   -1.4600e-01
    2.0161e-02
   -1.0658e-02

% compute using matlab's kalman
sys = ss(At, Bwt(:,1:2), Cyt, 0);
est = kalman(sys, Ww, Wv/r^2);
F = -est.b
F =
    2.5872e+06
   -7.7464e+01
    6.0001e+03
   -3.0004e+03

% error dynamics
eig(At + F * Cyt)
ans =
   -1.1596e-03
   -3.4789e-03
  -3.8730e+01 + 3.8730e+01i
  -3.8730e+01 - 3.8730e+01i

% simulate estimator
sys = ss(T * A / T, T * Bwa, Cy / T, Dywa);
filt = est * sys;

Tmax = 20000;
T = 0 : 0.1 : Tmax;
w = [randn(length(T),2)*sqrtm(Ww) randn(length(T),1)*sqrtm(Wv)];

% estimate stationary position
figure(1)

```

```

x0 = [0; 0; 0; 0]
x0 =
    0
    0
    0
    0
xhat0 = [0; 0; -6; 0]
xhat0 =
    0
    0
   -6
    0
[y,t,x] = lsim(sys, w, T, x0);
[yf,tf,xf] = lsim(filt, w, T, [xhat0; x0]);
xhat = xf(:,1:4);

figure(1)
plot(t, x(:,[1 2]), 'g', t, xhat(:,[1 2]), 'b'),
title('system and estimator trajectory (x_1 and x_2)')
xlim([0, Tmax])
grid

figure(2)
plot(t, x(:,[3 4]), 'g', t, xhat(:,[3 4]), 'b'),
title('system and estimator trajectory (x_3 and x_4)')
xlim([0, Tmax])
grid

figure(3)
plot([0 T(end)], [0 0], 'g', t, x(:,[1 2])-xhat(:,[1 2]), 'b'),
title('estimation error (x_1 and x_2)')
xlim([0, Tmax])
grid

figure(4)
plot([0 T(end)], [0 0], 'g', t, x(:,[3 4])-xhat(:,[3 4]), 'b'),
title('estimation error (x_3 and x_4)')
xlim([0, Tmax])
grid

diary off

```

4.4 Example: estimating position

Dynamics of particle of mass $m = 100$ Kg in a line:

$$m\ddot{x} = w_1,$$

where w_1 is a zero-mean white noise disturbances with variance $\delta_1 = (100 \text{ N})^2$. As a measurement we have a noisy measurement of position, say through GPS

$$y_1 = x + w_2$$

where w_2 is a white noise disturbances with variance $\delta_2 = (10 \text{ m})^2$.

In state space

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1/m & 0 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\end{aligned}$$

Problem #1: Estimate the state x .

Problem #2: Can an accelerometer improve the estimate?

```

% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% State estimation - Part II
%
m = 100;                % 100 kg

% linearized system matrices
A = [0 1; 0 0]
A =
    0    1
    0    0
Bw = [0; 1/m]
Bw =
    0
 1.0000e-02

% position
Cy = [1 0]
Cy =
    1    0
Dyw = 0
Dyw =
    0

% noise variances
Ww = 100^2
Ww =
 10000

% augmented system
Bwa = [Bw zeros(2,1)]
Bwa =
    0    0
 1.0000e-02    0
Dywa = [zeros(1,1) 1]
Dywa =
    0    1

% measurement noise variances
Wv = 10^2
Wv =
 100

Wa = [Ww zeros(1, 1); zeros(1, 1) Wv]
Wa =
 10000    0
    0    100

% compute using dual state feedback
[F,X,S] = lqr(A', Cy', Bwa * Wa * Bwa', Dywa * Wa * Dywa');
F = - F'
F =

```

```

-4.4721e-01
-1.0000e-01

% compute using matlab's kalman
sys = ss(A, Bw, Cy, Dyw);
est = kalman(sys, Ww, Wv);
F = -est.b
F =
-4.4721e-01
-1.0000e-01

% error dynamics
eig(A + F * Cy)
ans =
-2.2361e-01 + 2.2361e-01i
-2.2361e-01 - 2.2361e-01i

% try to simulate estimator
sys1 = ss(A, Bwa, Cy, Dywa);
filt1 = est * sys1;

Tmax = 240;
T = 0 : 0.1 : Tmax;
w = [randn(length(T),1)*sqrtm(Ww) randn(length(T),1)*sqrtm(Wv)];

% estimate stationary position (no noise)
figure(1)
x0 = [10; 0]
x0 =
    10
     0
xhat0 = [0; 0]
xhat0 =
     0
     0
[y,t,x] = initial(sys1, x0, T);
[yf,tf,xf] = initial(filt1, [xhat0; x0], T);
xhat = xf(:,1:2);

figure(1)
plot(t, x, t, xhat),
title('system and estimator trajectory')
xlim([0, Tmax])
grid

figure(2)
plot(t, x-xhat(:,1:2)),
title('estimation error')
xlim([0, Tmax])
grid
pause

% estimate stationary position (with measurement noise)
wm = w;

```

```

wm(:,1) = 0;
x0 = [10; 0]
x0 =
    10
     0
xhat0 = [5; 0]
xhat0 =
     5
     0
[y,t,x] = lsim(sys1, wm, T, x0);
[yf,tf,xf] = lsim(filt1, wm, T, [xhat0; x0]);
xhat = xf(:,1:2);

figure(1)
plot(t, x, 'g', t, xhat, 'b'),
title('system and estimator trajectory')
xlim([0, Tmax])
grid

figure(2)
plot([0 T(end)], [0 0], 'g', t, x-xhat(:,1:2), 'b'),
title('estimation error')
xlim([0, Tmax])
grid
pause

% estimate moving position (with measurement and process noise)
x0 = [0; 0]
x0 =
     0
     0
xhat0 = zeros(2,1)
xhat0 =
     0
     0
[y,t,x] = lsim(sys1, w, T, x0);
[yf,tf,xf] = lsim(filt1, w, T, [xhat0; x0]);
xhat = xf(:,1:2);

figure(1)
plot(t, x, 'g', t, xhat, 'b'),
title('system and estimator trajectory')
xlim([0, Tmax])
grid

figure(2)
plot([0 T(end)], [0 0], 'g', t, x-xhat(:,1:2), 'b'),
title('estimation error')
xlim([0, Tmax])
grid

diary off

```

```

% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% State estimation - Part III
%
m = 100;                % 100 kg

% linearized system matrices
A = [0 1; 0 0]
A =
    0    1
    0    0
Bw = [0; 1/m]
Bw =
    0
 1.0000e-02

% position
Cy1 = [1 0]
Cy1 =
    1    0
Dyw1 = 0
Dyw1 =
    0

% accelerometer
P = [0 1];
Cy2 = P*A
Cy2 =
    0    0
Dyw2 = P*Bw
Dyw2 =
 1.0000e-02

Cy = [Cy1; Cy2];
Dyw = [Dyw1; Dyw2];

% noise variances
Ww = 100^2
Ww =
 10000
Wv = diag([10^2 100]);

% augmented system matrices
Bwa = [Bw zeros(2,2)]
Bwa =
    0    0    0
 1.0000e-02    0    0
Dywa = [zeros(2,1) eye(2)]
Dywa =
    0    1    0
    0    0    1
Wa = [Ww zeros(1, 2); zeros(2, 1) Wv]

```

```

Wa =
    10000         0         0
         0        100         0
         0         0        100

% compute using matlab's kalman
sys = ss(A, Bw, Cy, Dyw);
est = kalman(sys, Ww, Wv);
F = -est.b
F =
   -4.4610e-01         0
   -9.9504e-02   -9.9010e-03

% error dynamics
eig(A + F * Cy)
ans =
   -2.2305e-01 + 2.2305e-01i
   -2.2305e-01 - 2.2305e-01i

% try to simulate estimator
sys2 = ss(A, Bwa, Cy, Dywa);
filt2 = est * sys2;

Tmax = 240;
T = 0 : 0.1 : Tmax;
w = [randn(length(T),1)*sqrtm(Ww) randn(length(T),2)*sqrtm(Wv)];

% estimate stationary position (with measurement noise)
wm = w;
wm(:,1) = 0;
x0 = [10; 0]
x0 =
    10
     0
xhat0 = [5; 0]
xhat0 =
     5
     0
[y,t,x] = lsim(sys2, wm, T, x0);
if exist('filt1')
end
[yf2,tf2,xf2] = lsim(filt2, wm, T, [xhat0; x0]);
xhat2 = xf2(:,1:2);

figure(1)
if exist('filt1')
else
    plot(t, x, 'g', t, xhat2, 'b'),
end
title('system and estimator trajectory')
xlim([0, Tmax])
grid

figure(2)

```

```

if exist('filt1')
else
    plot([0 T(end)], [0 0], 'g', t, x-xhat2(:,1:2), 'b'),
end
title('estimation error')
xlim([0, Tmax])
grid
pause

% estimate moving position (with measurement and process noise)
x0 = [10; 0]
x0 =
    10
     0
xhat0 = [5; 0]
xhat0 =
     5
     0
[y,t,x] = lsim(sys2, w, T, x0);
if exist('filt1')
end
[yf2,tf2,xf2] = lsim(filt2, w, T, [xhat0; x0]);
xhat2 = xf2(:,1:2);

figure(1)
if exist('filt1')
else
    plot(t, x, 'g', t, xhat2, 'b'),
end
title('system and estimator trajectory')
xlim([0, Tmax])
grid

figure(2)
if exist('filt1')
else
    plot([0 T(end)], [0 0], 'g', t, x-xhat2(:,1:2), 'b'),
end
title('estimation error')
xlim([0, Tmax])
grid

diary off

```