2 H_2 and H_∞ Spaces

2.1 Complex-valued Functions

Let $S \subset \mathbb{C}$ and $f(s) : S \to \mathbb{C}$.

f is *analytic at* $s_0 \in S$ if it is differentiable at s_0

If f is analytic at $s_0 \in S$ then it has continuous derivatives of all orders at s_0

If f is analytic at $s_0 \in S$ then it addmits a *power series expansion* at s_0; The converse is also true

f is *analytic in* S if it is analytic at all $s \in S$

Maximum Modulus Theorem: Let S be a closed and bounded set and f be analytic in the interior of S ($\operatorname{int} S$). The function $|f(s)|$ attain its maximum in $\operatorname{int} S$ if and only if $f(s)$ is constant.

Cauchy Integral Theorem: Let S be a simply connected region and f be analytic in $\operatorname{int} S$. The integral of f on any closed curve in S is zero.

Examples

1. e^s is analytic everywhere in $\mathbb{C}$
2. $e^s = \sum_{i=0}^{\infty} \frac{s^i}{i!}$ for all $s \in \mathbb{C}$
3. $\frac{1}{s-a}$ is analytic in $\mathbb{C} \backslash \{a\}$

2.2 Linear Vector Spaces

Things we should know:

1. What is a linear vector space?
2. What is a subspace?
3. What is the dimension of a linear space (subspace)?
4. Let X be a vector space over $\mathbb{C}$
   (a) $\|x\| : X \to \mathbb{R}$ is a *norm* if
      i. $\|x\| \geq 0 \quad \forall x \in X, \quad \|x\| = 0 \Leftrightarrow x = 0$
      ii. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$ (triangle inequality)
      iii. $\|\alpha x\| = |\alpha|\|x\| \quad \forall \alpha \in \mathbb{C}, x \in X$
   (b) $\langle x, y \rangle : X \times X \to \mathbb{C}$ is an *inner product* if
      i. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
      ii. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
      iii. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \forall \alpha \in \mathbb{C}$
      iv. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
5. Normed Spaces: vector space + norm

Examples (Linear Vector Spaces):

1. $\mathbb{C}^n$ and $\mathbb{C}^{m \times n}$
2. $x = p(t) \in \mathbb{C}^{m \times n} \quad \forall\, t \in [a, b] \quad p(\cdot)$ is a function in $[a, b]$
3. $x = p(s) \in \mathbb{C}^{m \times n} \quad \forall\, s \in \mathbb{C} \quad p(\cdot)$ is a polynomial

Examples (Norms)

1. $\mathbb{C}^n: \quad \|x\| = \sum_i |x_i|, \quad \|x\| = \max_i |x_i|$
2. $\mathbb{C}^{m \times n}: \quad \|x\| = \max_i \sum_j |x_{ij}|, \quad \|x\| = \max_j \sum_i |x_{ij}|$
3. $\|x\| = \sup_{t \in [a,b]} \|x(t)\|$

2.3 $L_\infty[a,b]$ Space

Linear Vector Space: real-valued functions

Norm: $\|f\| = \sup_{t\in[a,b]} \overline{\sigma}[f(t)]$

$\overline{\sigma}(X)$ is the maximum singular value of the complex matrix X

L_∞ Space: all f such that $\|f\| < \infty$

Examples:

1. $f(t) = t^2 - t + 3 \in L_\infty[0,1]$
2. $f(t) = t^{-1} \notin L_\infty(-\infty,\infty)$, $f \in L_\infty[1,\infty)$
3. $f(t) = t^{-1/2} \in L_\infty[1,\infty)$
4. $f(t) = e^{-t} \in L_\infty[0,\infty)$
5. $f(t) = e^{t} \notin L_\infty[0,\infty)$
6. $f(t) = e^{-|t|} \in L_\infty(-\infty,\infty)$

2.4 $L_\infty(j\mathbb{R})$ Space

Linear Vector Space: complex-valued functions

Norm: $\|F\| = \sup_{\omega\in\mathbb{R}} \overline{\sigma}[F(j\omega)]$

$\overline{\sigma}(X)$ is the maximum singular value of the complex matrix X

$L_\infty(j\mathbb{R})$ Space: all F such that $\|F\| < \infty$

$RL_\infty(j\mathbb{R})$ Space: F is rational and $F \in L_\infty(j\mathbb{R})$; $RL_\infty(j\mathbb{R}) \subset L_\infty(j\mathbb{R})$

SISO $RL_\infty(j\mathbb{R})$ = rational, proper, no poles on the imaginary axis

Examples:

1. $F(s) = \frac{1}{s+2} \in RL_\infty(j\mathbb{R})$
2. $F(s) = \frac{1}{s-2} \in RL_\infty(j\mathbb{R})$
3. $F(s) = \frac{s+1}{s} \notin L_\infty(j\mathbb{R})$
4. $F(s) = \frac{e^{-s}}{s+2} \in L_\infty(j\mathbb{R}), \quad F \notin RL_\infty(j\mathbb{R})$
5. $F(s) = \begin{bmatrix} \frac{(s-3)(s+1)}{(s-5)(s-2)} \\ \frac{(s-3)}{(s-5)} \end{bmatrix} \in RL_\infty(j\mathbb{R})$

2.5 H_∞ Space

Linear Vector Space: complex-valued functions, analytic in $\mathrm{Re}(s) > 0$

Norm:

$$\begin{aligned} \|F\| &= \sup_{\mathrm{Re}(s)>0} \overline{\sigma}[F(s)] \\ &= \sup_{\omega \in \mathbb{R}} \overline{\sigma}[F(j\omega)] \end{aligned}$$

H_∞ Space: F analytic in $\mathrm{Re}(s) > 0$ and $\|F\| < \infty$; $H_\infty \subset L_\infty(j\mathbb{R})$

RH_∞ Space: F is rational and $F \in H_\infty$; $RH_\infty \subset H_\infty$

SISO RH_∞ : rational, proper and asymptotically stable

Examples:

1. $F(s) = \frac{1}{s+2} \in RH_\infty$
2. $F(s) = \frac{1}{s-2} \notin H_\infty$
3. $F(s) = \frac{e^{-s}}{s+2} \in H_\infty, \quad F \notin RH_\infty$
4. $F(s) = \begin{bmatrix} \frac{(s-3)(s+1)}{(s+5)(s+2)} \\ \frac{(s-3)}{(s+5)} \end{bmatrix} \in RH_\infty$

2.6 Pre-Hilbert and Hilbert Spaces

Pre-Hilbert Spaces: vector space + inner product

The function $\|x\| := \sqrt{\langle x, x \rangle}$ is a norm

(Pre-)Hilbert Spaces are normed spaces

Pre-Hilbert Space + completeness = Hilbert Space

Some facts:

1. x, y are *orthogonal* if $\langle x, y \rangle = 0$. $(x \perp y)$
2. $| \langle x, y \rangle | \leq \|x\| \|y\|$; (Cauchy-Schwarz Inequality)
   $| \langle x, y \rangle | = \|x\| \|y\|$ iff $x = \alpha y$, $\alpha \in \mathbb{C}$, or $x(y) = 0$
3. $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (Parallelogram Law)
4. $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if $x \perp y$ (Pythagorean Theorem)

Projection Theorem: Let S be a closed subspace of a Hilbert space X. For a given $x \in X$ there is a *unique* vector $\bar{s} \in S$ such that $\|x - \bar{s}\| \leq \|x - s\|, \forall s \in S$, or, in other words, $\bar{s} = \arg\inf_{s \in S} \|x - s\|$. This vector is such that $(x - \bar{s}) \perp S$.

Examples (Inner Products)

1. $\mathbb{C}^n : \quad \langle x, y \rangle = x^* y$
2. $\mathbb{C}^{m \times n} : \quad \langle x, y \rangle = \mathrm{trace}(x^* y)$
3. $L_2[a, b] : \quad \langle x, y \rangle = \int_a^b \mathrm{trace}[x^*(t) y(t)] \, dt$

2.7 $L_2[a,b]$ Space

Linear Vector Space: real-valued functions in $[a,b]$

Inner product:

$$\langle f, g\rangle = \int_a^b \mathrm{trace}[f^T(t)g(t)]\,dt$$

Norm: induced by the inner product (Hilbert space)

$L_2[a,b]$ Space: f such that $\|f\| < \infty$

Examples:

1. $f(t) = t^2 - t + 3 \in L_2[0,1]$
2. $f(t) = t^{-1} \notin L_2[0,\infty)$, $f \in L_2[1,\infty)$, $(\int_1^\infty t^{-2}\,dt = -t^{-1}\big|_1^\infty = 1)$
3. $f(t) = t^{-1/2} \notin L_2[1,\infty)$ $(\int_1^\infty t^{-1}\,dt = \log t|_1^\infty = \infty)$
4. $f(t) = e^{-t} \in L_2(0,\infty)$
5. $f(t) = e^{t} \notin L_2(0,\infty)$
6. $f(t) = e^{-|t|} \in L_2(-\infty,\infty)$

2.8 $L_2(j\mathbb{R})$ Space

Linear Vector Space: complex-valued functions

Inner product:

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{trace}[F^{\sim}(j\omega) G(j\omega)]\, d\omega$$

Norm: induced by the inner product (Hilbert space)

$L_2(j\mathbb{R})$ Space: F such that $\|F\| < \infty$

$RL_2(j\mathbb{R})$ Space: F is rational and $F \in L_2(j\mathbb{R})$; $RL_2(j\mathbb{R}) \subset L_2(j\mathbb{R})$

For SISO $RL_2(j\mathbb{R}) =$ strictly proper, no poles on the imaginary axis

Parseval's Theorem: Let $f \in L_2(-\infty, \infty)$ and $F(s)$ be its Laplace Transform. Then $F \in L_2(j\mathbb{R})$ and $\|f\| = \|F\|$.

Examples:

1. $F(s) = \frac{1}{s+2} \in RL_2(j\mathbb{R})$
2. $F(s) = \frac{1}{s-2} \in RL_2(j\mathbb{R})$
3. $F(s) = \frac{1}{s(s+1)} \notin RL_2(j\mathbb{R})$
4. $F(s) = \frac{s+1}{s+2} \notin L_2(j\mathbb{R})$ $(F(j\infty) \to 1!)$
5. $F(s) = e^{-s} \notin L_2(j\mathbb{R})$ $(F^{\sim}(j\omega)F(j\omega) = 1 \quad \forall \omega!)$
6. $F(s) = \frac{e^{-s}}{s-2} \in L_2(j\mathbb{R})$

2.9 H_2 Space

Linear Vector Space: complex-valued functions, analytic in $\mathrm{Re}(s) > 0$

Inner product:

$$\begin{aligned} \langle F, G \rangle &= \frac{1}{2\pi} \sup_{\sigma > 0} \left\{ \int_{-\infty}^{\infty} \mathrm{trace}[F^{\sim}(\sigma + j\omega) G(\sigma + j\omega)] \, d\omega \right\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{trace}[F^{\sim}(j\omega) G(j\omega)] \, d\omega \end{aligned}$$

Norm: induced by the inner product (Hilbert space)

H_2 Space: F analytic in $\mathrm{Re}(s) > 0$ and $\|F\| < \infty$; $H_2 \subset L_2(j\mathbb{R})$

RH_2 Space: F is rational and $F \in H_2$; $RH_2 \subset H_2$

For SISO $RH_2 =$ rational, strictly proper and asymptotically stable

Parseval's Theorem: Let $f \in L_2[0, \infty)$ and $F(s)$ be its Laplace Transform. Then $F \in H_2$ and $\|f\| = \|F\|$.

Examples:

1. $F(s) = \frac{1}{s+2} \in RH_2$
2. $F(s) = \frac{1}{s-2} \notin RH_2$ (not stable!)
3. $F(s) = \frac{s+1}{s+2} \notin H_2$ $(F(j\infty) \to 1!)$
4. $F(s) = e^{-s} \notin H_2$ $(F^{\sim}(j\omega)F(j\omega) = 1 \quad \forall \omega!)$
5. $F(s) = \frac{e^{-s}}{s+2} \in H_2$
6. $F(s) = \begin{bmatrix} \frac{(s+1)}{(s+2)} \\ 1 \end{bmatrix} \notin RH_2,$ $\left(F^{\sim}(j\omega)F(j\omega) = 1 + \frac{|j\omega+1|^2}{|j\omega+2|^2} \geq 1 \quad \forall \omega! \right)$

2.10 $H_2^\perp$ Space

Linear Vector Space: complex-valued functions, analytic in $\mathrm{Re}(s) < 0$

Inner product:

$$\begin{aligned}\langle F, G\rangle &= \frac{1}{2\pi}\sup_{\sigma<0}\left\{\int_{-\infty}^{\infty}\mathrm{trace}[F^{\sim}(\sigma+j\omega)G(\sigma+j\omega)]\,d\omega\right\}\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{trace}[F^{\sim}(j\omega)G(j\omega)]\,d\omega\end{aligned}$$

Norm: induced by the inner product (Hilbert space)

$H_2^\perp$ Space: F analytic in $\mathrm{Re}(s) < 0$ and $\|F\| < \infty$; $H_2^\perp \subset L_2(j\mathbb{R})$

$RH_2^\perp$ Space: F is rational and $F \in H_2^\perp$; $RH_2^\perp \subset H_2^\perp$

For SISO $RH_2^\perp =$ rational, strictly proper and asymptotically anti-stable

Parseval's Theorem: Let $f \in L_2(-\infty, 0]$ and $F(s)$ be its Laplace Transform. Then $F \in H_2^\perp$ and $\|f\| = \|F\|$.

Examples:

1. $F(s) = \frac{1}{s+2} \notin RH_2^\perp$ (stable!)
2. $F(s) = \frac{1}{s-2} \in RH_2^\perp$
3. $F(s) = \frac{s+1}{s-2} \notin H_2^\perp$ $(F(j\infty) \to 1!)$
4. $F(s) = e^{-s} \notin H_2^\perp$ $(F^{\sim}(j\omega)F(j\omega) = 1 \quad \forall\omega!)$
5. $F(s) = \frac{e^{-s}}{s+2} \in H_2^\perp$
6. $F(s) = \begin{bmatrix}\frac{(s+1)}{(s-2)}\\ 1\end{bmatrix} \notin RH_2,$ $\left(F^{\sim}(j\omega)F(j\omega) = 1 + \frac{|j\omega+1|^2}{|j\omega-2|^2} \geq 1 \quad \forall\omega!\right)$

2.11 Relations between H_2, $H_2^\perp$ and $L_2(j\mathbb{R})$ spaces

Lemma: Let $G(s) \in RH_2^\perp$. Then $G^\sim(s) \in RH^2$.
Proof: Because $G^\sim(s) = G^T(-s)$, if $\bar{s}$ is a pole of $G(s)$ then $-\bar{s}$ is a pole of $G^\sim(s)$.

Lemma: Let $F(s) \in RH_2$ and $G(s) \in RH_2^\perp$. Then $F(s) \perp G(s)$.
Proof: Consider the semicircle C_ρ of radius ρ centered at the origin and entirely contained in the right hand side of the complex plane. Because $G^\sim(s) \in RH_2$, the rational scalar function

$$T(s) = \mathrm{trace}[F(s)G^\sim(s)] \in RH_2.$$

Therefore $T(s)$ is analytic in C_ρ for any value of $\rho > 0$, and the Cauchy Integral Theorem states that

$$\oint_{C_\rho} T(s)\, ds = \int_{-\rho}^{\rho} T(j\omega)\, d\omega - \int_{-\pi/2}^{\pi/2} T(\rho e^{j\theta})\, d\theta = 0$$

Since $T(s)$ is in RH_2 it is strictly proper, hence

$$\lim_{\rho \to \infty} \int_{-\pi/2}^{\pi/2} T(\rho e^{j\theta})\, d\theta = 0.$$

Consequently

$$\langle F(s), G(s) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(j\omega)\, d\omega = \oint_{C_\infty} T(s)\, ds = 0$$

Corollary: $RL_2(j\mathbb{R}) = RH_2 \oplus RH_2^\perp$
Proof: Using *partial fractions expansion*, any $H(s) \in RL_2(j\mathbb{R})$ can be written

$$H(s) = H_\mathrm{s}(s) + H_\mathrm{a}(s)$$

where $H_\mathrm{s}(s) \in RH_2$ and $H_\mathrm{a}(s) \in RH_2^\perp$, so that $H_\mathrm{s}(s) \perp H_\mathrm{a}(s)$.

Corollary: Let $H(s) = H_\mathrm{s}(s) + H_\mathrm{a}(s) \in RL_2(j\mathbb{R})$ where $H_\mathrm{s}(s) \in RH_2$ and $H_\mathrm{a}(s) \in RH_2^\perp$. Then $\|H\|^2 = \|H_\mathrm{s}(s)\|^2 + \|H_\mathrm{a}(s)\|^2$

Example:

$$H(s) = \frac{2s}{(s+1)(s-1)} = \underbrace{\frac{1}{s+1}}_{H_\mathrm{s}(s)} \quad + \quad \underbrace{\frac{1}{s-1}}_{H_\mathrm{a}(s)}$$

Note that

$$\begin{aligned}
\|H_\mathrm{s}\|^2 = \left\|(s+1)^{-1}\right\|^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2+1}\, d\omega \\
&= \frac{1}{2\pi} \arctan \omega|_{-\infty}^{\infty} = \frac{1}{2\pi}(\frac{\pi}{2} - \frac{-\pi}{2}) = \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\|H\|^2 &= \|H_\mathrm{s}\|^2 + \|H_\mathrm{a}\|^2 \\
&= \|H_\mathrm{s}\|^2 + \|H_\mathrm{a}^\sim\|^2 \\
&= \left\|(s+1)^{-1}\right\|^2 + \left\|-(s+1)^{-1}\right\|^2 \\
&= 2\left\|(s+1)^{-1}\right\|^2 = 1
\end{aligned}$$

2.12 Linear Operators

Definition: A linear map between two linear spaces. $T : X \to Y$

Notation: $T : X \to Y$, $y = Tx$

Example: A linear time-invariant system H! Takes a signal $u \in L_2[0, \infty)$ and produces a $y = Hu \in L_2[0, \infty)$.

Operator norm: When X and Y are normed spaces

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|$$

Submultiplicative property: $\|AB\| \leq \|A\|\|B\|$
Proof: Note that for any $x \in X$

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \geq \frac{\|Tx\|}{\|x\|} \quad \Rightarrow \quad \|Tx\| \leq \|T\|\|x\|$$

Therefore

$$\begin{aligned} \|AB\| &= \sup_{x \neq 0} \frac{\|ABx\|}{\|x\|} \\ &\leq \sup_{x \neq 0} \frac{\|A\|\|Bx\|}{\|x\|} \\ &\leq \|A\| \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} \\ &\leq \|A\|\|B\| \end{aligned}$$

Example: $T : \mathbb{C}^n \to \mathbb{C}^n$, $y = Tx$
Since

$$\|Tx\|^2 = x^* T^* T x \leq \max_i \lambda_i(T^* T)\, x^* x = \overline{\sigma}^2(T)\, \|x\|^2$$

with equality holding when x is an eigenvector of T

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \overline{\sigma}(T)$$

2.13 H_∞ norm as an operator norm

Theorem: Let $H(s)$ be the transfer function of a linear time-invariant system. Let the input $u \in L_2[0,\infty)$ and assume the output $y \in L_2[0,\infty)$. The norm of the linear operator $H : L_2[0,\infty) \to L_2[0,\infty)$ is equal to the H_∞ norm of the transfer function $H(s)$.

Half proof: First note that

$$\|H\| = \sup_{\|u\|=1} \|y\|, \quad u \in L_2[0,\infty), \quad y = Hu \in L_2[0,\infty)$$

As u and y are in $L_2[0,\infty)$ compute the corresponding pair of unilateral Laplace transform U, $Y \in H_2$. Then use Parseval's theorem to compute

$$\|H\| = \sup_{\|U\|=1} \|Y\|, \quad U(s) \in H_2, \quad Y(s) = H(s)U(s) \in H_2$$

Now note that for any U

$$\begin{aligned}
\|Y\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}[U^\sim(j\omega)H^\sim(j\omega)H(j\omega)U(j\omega)]\, d\omega, \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}[U^\sim(j\omega) \sup_{\omega\in\mathbb{R}} [H^\sim(j\omega)H(j\omega)]\, U(j\omega)]\, d\omega, \\
&\leq \sup_{\omega\in\mathbb{R}} \overline{\sigma}^2[H(j\omega)]\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}[U^\sim(j\omega)U(j\omega)]\, d\omega, \\
&\leq \|H\|_\infty^2 \|U\|_2^2
\end{aligned}$$

which implies that

$$\|H\| = \sup_{U\neq 0} \frac{\|Y\|_2}{\|U\|_2} \leq \|H\|_\infty.$$

The second part is to prove that $\|H\| \geq \|H\|_\infty$. See Zhou p. 52 or Colaneri p. 53.

2.14 Inner Functions

Definition: A transfer function $F(s) \in RL_\infty$ is called *all-pass* if F is square and $F^\sim(s)F(s) = I$.

Definition: A transfer function $F(s) \in RH_\infty$ is called *inner* if $F^\sim(s)F(s) = I$.

Definition: A transfer function $F(s) \in RH_\infty$ is called *co-inner* if $F(s)F^\sim(s) = I$.

Definition: A transfer function $F(s) \in RH_\infty$ is called *outer* if there exists $X(s)$ analytic in $\mathrm{Re}(s) > 0$ such that $F(s)X(s) = I$.

Lemma: Let $F(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RH_\infty$ and let $X = X^T$ solve the Lyapunov equation

$$A^*X + XA + C^*C = 0$$

If $D^TC + B^TX = 0$ then $F^\sim(s)F(s) = D^TD$. If (A, B) is controlable and $F^\sim(s)F(s) = D^TD$ then $D^TC + B^TX = 0$.

Proof: Note that

$$\begin{aligned} F^\sim(s)F(s) &= \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\ &= \left[\begin{array}{cc|c} -A^T & -C^TC & -C^TD \\ 0 & A & B \\ \hline B^T & D^TC & D^TD \end{array}\right] \end{aligned}$$

Applying the similarity transformation defined by

$$T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

we obtain

$$F^{\sim}(s)F(s) = \left[\begin{array}{cc|c} -A^T & -A^TX - XA - C^TC & -C^TD - BX \\ 0 & A & B \\ \hline B^T & B^TX + D^TC & D^TD \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -A^T & 0 & -C^TD - BX \\ 0 & A & B \\ \hline B^T & B^TX + D^TC & D^TD \end{array}\right]$$

Corollary: Let $F(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RH_\infty$ be a minimal realization and $X = X^T$ solve the Lyapunov equation

$$A^*X + XA + C^*C = 0$$

$F(s)$ is inner if and only if $D^TC + B^TX = 0$ and $D^TD = I$.

Lemma: Let $F(s) \in RH_\infty$ be a scalar transfer function. There exists an inner function $F_i(s)$ and an outer function $F_o(s)$ such that $F(s) = F_i(s)F_o(s)$. If $F(j\omega) \neq 0$ then $F_o^{-1}(s) \in RH_\infty$.
Proof: Assume $F(s)$ has $m_r + 2m_c$ non-minimum phase zeros

$$\{\alpha_1, \ldots, \alpha_{m_r}, \quad \alpha_{m_r+1} \pm j\beta_{m_r+1}, \ldots, \alpha_{m_r+m_c} \pm j\beta_{m_r+m_c}\}$$

where $\alpha_i > 0$, $i = 1, \ldots, m_r + m_c$.
Define

$$F_i(s) = \prod_{i=1}^{m_r} \frac{s - \alpha_i}{s + \alpha_i} \prod_{i=m_r+1}^{m_r+m_c} \frac{s - \alpha_i \mp j\beta_i}{s + \alpha_i \pm j\beta_i}$$

Note that $F_i(s)$ is inner, i.e.,

$$F_i^{\sim}(s)F_i(s) = \prod_{i=1}^{m_r} \frac{s + \alpha_i}{s - \alpha_i}\frac{s - \alpha_i}{s + \alpha_i} \prod_{i=m_r+1}^{m_r+m_c} \frac{s + \alpha_i \pm j\beta_i}{s - \alpha_i \mp j\beta_i}\frac{s - \alpha_i \mp j\beta_i}{s + \alpha_i \pm j\beta_i} = 1$$

Now define

$$F_o(s) = F(s)F_i^{-1}(s).$$

All zeros of $F_o(s)$ have nonpositive real part. Therefore $F_o^{-1}(s)$ is analytic in $\mathrm{Re}(s) > 0$. If $F(j\omega) \neq 0$ then $F_o(s)$ has no zeros on the imaginary axis so that $F_o(s)^{-1} \in RH_\infty$.