

5 Optimal Linear Quadratic Gaussian (LQG) Control

Problem:

Given the LTI system

$$\begin{aligned}\dot{x} &= Ax + B_u u + B_w w, \\ y &= C_y x + D_{yw} w, \\ z &= C_z x + D_{zu} u,\end{aligned}$$

and the *observer-based controller*

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + B_u u + F(\hat{y} - y), \\ \hat{y} &= C_y \hat{x}, \\ u &= K\hat{x}.\end{aligned}$$

compute (K, F) that stabilize the closed loop system and minimize the cost function

$$J := \lim_{t \rightarrow \infty} E [z(t)^T z(t)]. \quad (3)$$

Assumptions:

1. (A, B_u) stabilizable,
2. (A, C_y) detectable,
3. $w(t)$ is a Gaussian zero mean white noise with variance $W \succ 0$,
4. $C_z^T D_{zu} = 0$ and $D_{zu}^T D_{zu} \succ 0$,
5. $B_w^T W D_{yw}^T = 0$ and $D_{yw}^T W D_{yw}^T \succ 0$.

Remark 1: 4 and 5 can be relaxed.

Remark 2: 4 implies that

$$J = \lim_{t \rightarrow \infty} E [x(t)^T Q x(t) + u(t)^T R u(t)],$$

where

$$Q = C_z^T C_z, \quad R = D_{zu}^T D_{zu} \succ 0.$$

5.1 The Closed Loop System

The closed loop system is

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} &= \begin{bmatrix} A & B_u K \\ -FC_y & A + B_u K + FC_y \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{bmatrix} B_w \\ -FD_{yw} \end{bmatrix} w, \\ z &= [C_z \quad D_{zu}K] \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \end{aligned} \quad (4)$$

As in the estimation problem, we write the dynamics in terms of $e := x - \hat{x}$. Note that

$$\begin{pmatrix} e \\ \hat{x} \end{pmatrix} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \quad \begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix},$$

and

$$\begin{aligned} \begin{pmatrix} \dot{e} \\ \dot{\hat{x}} \end{pmatrix} &= \begin{bmatrix} A + FC_y & 0 \\ -FC_y & A + B_u K \end{bmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix} + \begin{bmatrix} B_w + FD_{yw} \\ -FD_{yw} \end{bmatrix} w, \\ z &= [C_z \quad C_z + D_{zu}K] \begin{pmatrix} e \\ \hat{x} \end{pmatrix} \end{aligned} \quad (5)$$

5.2 Stability of the Closed Loop System

First fantastic result:

*The closed loop system (4) is stable
if and only if
K is a stabilizing state feedback gain and
F is a stabilizing state estimation gain.*

Proof: Systems (4) and (5) are **similar**, therefore the closed loop poles are the roots of the polynomial equation

$$\begin{aligned} 0 &= \det \begin{bmatrix} sI - A & -B_u K \\ FC_y & sI - (A + B_u K + FC_y) \end{bmatrix} \\ &= \det \begin{bmatrix} sI - (A + FC_y) & 0 \\ FC_y & sI - (A + B_u K) \end{bmatrix} \\ &= \det [sI - (A + FC_y)] \det [sI - (A + B_u K)] \end{aligned}$$

5.3 The Cost Function

Now write the cost function (1) in terms of system (5)

$$J = \lim_{t \rightarrow \infty} E [z(t)^T z(t)] .$$

Computing J using the Controllability Gramian of system (5) we have

$$J = \text{trace} \left([C_z \ C_z + D_{zu}K] \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \begin{bmatrix} C_z^T \\ C_z^T + K^T D_{zu}^T \end{bmatrix} \right) , \quad (6)$$

where Y_1 , Y_2 and Y_3 satisfy the Lyapunov equation

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} A + FC_y & 0 \\ -FC_y & A + B_u K \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \\ &+ \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \begin{bmatrix} A + FC_y & 0 \\ -FC_y & A + B_u K \end{bmatrix}^T \\ &+ \begin{bmatrix} B_w + FD_{yw} \\ -FD_{yw} \end{bmatrix} W [(B_w + FD_{yw})^T \ - (FD_{yw})^T] \end{aligned}$$

Expanding these equations we have

$$0 = (A + FC_y)Y_1 + Y_1(A + FC_y)^T + (B_w + FD_{yw})W(B_w + FD_{yw})^T, \quad (7)$$

$$0 = (A + FC_y)Y_2 + Y_2(A + B_u K)^T - Y_1 C_y^T F^T - FD_{yw} W D_{yw}^T F^T, \quad (8)$$

$$\begin{aligned} 0 &= (A + B_u K)Y_3 + Y_3(A + B_u K)^T + FD_{yw} W D_{yw}^T F^T \\ &\quad - FC_y Y_2 - Y_2^T C_y^T F^T. \end{aligned} \quad (9)$$

In (8) we have used the assumption that $B_w W D_{yw}^T = 0$.

5.4 The Optimal Estimator Gain

The closed loop is stable if and only if F is stabilizing. Furthermore

$$J = J_1(Y_1) + J_2(Y_2, Y_3, K)$$

where

$$\begin{aligned} J_1(Y_1) &= \text{trace}(C_z Y_1 C_z^T), \\ J_2(Y_2, Y_3, K) &= \text{trace}([C_z + D_{zu}K]Y_3[C_z + D_{zu}K]^T) \\ &\quad + 2 \text{trace}(C_z Y_2 [C_z + D_{zu}K]^T). \end{aligned}$$

Note that Y_1 is determined by solving (7)

$$(A + FC_y)Y_1 + Y_1(A + FC_y)^T + (B_w + FD_{yw})W(B_w + FD_{yw})^T = 0,$$

which does not depend on K . Therefore the cost J is minimum if $J_1(Y_1)$ is also minimum. But this is the case if Y_1 is minimum in the sense of the Comparison Lemma¹. That is, with $Y_1 = Y^*$ satisfying the ARE

$$AY^* + Y^*A^T - Y^*C_y^T(D_{yw}WD_{yw}^T)^{-1}C_yY^* + B_wWB_w^T = 0. \quad (10)$$

The optimal observer gain is then

$$F^* = -Y^*C_y^T(D_{yw}WD_{yw}^T)^{-1}.$$

5.5 A Temporary Solution For Y_2

Knowing that, we rewrite (8) as

$$0 = AY_2 + Y_2(A + B_uK)^T - Y^*C_y^T(D_{yw}WD_{yw}^T)^{-1}C_yY_2.$$

Note that the above equation is trivially satisfied with $Y_2^* = 0$. Assume this choice of Y_2^* as a matter of faith for the moment.

¹See section "What if $Y_2 \neq 0$ " for a formal proof of a similar fact.

5.6 The Optimal Control Gain

With $Y_1 = Y^*$ and $Y_2 = Y_2^* = 0$, the cost function (20) becomes

$$J = \text{trace} [C_z Y^* C_z^T] + \text{trace} \left[(C_z + D_{zu} K) Y_3 (C_z + D_{zu} K)^T \right].$$

In this form, it is clear that the optimal K is the one that minimizes

$$J_2 = \text{trace} \left[(C_z + D_{zu} K) Y_3 (C_z + D_{zu} K)^T \right].$$

subject to equation (9), which for $Y_2^* = 0$ is simply

$$(A + B_u K) Y_3 + Y_3 (A + B_u K)^T + Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1} C_y Y^* = 0. \quad (11)$$

Using our previous duality results, the optimal K is the one that minimizes

$$J_2 = \text{trace} [\bar{B}^T X \bar{B}], \quad \bar{B} := Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1/2}$$

where

$$\begin{aligned} 0 &= (A + B_u K)^T X + X(A + B_u K) + (C_z + D_{zu} K)^T (C_z + D_{zu} K) \\ &= (A + B_u K)^T X + X(A + B_u K) + C_z^T C_z + K^T D_{zu}^T D_{zu} K. \end{aligned}$$

Using what we know about the LQR problem, the solution is

$$K^* = -(D_{zu}^T D_{zu})^{-1} B_u^T X^*,$$

where X^* satisfies the ARE

$$A^T X^* + X^* A - X^* B_u (D_{zu}^T D_{zu})^{-1} B_u^T X^* + C_z^T C_z = 0.$$

Matrix Y_3^* can be computed from (9) upon substitution of the optimal K^* .

5.7 Summary of LQG Control

Given the LTI system

$$\begin{aligned}\dot{x} &= Ax + B_u u + B_w w, \\ y &= C_y x + D_{yw} w, \\ z &= C_z x + D_{zu} u,\end{aligned}$$

and the *observer-based controller*

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + B_u u + F(\hat{y} - y), \\ \hat{y} &= C_y \hat{x}, \\ u &= K\hat{x},\end{aligned}$$

if

1. (A, B_u) stabilizable,
2. (A, C_y) detectable,
3. $w(t)$ is a Gaussian zero mean white noise with variance $W \succ 0$,
4. $C_z^T D_{zu} = 0$ and $D_{zu}^T D_{zu} \succ 0$,
5. $B_w W D_{yw}^T = 0$ and $D_{yw} W D_{yw}^T \succ 0$,

then the choice of (K, F) that stabilize the closed loop system and minimize the cost function

$$J := \lim_{t \rightarrow \infty} E [z(t)^T z(t)].$$

is given by

$$K^* = -(D_{zu}^T D_{zu})^{-1} B_u^T X^*,$$

where X^* satisfies the ARE

$$A^T X^* + X^* A - X^* B_u (D_{zu}^T D_{zu})^{-1} B_u^T X^* + C_z^T C_z = 0,$$

and

$$F^* = -Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1},$$

where Y^* satisfies the ARE

$$A Y^* + Y^* A^T - Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1} C_y Y^* + B_w W B_w^T = 0.$$

5.8 What if $Y_2 \neq 0$?

The optimal covariance matrix has the form

$$\begin{bmatrix} Y_1^* & Y_2^* \\ Y_2^{*T} & Y_3^* \end{bmatrix} = \begin{bmatrix} Y^* & 0 \\ 0 & Y_3^* \end{bmatrix} \succ 0$$

We claim that this matrix is minimal in the sense of the comparison lemma.

Proof: Assume it is not. Then there exists

$$\begin{bmatrix} Y^* & 0 \\ 0 & Y_3^* \end{bmatrix} \succeq \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \succ 0$$

This implies that

$$\begin{bmatrix} Y^* - Y_1 & -Y_2 \\ -Y_2^T & Y_3^* - Y_3 \end{bmatrix} \succeq 0 \quad (12)$$

Multiplying (12) on the left by $\begin{bmatrix} I & 0 \end{bmatrix}$ and on the right by $\begin{bmatrix} I \\ 0 \end{bmatrix}$ we obtain that

$$Y^* \succeq Y_1.$$

But since we have already proved that Y^* is minimal

$$Y_1 \succeq Y^*.$$

Therefore, $Y_1 = Y^*$. Using this fact (12) becomes

$$\begin{bmatrix} 0 & -Y_2 \\ -Y_2^T & Y_3^* - Y_3 \end{bmatrix} \succeq 0.$$

Multiplying (12) on the left by $\begin{bmatrix} 0 & I \end{bmatrix}$ and on the right by $\begin{bmatrix} 0 \\ I \end{bmatrix}$ we conclude that

$$Y_3^* \succeq Y_3.$$

Now, matrix (12) is positive semidefinite if and only if

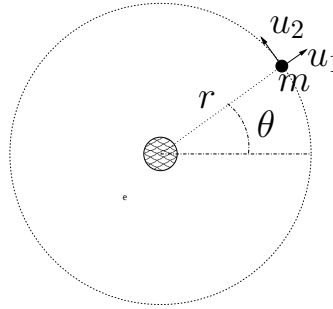
$$\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{bmatrix} 0 & -Y_2 \\ -Y_2^T & Y_3^* - Y_3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -2x^T Y_2 y + y^T (Y_3^* - Y_3) y \geq 0, \quad \forall x, y$$

Because, $Y_3^* - Y_3 \succeq 0$, this is certainly true for $Y_2 = 0$. On the other hand, if $Y_2 \neq 0$, all vectors in the form $x = \alpha Y_2 y$ are such that

$$y^T (Y_3^* - Y_3) y - 2\alpha y^T Y_2^T Y_2 y \geq 0, \quad \forall y.$$

As $Y_2^T Y_2 \succeq 0$, for any y such that $Y_2 y \neq 0$ there is a sufficiently big $\alpha > 0$ such that the above inequality is violated, proving that Y_2 must indeed be zero.

5.9 Example: controlling a satellite in circular orbit



Satellite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2 . Continuing...

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= u_1 - \frac{km}{r^2} + w_1, \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= u_2 + w_2, \end{aligned}$$

where w_1 and w_2 are independent white noise disturbances with variances δ_1 and δ_2 .

As before, putting in state space and linearizing

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Consider that you have a noisy measurement of θ (x_2)

$$y = [0 \ 1 \ 0 \ 0] + v$$

where $Evv^T = \delta_3$.

Problem: Given

$$m = 100 \text{ kg}, \quad \bar{r} = R + 300 \text{ km}, \quad \bar{k} = GM$$

where $G \approx 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2$ is the universal gravitational constant, and $M \approx 5.98 \times 10^{24} \text{ kg}$ and $R \approx 6.37 \times 10^3 \text{ km}$ are the mass and radius of the earth. If the variances $\delta_1 = \delta_2 = \delta_3 = 0.1N$ find solutions to the LQR control problem where

$$Q = I, \quad R = I,$$

using u_2 only first, then using u_1 and u_2 .

```

% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% LQG Control - Part II
%
m = 100;                % 100 kg
r = 300E3;              % 300 km
R = 6.37E6;             % 6.37 10^3 km
G = 6.673E-11;          % 6.673 N m^2/kg^2
M = 5.98E24;            % 5.98 10^24 kg
k = G * M;              % gravitational force constant
w = sqrt(k/((R+r)^3)); % angular velocity (rad/s)
v = w * (R + r);        % "ground" velocity (m/s)

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0];
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*(r+R))];
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*(r+R))];

% noise variances
W = 0.1 * eye(2)
W =
    1.0000e-01    0
           0    1.0000e-01

% scale
T = diag([1 r+R 1 r+R])
T =
         1         0         0         0
         0    6670000         0         0
         0         0         1         0
         0         0         0    6670000

% similarity transformation
At = T * A / T
At =
         0         0    1.0000e+00         0
         0         0         0    1.0000e+00
    4.0343e-06         0         0    2.3193e-03
         0         0   -2.3193e-03         0
But = T * Bu
But =
         0         0
         0         0
    1.0000e-02         0
         0    1.0000e-02
Bwt = T * Bw
Bwt =
         0         0
         0         0
    1.0000e-02         0
         0    1.0000e-02

```

```

% measuring x2 (theta)
Cy = [0 r+R 0 0];
Cyt = Cy / T
Cyt =
    0    1    0    0

% augment noise matrices
Bwt = [Bwt zeros(4,1)]
Bwt =
         0         0         0
         0         0         0
    1.0000e-02         0         0
         0    1.0000e-02         0
Dywt = [zeros(1,2) 1]
Dywt =
    0    0    1

Ww = W
Ww =
    1.0000e-01         0
         0    1.0000e-01
Wv = 0.1
Wv =
    1.0000e-01

Wt = [Ww zeros(2, 1); zeros(1, 2) Wv]
Wt =
    1.0000e-01         0         0
         0    1.0000e-01         0
         0         0    1.0000e-01

% optimal state feedback control (using u2 only)
rho = 1
rho =
    1
Czt = [eye(4); zeros(1,4)];
Dzut = sqrt(rho)*[zeros(4,1); eye(1)];
Czt'*Dzut
ans =
    0
    0
    0
    0
[K,X,S] = lqr(At, But(:,2), Czt'*Czt, Dzut'*Dzut);
K = - K
K =
   -2.0501e+00    1.0000e+00   -8.9080e+02   -1.4636e+01

Acl = At + But(:,2)*K;
eig(Acl)
ans =
   -7.0859e-02 + 7.0562e-02i
   -7.0859e-02 - 7.0562e-02i
   -3.4789e-03

```

```

-1.1596e-03

% optimal state estimation
[F,X,S] = lqr(At', Cyt', Bwt * Wt * Bwt', Dywt * Wt * Dywt');
F = - F'
F =
    8.9073e+00
   -1.4600e-01
    2.0161e-02
   -1.0658e-02

Acl = At + F*Cy;
eig(Acl)
ans =
   -1.1595e-03
   -3.4863e-03
   -6.8356e-02
   -9.7384e+05

% optimal controller
ctrl = ss(At + But(:,2) * K + F * Cyt, -F, K, 0)

a =
           x1           x2           x3           x4
x1           0          8.907            1            0
x2           0         -0.146            0            1
x3    4.034e-06         0.02016            0        0.002319
x4       -0.0205    -0.0006584        -8.91       -0.1464

b =
           u1
x1       -8.907
x2         0.146
x3      -0.02016
x4       0.01066

c =
           x1           x2           x3           x4
y1       -2.05            1      -890.8      -14.64

d =
           u1
y1         0

Continuous-time model.

% controller transfer-function
zpk(ctrl)

Zero/pole/gain:
36.2105 (s+0.0005704) (s^2 + 0.001122s + 1.953e-06)
-----
(s+0.8036) (s+0.002027) (s^2 - 0.5132s + 0.4545)

```

```

% compute using matlab's lqg
sys = ss(At, But(:,2), Cyt, 0);
ctr2 = lqg(sys, [Czt Dzut]' * [Czt Dzut], ...
            [Bwt; Dywt] * Wt * [Bwt; Dywt]')

a =
           x1_e      x2_e      x3_e      x4_e
x1_e          0      8.907          1          0
x2_e          0     -0.146          0          1
x3_e  4.034e-06    0.02016          0    0.002319
x4_e   -0.0205  -0.0006584     -8.91   -0.1464

b =
           y1
x1_e   -8.907
x2_e    0.146
x3_e  -0.02016
x4_e   0.01066

c =
           x1_e      x2_e      x3_e      x4_e
u1   -2.05          1  -890.8  -14.64

d =
           y1
u1    0

Input groups:
      Name      Channels
Measurement      1

Output groups:
      Name      Channels
Controls        1

Continuous-time model.

% compute closed loop system
Acl = [At But(:,2)*K; -F*Cyt, At+But(:,2)*K+F*Cyt];
Bcl = [Bwt; -F*Dywt];
Ccl = [Czt; Dzut*K];

eig(Acl)
ans =
-7.0682e-02 + 7.0739e-02i
-7.0682e-02 - 7.0739e-02i
-7.0859e-02 + 7.0562e-02i
-7.0859e-02 - 7.0562e-02i
-3.4789e-03 + 5.1440e-08i
-3.4789e-03 - 5.1440e-08i
-1.1596e-03 + 3.4817e-08i
-1.1596e-03 - 3.4817e-08i

```

```

% state variance
Y = lyap(Acl, Bcl * Wt * Bcl')
Y =
Columns 1 through 6
    4.4534e+03    5.3438e+02   -1.7990e-09   -3.1304e+01    2.2072e+03    5.3527e+02
    5.3438e+02    2.8543e+04    3.1304e+01   -2.7380e-09    5.3527e+02    2.8543e+04
    1.7546e-09    3.1304e+01    5.4635e-02   -2.1558e-03   -3.9670e+00    3.1306e+01
   -3.1304e+01    2.7388e-09   -2.1558e-03    4.4805e+00   -3.1175e+01   -1.0658e-03
    2.2072e+03    5.3527e+02   -3.9670e+00   -3.1175e+01    2.2072e+03    5.3527e+02
    5.3527e+02    2.8543e+04    3.1306e+01   -1.0658e-03    5.3527e+02    2.8543e+04
   -3.9670e+00    3.1306e+01    4.5442e-02   -1.8626e-03   -3.9670e+00    3.1306e+01
   -3.1175e+01   -1.0658e-03   -1.8626e-03    4.4803e+00   -3.1175e+01   -1.0658e-03
Columns 7 through 8
   -3.9670e+00   -3.1175e+01
    3.1306e+01   -1.0658e-03
    4.5442e-02   -1.8626e-03
   -1.8626e-03    4.4803e+00
   -3.9670e+00   -3.1175e+01
    3.1306e+01   -1.0658e-03
    4.5442e-02   -1.8626e-03
   -1.8626e-03    4.4803e+00
sqrt(trace(Y))
ans =
    2.5250e+02
sqrt(diag(Y))
ans =
    6.6734e+01
    1.6895e+02
    2.3374e-01
    2.1167e+00
    4.6981e+01
    1.6895e+02
    2.1317e-01
    2.1167e+00

% optimal state feedback control (using u1 and u2)
rho = 1
rho =
    1
Czt = [eye(4); zeros(2,4)];
Dzut = sqrt(rho)*[zeros(4,2); eye(2)];
Czt'*Dzut
ans =
    0    0
    0    0
    0    0
    0    0
[K,X,S] = lqr(At, But, Czt'*Czt, Dzut'*Dzut);
K = - K
K =
   -1.0003e+00    1.6359e-02   -1.4179e+01   -2.3277e-05
   -1.6359e-02   -9.9987e-01   -2.3277e-05   -1.4177e+01
Acl = At + But*K;

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```

eig(Acl)
ans =
    -7.0895e-02 + 7.1691e-02i
    -7.0895e-02 - 7.1691e-02i
    -7.0884e-02 + 6.9372e-02i
    -7.0884e-02 - 6.9372e-02i

% optimal state estimation
[F,X,S] = lqr(At', Cyt', Bwt * Wt * Bwt', Dywt * Wt * Dywt');
F = - F'
F =
    8.9073e+00
   -1.4600e-01
    2.0161e-02
   -1.0658e-02

Acl = At + F*Cy;
eig(Acl)
ans =
   -1.1595e-03
   -3.4863e-03
   -6.8356e-02
  -9.7384e+05

% optimal controller
ctrl = ss(At + But * K + F * Cyt, -F, K, 0)

a =
           x1           x2           x3           x4
x1           0         8.907           1           0
x2           0        -0.146           0           1
x3   -0.0099999    0.02032    -0.1418    0.002319
x4   -0.0001636   -0.02066   -0.00232   -0.1418

b =
           u1
x1   -8.907
x2    0.146
x3   -0.02016
x4    0.01066

c =
           x1           x2           x3           x4
y1         -1    0.01636    -14.18   -2.328e-05
y2   -0.01636   -0.9999   -2.328e-05   -14.18

d =
           u1
y1     0
y2     0

Continuous-time model.

% controller transfer-function

```

```

zpk(ctrl1)

Zero/pole/gain from input to output...
      9.1979 (s+0.002286) (s^2 + 0.1417s + 0.009996)
#1:  -----
      (s+0.08516) (s+0.1441) (s^2 + 0.2003s + 0.03397)

      -0.15136 (s-0.03449) (s^2 + 0.2489s + 0.01916)
#2:  -----
      (s+0.08516) (s+0.1441) (s^2 + 0.2003s + 0.03397)

% compute using matlab's lgg
sys = ss(At, But, Cyt, 0);
ctr2 = lgg(sys, [Czt Dzut]' * [Czt Dzut], ...
            [Bwt; Dywt] * Wt * [Bwt; Dywt]')

a =
      x1_e      x2_e      x3_e      x4_e
x1_e          0      8.907          1          0
x2_e          0     -0.146          0          1
x3_e    -0.009999      0.02032     -0.1418      0.002319
x4_e    -0.0001636     -0.02066     -0.00232     -0.1418

b =
      y1
x1_e    -8.907
x2_e      0.146
x3_e    -0.02016
x4_e      0.01066

c =
      x1_e      x2_e      x3_e      x4_e
u1         -1      0.01636     -14.18    -2.328e-05
u2     -0.01636     -0.9999    -2.328e-05     -14.18

d =
      y1
u1      0
u2      0

Input groups:
      Name      Channels
Measurement      1

Output groups:
      Name      Channels
Controls      1,2

Continuous-time model.

% compute closed loop system
Acl = [At But*K; -F*Cyt, At+But*K+F*Cyt];
Bcl = [Bwt; -F*Dywt];

```

```

Ccl = [Czt; Dzut*K];

eig(Acl)
ans =
    -3.4789e-03
    -1.1596e-03
    -7.0895e-02 + 7.1691e-02i
    -7.0895e-02 - 7.1691e-02i
    -7.0682e-02 + 7.0739e-02i
    -7.0682e-02 - 7.0739e-02i
    -7.0884e-02 + 6.9372e-02i
    -7.0884e-02 - 6.9372e-02i

% state variance
Y = lyap(Acl, Bcl * Wt * Bcl')
Y =
Columns 1 through 6
    2.3322e+03    -3.0118e+00    7.6599e-14    -1.1265e-01    8.6019e+01    -2.1211e+00
   -3.0118e+00     7.0759e-02     1.1265e-01     9.3445e-18   -2.1211e+00     5.6159e-02
   -8.7764e-13     1.1265e-01     2.8913e-01    -4.9551e-03   -3.9670e+00     1.1466e-01
   -1.1265e-01    -1.8079e-17    -4.9551e-03     3.2470e-04     1.5387e-02    -1.0658e-03
    8.6019e+01    -2.1211e+00   -3.9670e+00     1.5387e-02     8.6019e+01    -2.1211e+00
   -2.1211e+00     5.6159e-02     1.1466e-01    -1.0658e-03   -2.1211e+00     5.6159e-02
   -3.9670e+00     1.1466e-01     2.7993e-01    -4.6618e-03   -3.9670e+00     1.1466e-01
     1.5387e-02    -1.0658e-03    -4.6618e-03     1.7376e-04     1.5387e-02    -1.0658e-03
Columns 7 through 8
   -3.9670e+00     1.5387e-02
     1.1466e-01    -1.0658e-03
     2.7993e-01    -4.6618e-03
   -4.6618e-03     1.7376e-04
   -3.9670e+00     1.5387e-02
     1.1466e-01    -1.0658e-03
     2.7993e-01    -4.6618e-03
   -4.6618e-03     1.7376e-04
sqrt(trace(Y))
ans =
    4.9182e+01
sqrt(diag(Y))
ans =
    4.8293e+01
    2.6601e-01
    5.3771e-01
    1.8019e-02
    9.2746e+00
    2.3698e-01
    5.2909e-01
    1.3182e-02

diary off

```