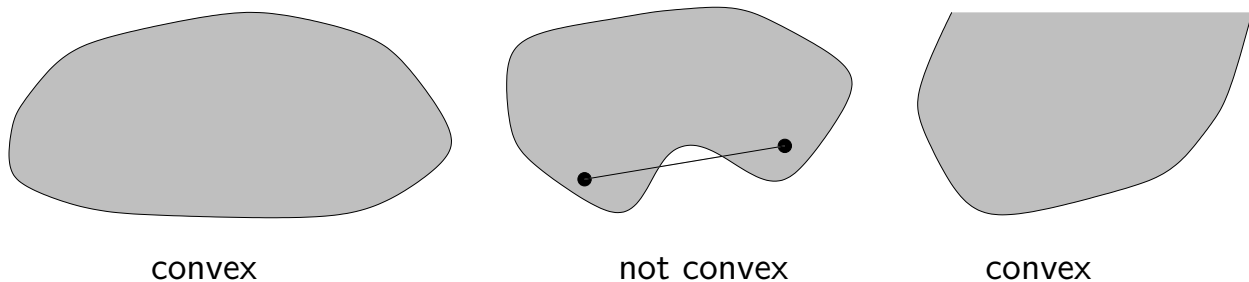


6 Linear Matrix Inequalities

6.1 Convex Sets

Definition: The set $\Omega \subseteq X$, where X is a real linear vector space, is convex if for any $x_1, x_2 \in \Omega$ and $\alpha \in [0, 1]$ the vector $x = \alpha x_1 + (1 - \alpha)x_2 \in \Omega$.

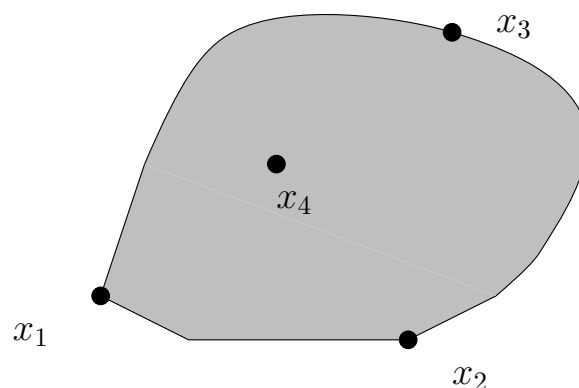


Convex combination: Given the convex set Ω and the vectors $x_i \in \Omega$, $i = 1, \dots, N$, then

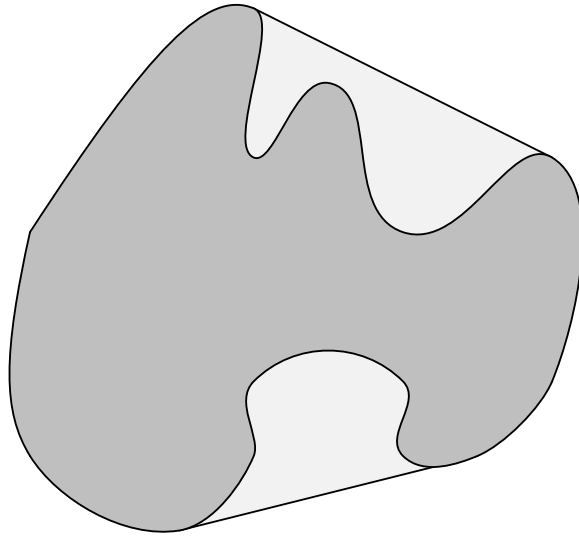
$$x = \sum_{i=1}^N \xi_i x_i \in \Omega, \quad \text{for all } \xi_i \geq 0, \quad \sum_i \xi_i = 1.$$

Extreme Points (x_1, x_2, x_3): A vector $x \in \Omega$, where Ω is a convex set, is an *extreme point* if it cannot be generated by a convex combination of any other vector in Ω .

Interior Points (x_4): A vector $x \in \Omega$, where Ω is a convex set, is an *interior point* if it is not an extreme point.



Convex Hull: Given an arbitrary set Γ , its *convex hull* is the smallest convex set containing Γ . (Notation: $\text{co}(\Gamma)$)



Proposition: Let Ω and Γ be convex sets.

- a) $\alpha\Omega := \{x : x = \alpha x, x \in \Omega\}$ is a convex set.
- b) $\Omega + \Gamma := \{z : z = x + y, x \in \Omega, y \in \Gamma\}$ is a convex set.
- c) $\Omega \cap \Gamma$ is a convex set.

6.2 Convex Functions

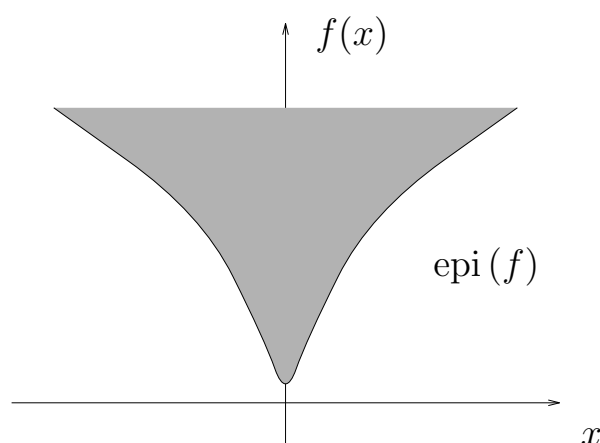
Definition: The real valued function $f : \Omega \rightarrow \mathbb{R}$, where $\Omega \subseteq X$ is a convex set, is said to be a *convex function* if

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2), \forall x_1, x_2 \in \Omega, \alpha \in [0, 1].$$

Epigraph: Given a function $f : \Omega \rightarrow \mathbb{R}$ the set

$$\text{epi}(f) := \{(x, \gamma) : x \in \Omega, \gamma \in \mathbb{R}, f(x) \leq \gamma\}$$

is the *epigraph of f* .



Convex function: The function $f : \Omega \rightarrow \mathbb{R}$ is convex iff $\text{epi}(f)$ is a convex set.

Proposition: All convex functions are continuous.

Proposition: All norms are convex functions.

Concave function: The function $f : \Omega \rightarrow \mathbb{R}$ is concave iff $(-f)$ is convex.

Convexity (twice differentiable real function): Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a twice differentiable function. The following are equivalent

- a) f is convex.
- b) $f(y) \geq f(x) + \nabla f(x)(y - x) \quad \forall x, y \in \mathbb{R}^n$.
- c) $h^T \nabla^2 f(x) h \geq 0 \quad \forall x, h \in \mathbb{R}^n$.

6.3 Cones

Definition: The set $\Gamma \subset X$, where X is a real vector space is a *cone with vertex at the origin* if $\alpha x \in \Gamma$ for all $x \in \Gamma$, $\alpha \geq 0$.

Convex Cone: $\Gamma \subset X$ is a *convex cone* if it is a convex cone. :)

Examples (Convex cones)

$$1. \Gamma := \{x : x \in \mathbb{R}^n, \quad x_i \geq 0, \quad i = 1, \dots, n\}.$$

$$x, y \in \Gamma \Rightarrow \begin{cases} \forall \alpha \geq 0, & \alpha x_i \geq 0 \Rightarrow \alpha x \in \Gamma. \\ \forall \alpha \in [0, 1], & \alpha x_i + (1 - \alpha)y_i \geq 0 \Rightarrow \alpha x + (1 - \alpha)y \in \Gamma. \end{cases}$$

$$2. \Gamma := \{X : X \in \mathbb{S}^n, \quad x^T X x \geq 0, \quad \forall x \in \mathbb{R}^n\}.$$

$$X, Y \in \Gamma \Rightarrow \begin{cases} \forall \alpha \geq 0, & \alpha x^T X x \geq 0, \quad \forall x \in \mathbb{R}^n \Rightarrow \alpha X \in \Gamma. \\ \forall \alpha \in [0, 1], & \alpha x^T X x + (1 - \alpha)x^T Y x \geq 0 \Rightarrow \alpha X + (1 - \alpha)Y \in \Gamma. \end{cases}$$

Example (Nonconvex cone)

$$3. \Gamma := \{X : X \in \mathbb{R}^{n \times n}, \quad \mathbb{R}\{\lambda_i(X)\} \leq 0, \quad \forall i\}.$$

Remembering that $\lambda(\alpha X) = \alpha \lambda(X)$

$$X \in \Gamma \Rightarrow \forall i, \alpha \in \mathbb{R}, \alpha \geq 0, \quad \alpha \mathbb{R}\{\lambda_i(X)\} = \mathbb{R}\{\lambda_i(\alpha X)\} \leq 0 \Rightarrow \alpha X \in \Gamma.$$

For

$$X_1 = \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix}, \quad X_2 = X_1^T = \begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix},$$

notice that $\lambda_i(X_1) = \lambda_i(X_2) = -1, \quad i = 1, 2 \Rightarrow X_1, X_2 \in \Gamma$. However, for

$$X = \frac{1}{2}X_1 + \frac{1}{2}X_2 = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix},$$

$$\lambda_1(X) = -3, \lambda_2(X) = 1 \Rightarrow X \notin \Gamma.$$

6.4 Convex Functionals

Comparing vectors: Let Γ be a *convex cone* defined in X . Given $x, y \in X$ we say that $x \succeq y$ if $(x - y) \in \Gamma$ (Particular case: $x \succeq 0 \Leftrightarrow x \in \Gamma$). The convex cone Γ is usually referred to as a *positive cone*.

Definition: Let X and Z be vector spaces with $F = \mathbb{R}$. Assume that Z is equipped with a positive cone Γ . The mapping $f : \Omega \rightarrow Z$, where $\Omega \subseteq X$ is a convex set, is said to be *convex* if

$$f(\alpha x_1 + (1 - \alpha)x_2) \preceq \alpha f(x_1) + (1 - \alpha)f(x_2), \forall x_1, x_2 \in \Omega, \alpha \in [0, 1].$$

Proposition: If the functional $f : \Omega \rightarrow Z$ is convex then the set

$$\{x : x \in \Omega, f(x) \preceq y\}$$

is a convex set for all $y \in Z$.

Examples:

1. $f : \mathbb{S}^n \rightarrow \mathbb{S}^n, f(X) := A^T X + X A + Q$.
2. $f : \mathbb{S}^n \rightarrow \mathbb{S}^n, f(X) := X R X - A^T X - X A - Q$.
3. $f : \Omega \times \mathbb{R}^{m \times n} \rightarrow \mathbb{S}^m, \Omega := \{X : X \in \mathbb{S}^n, X \succ 0\}, f(X) := Y X^{-1} Y^T$.
4. $f : \Omega \times \mathbb{R}^{n \times n} \rightarrow \mathbb{S}^m, \Omega := \{(X, Z) : X, Z \in \mathbb{S}^n, X \succ 0, Z \succ Y X^{-1} Y^T\},$

$$f(X) := Z + (Y - X) (Z - Y X^{-1} Y^T)^{-1} (Y - X)^T.$$

6.5 Positivity for Symmetric Matrices

$\mathbb{S}^n$: symmetric matrices of dimensions $n \times n$

Γ : Positive Cone on $\mathbb{S}^n$

$$\begin{aligned}\Gamma &:= \{X : X \in \mathbb{S}^n, \quad x^T X x \geq 0, \forall x \in \mathbb{R}^n\}, \\ \text{int } \Gamma &= \{X : X \in \mathbb{S}^n, \quad x^T X x > 0, \forall x \in \mathbb{R}^n, x \neq 0\}.\end{aligned}$$

Positiveness on $\mathbb{S}^n$:

Given $X \in \mathbb{S}^n$ the following statements are equivalent:

- | | |
|---|---|
| a) X is <i>positive semidefinite</i> . | a) X is <i>positive definite</i> . |
| b) $X \in \Gamma \quad (X \succeq 0)$. | b) $X \in \text{int } \Gamma \quad (X \succ 0)$. |
| c) $\lambda_{\min}(X) \geq 0 \quad (\lambda_{\max}(-X) \leq 0)$. | c) $\lambda_{\min}(X) > 0 \quad (\lambda_{\max}(-X) < 0)$. |

Negativeness on $\mathbb{S}^n$:

A matrix $X \in \mathbb{S}^n$ is *negative (semi)definite* if $(-X \succeq 0) \rightarrow -X \succ 0$;

Notation: $X \prec 0$ ($X \preceq 0$).

6.6 Linear Matrix Inequalities

Affine functional:

The functional $f : X \rightarrow \mathbb{S}^n$ is *affine* if

$$f(\alpha x + (1 - \alpha)y) = \alpha f(x) + (1 - \alpha)f(y), \quad \forall x, y \in X, \alpha \in \mathbb{R},$$

Proposition: The set $\{x : f(x) \preceq 0\}$ (or $\{x : f(x) \prec 0\}$), where $f : X \rightarrow \mathbb{S}^n$ is an affine functional, is a convex set.

Proposition: The set $\{x : f(x) \succeq 0\}$ (or $\{x : f(x) \succ 0\}$), where $f : X \rightarrow \mathbb{S}^n$ is an affine functional, is a convex set.

Proof: An affine functional is simultaneously convex and concave.

Definition: Let $f : X \rightarrow \mathbb{S}^n$ be an affine functional. The functional inequalities

$$f(x) \preceq 0 \quad \text{or} \quad f(x) \succeq 0$$

are known as *Linear Matrix Inequalities* (LMI). The functional inequalities

$$f(x) \prec 0 \quad \text{or} \quad f(x) \succ 0$$

are *strict* LMIs.

Examples:

1. $f : \mathbb{S}^n \rightarrow \mathbb{S}^n, \quad f(X) := A^T X + X A + Q \preceq 0.$
2. $f : \mathbb{S}^n \rightarrow \mathbb{S}^n, \quad f(X) := A^T X + X A \prec 0.$
3. $f : \mathbb{S}^n \rightarrow \mathbb{S}^n, \quad f(X) := A^T X A - X + Q \preceq 0.$
4. $f : \mathbb{S}^n \rightarrow \mathbb{S}^{2n}, \quad f(X) := \begin{bmatrix} X & A^T X \\ X A & X \end{bmatrix} \succ 0.$

6.7 Semidefinite Programming (SDP)

Proposition: Any affine functional $F : X \rightarrow \mathbb{S}^n$ where X is a real *finite dimensional space* can be put in the canonical form

$$F(x) \succeq 0, \quad F(x) := F_0 + \sum_{i=1}^m F_i x_i, \quad F_i \in \mathbb{S}^n, x \in \mathbb{R}^m.$$

Examples:

$$1. G(X) := A^T X + X A + Q \preceq 0, \quad X \in \mathbb{S}^2.$$

$$\begin{aligned} X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x_3 = \sum_{i=1}^3 X_i x_i, \quad x \in \mathbb{R}^3. \\ \implies -G(X) &= \underbrace{-Q}_{F_0} + \sum_{i=1}^3 \underbrace{-(A^T X_i + X_i A)}_{F_i} x_i = F(x) \end{aligned}$$

$$2. G(\mu, X) := \mu - \text{trace}(CXC^T) \succeq 0, \quad \mu \in \mathbb{R}, X \in \mathbb{S}^2.$$

$$G(\mu, X) = \mu - \text{trace}(CXC^T) = \underbrace{1}_{F_0} \mu + \sum_{i=1}^3 \underbrace{-\text{trace}(CX_i C^T)}_{F_i} x_i = F(\mu, x)$$

Canonical Semidefinite Program:

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \quad & c^T x \\ \text{s.t.} \quad & F(x) \succeq 0 \end{aligned}$$

Remarks:

- a) Several LMI $F_j(x) \succeq 0$, $j = 1, \dots, N$ can be handled by stacking $F_j(x)$ as a block diagonal constraint

$$F(x) = \text{blockdiag}(F_1(x), \dots, F_j(x), \dots, F_N) \succeq 0.$$

- b) Some LMI may have equality constraints (LME). Examples

1. $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{S}^n$, $f(X) := A^T X + X A + Q \preceq 0$, $X = X^T$.
2. $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{S}^n$, $f(X) := A^T X + X A \prec 0$, $CX = B^T$.

- c) Use a parser to generate the F_i s, e.g. YALMIP.

- d) WARNING: Do not use Matlab's LMIToolbox :(

6.8 SDP with equality constraints

Canonical SDP with equality constraints:

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \quad & c^T x \\ \text{s.t.} \quad & F(x) \succeq 0 \\ & Ax = b \end{aligned}$$

Conceptual elimination of equality constraints: Solve $Ax = b$

$$\tilde{x}(y) = A^\dagger b + A^\perp y, \quad y \in \mathbb{R}^p \quad (p = \dim A^\perp)$$

then

$$\begin{aligned} c^T \tilde{x}(y) &= \tilde{c}_0 + \tilde{c}^T y, & \tilde{c}_0 &:= c^T A^\dagger b, & \tilde{c} &:= A^{\perp T} c, \\ \tilde{F}(y) &:= F(\tilde{x}(y)), \end{aligned}$$

Notice that

$$\min_{x \in \mathbb{R}^n, Ax=b} c^T x = \min_{y \in \mathbb{R}^p} c^T \tilde{x}(y) = \min_{y \in \mathbb{R}^p} (\tilde{c}_0 + \tilde{c}^T y) = \tilde{c}_0 + \min_{y \in \mathbb{R}^p} \tilde{c}^T y$$

This provides the standard SDP

$$\begin{aligned} \min_{y \in \mathbb{R}^p} \quad & \tilde{c}^T y \\ \text{s.t.} \quad & \tilde{F}(y) \succeq 0. \end{aligned}$$

A good solver/parser will do that for you!

6.9 Some useful tools

6.9.1 Congruence Transformation

Definition: Two matrices $X, Y \in \mathbb{S}^n$ are said to be *congruent* if there exists a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ such that $Y = T^T X T$.

Proposition: If X and Y are congruent then $Y \succ 0$ if, and only if, $X \succ 0$.

Proof: If $X \succ 0$ then $x^T X x > 0, \forall x \in \mathbb{R}^n, x \neq 0$. Since X and Y are congruent there exists T nonsingular such that $Y = T^T X T$. Hence, using the fact that T is nonsingular, for all $x \neq 0$, the vector $y := T^{-1}x \neq 0$ and

$$X \succ 0 \Leftrightarrow x^T X x = y^T T^T X T y = y^T Y y > 0 \Leftrightarrow Y \succ 0$$

Generalization: For any $T \in \mathbb{R}^{m \times n}$, the matrix $Y = T^T X T \succeq 0$ if $X \succeq 0$. The converse is false when T does not have full row rank; example: $t = 0, x = -1, y = t^2 x \geq 0$.

6.9.2 Schur Complement

Lemma: For all $X \in \mathbb{S}^n, Y \in \mathbb{R}^{m \times n}, Z \in \mathbb{S}^m$, the following statements are equivalent:

- a) $Z \succ 0, \quad X - Y^T Z^{-1} Y \succ 0.$ a) $Z \succ 0, \quad X - Y^T Z^{-1} Y \succeq 0.$
- b) $\begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succ 0.$ b) $Z \succ 0, \quad \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succeq 0.$

Proof: Assume $Z \succ 0$. The *nonsingular matrix*

$$T = \begin{bmatrix} I & 0 \\ -Z^{-1}Y & I \end{bmatrix}$$

establishes the congruence transformation

$$T^T \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} T = \begin{bmatrix} X - Y^T Z^{-1} Y & 0 \\ 0 & Z \end{bmatrix} \succ 0 (\succeq 0).$$

6.10 Scope of Semidefinite Programming

Problem: Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \quad & c^T x \\ \text{s.t.} \quad & Ax \succeq b \end{aligned}$$

Equivalent SDP:

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \quad & c^T x \\ \text{s.t.} \quad & F(x) \succeq 0, \end{aligned}$$

where $F(x) := \text{blockdiag}(A_i x - b_i)$, A_i is the i th row of A .

Problem: Quadratic Programming (convex)

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \quad & x^T Q^T Q x + c^T x \\ \text{s.t.} \quad & Ax \succeq b \end{aligned}$$

Using Schur complement

$$\gamma \geq x^T Q^T Q x + c^T x \quad \Leftrightarrow \quad F_Q(\gamma, x) := \begin{bmatrix} \gamma - c^T x & x^T Q^T \\ Qx & I \end{bmatrix} \succeq 0$$

Equivalent SDP:

$$\begin{aligned} \min_{\gamma \in \mathbb{R}, x \in \mathbb{R}^m} \quad & \gamma \\ \text{s.t.} \quad & F(x) \succeq 0, \end{aligned}$$

where $F(\gamma, x) = \text{blockdiag}(F_Q(\gamma, x), A_i x - b_i)$, A_i is the i th row of A .

Problem: Minimum maximum eigenvalue problem

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \max_i \lambda_i(F(x)) \\ \text{s.t. } F(x) \in \mathbb{S}^n \end{aligned}$$

For any $X \in \mathbb{S}^n$

$$\min_i \lambda_i(X)I \preceq X \preceq \max_i \lambda_i(X)I$$

hence any γ such that

$$\gamma I \succeq F(x)$$

is also such that $\gamma \geq \max_i \lambda_i(F(x))$.

Equivalent SDP:

$$\begin{aligned} \min_{\gamma \in \mathbb{R}, x \in \mathbb{R}^m} \gamma \\ \text{s.t. } \gamma I \succeq F(x). \end{aligned}$$

Problem: Minimum Matrix Norm

$$\min_{X \in \mathbb{R}^{m \times n}} \|A - BXC\|^2, \quad \text{where } \|M\| := \lambda_{\max}^{1/2}[M^T M]$$

Notice that

$$\gamma \geq \lambda_{\max} [(A - BXC)^T (A - BXC)] \Leftrightarrow \gamma I \succeq (A - BXC)^T (A - BXC)$$

Using Schur complement

$$\gamma I \succeq (A - BXC)^T (A - BXC) \Leftrightarrow F(\gamma, X) := \begin{bmatrix} \gamma I & (A - BXC)^T \\ (A - BXC) & I \end{bmatrix} \succeq 0.$$

Equivalent SDP

$$\begin{aligned} \min_{\gamma \in \mathbb{R}, X \in \mathbb{R}^{m \times n}} \gamma \\ \text{s.t. } F(\gamma, X) \succeq 0, \end{aligned}$$

6.11 Feasibility Semidefinite Program

Problem: Find $x \in \mathbb{R}^m$ such that $F(x) \succeq 0$.

Solution: Introduce the scalar $\gamma \in \mathbb{R}$ and solve the *Optimization* SDP

$$\begin{aligned} \min_{\gamma \in \mathbb{R}, x \in \mathbb{R}^m} \quad & \gamma \\ \text{s.t.} \quad & F(x) + \gamma I \succeq 0 \end{aligned} \tag{13}$$

Given x_0 not feasible then compute $\max_i \lambda_i(F(x_0))I$. Then

$$\mu_0 := \min_i \lambda_i(F(x_0)) - 1 \quad \implies \quad \mu_0 I \prec \min_i \lambda_i(F(x_0)) \preceq F(x_0)$$

Therefore $\gamma_0 = -\mu_0$, x_0 are such that

$$F(x_0) + \gamma_0 I \succ 0$$

i.e., (γ_0, x_0) are strictly feasible solutions to the SDP. Most algorithms for SDP (interior-point methods) require a strictly feasible initial solution.

Let $(\tilde{\gamma}, \tilde{x})$ be the optimal solution to SDP (13). Then

- a) If $\tilde{\gamma} > 0$ then there is no feasible solution to the original problem.
- b) If $\tilde{\gamma} = 0$ then $\tilde{x}$ is a feasible solution to the original problem ($F(x) \succeq 0$).
- c) If $\tilde{\gamma} < 0$ then $\tilde{x}$ is a *strictly feasible* solution to the original problem ($F(x) \succ 0$).