3 Computing RH_2 and RH_∞ Norms

3.1 Sylvester and Lyapunov Equations

Lemma 1: Consider the Sylvester Equation

$$AX + XB = C$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times m}$ and $C \in \mathbb{C}^{n\times m}$. A solution $X \in \mathbb{C}^{n\times m}$ exists and is unique if and only if $\lambda_i(A) + \lambda_j(B) \neq 0$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Proof: The Sylvester Equation is a linear equation and it has a unique solution if and only if the homogeneous equation associated with the Sylvester equation admits only the trivial solution. Assume it does not, that is, there $\bar{X} \neq 0$ such that

$$A\bar{X} + \bar{X}B = 0$$

Then, multiplication of the above on the left by $v_i^* \neq 0$, the ith left eigenvector of A, and on the right by $u_j \neq 0$, the jth right eigenvector of B, yields

$$0 = v_i^* A\bar{X}u_j + v_i^*\bar{X}Bu_j = [\lambda_i(A) + \lambda_j(B)]\, v_i^*\bar{X}u_j.$$

Since $\lambda_i(A) + \lambda_j(B) \neq 0$ by hypothesis we must have $v_i^*\bar{X}u_j$ for all i, j. One can show that this indeed implies $\bar{X} = 0$, establishing a contradiction.

Lemma 2: Consider the Lyapunov Equation

$$A^T X + XA + C^T C = 0$$

where $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$.

1. A solution $X \in \mathbb{C}^{n\times n}$ exists and is unique if and only if $\lambda_i(A) + \lambda_j^*(A) \neq 0$ for all $i, j = 1, \ldots, n$. Furthermore X is symmetric.
2. If A is Hurwitz then X is positive semidefinite.
3. If (A, C) is detectable and X is positive semidefinite then A is Hurwitz.
4. If (A, C) is observable and A is Hurwitz then X is positive definite.

Proof:
Item 1. is a corollary of Lemma 1. That X is symmetric follows from unicity since

$$\begin{aligned} 0 &= (A^T X + XA + C^T C)^T - (A^T X + XA + C^T C) \\ &= A^T(X^T - X) + (X^T - X)A \end{aligned}$$

so that $X^T - X = 0$.

Item 2. If A is Hurwitz then

$$X = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt \succeq 0$$

and

$$A^T X + XA = \lim_{t\to\infty} \int_0^\infty \frac{d}{dt} e^{A^T t} C^T C e^{At}\, dt = e^{A^T t} C^T C e^{At}\Big|_0^\infty = -C^T C$$

since $\lim_{t\to\infty} e^A t = 0$.

Item 3. Assume (A, C) is detectable, $X \succeq 0$ and that A is not Hurwitz. Then there exists λ and $x \neq 0$ such that $Ax = \lambda x$ with $\lambda + \lambda^* \geq 0$. But if X solves the Lyapunov equation

$$-x^* C^T C x = x^* A^T X x + x^* X A x = (\lambda + \lambda^*)\, x^* X x \geq 0$$

which implies $Cx = 0$, hence (A, C) not detectable.

Item 4. Assume that (A, C) is observable and A is Hurwitz. From Item 2. $X \succeq 0$. Assume X is not positive definite, that is, there exists $\bar{x} \neq 0$ such that $X\bar{x} = 0$. It follows that

$$0 = \bar{x}^* X \bar{x} = \int_0^\infty \bar{x}^* e^{A^T t} C^T C e^{At} \bar{x}\, dt = \int_0^\infty y^*(t) y(t)\, dt$$

which implies that response $y(t) = Ce^{At}\bar{x} = 0$ to a non null initial condition $x(0) = \bar{x}$ is null, which contradicts the hypothesis that (A, C) is observable.

3.2 Computing the RH_2 norm

Let $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a realization of $H(s) \in RH_2$. It follows that A can be assumed to be Hurwitz and $D = 0$. From Parseval's Theorem

$$\|H\| = \|h\|$$

where $h = h(t) \in L_2[0, \infty)$ is the impulse response

$$h(t) = \begin{cases} 0, & t < 0 \\ Ce^{At}B, & t \geq 0 \end{cases}$$

Then

$$\begin{aligned} \|H\|^2 &= \|h\|^2 \\ &= \int_0^\infty \mathrm{trace}[h^T(t)h(t)]\, dt \\ &= \int_0^\infty \mathrm{trace}[B^T e^{A^T t} C^T C e^{At} B]\, dt, \\ &= \mathrm{trace}\left[B^T \left(\int_0^\infty e^{A^T t} C^T C e^{At}\, dt\right) B\right], \\ &= \mathrm{trace}\left[B^T X B\right], \qquad X = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt \succeq 0 \end{aligned}$$

Since A is Hurwitz X can be obtained by computing the unique[1] solution to the Lyapunov equation

$$A^T X + XA + C^T C = 0$$

[1]Because A is Hurwitz, $\mathrm{Re}[\lambda_i(A)] < 0$ so that $\lambda_i(A) + \lambda_j^*(A) < 0$ for any $i, j = 1, \ldots, n$.

Alternatively

$$
\begin{aligned}
\|H\|^2 &= \|h\|^2 \\
&= \int_0^\infty \operatorname{trace}[h(t)h^T(t)]\,dt \\
&= \operatorname{trace}\left[C\left(\int_0^\infty e^{At}BB^Te^{A^Tt}\,dt\right)C^T\right], \\
&= \operatorname{trace}\left[CYC^T\right], \qquad Y = \left(\int_0^\infty e^{At}BB^Te^{A^Tt}\,dt\right) \succeq 0
\end{aligned}
$$

and

$$AY + YA^T + BB^T = 0$$

3.3 Computing the RL_2 and $RH_2^\perp$ norms

If $H(s) \in RH_2^\perp$ then $H^\sim(s) \in RH_2$ and $\|H\| = \|H^\sim\|$

If $H(s) \in RL_2$ then find

$$H(s) = H_\mathrm{s}(s) + H_\mathrm{a}(s)$$

where $H_\mathrm{s}(s) \in RH_2$ and $H_\mathrm{a}(s) \in RH_2^\perp$.

$$\|H\| = \sqrt{\|H_\mathrm{s}\|^2 + \|H_\mathrm{a}\|^2} = \sqrt{\|H_\mathrm{s}\|^2 + \|H_\mathrm{a}^\sim\|^2}$$

3.4 Computing the RL_∞ norm

Theorem: Let $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a realization of $H(s)$. Assume A is Hurwitz and define

$$R_\gamma := \gamma^2 I - D^T D, \qquad\qquad F_\gamma := A + B R_\gamma^{-1} D^T C$$

The following statements are equivalent:

1. $\|H\| < \gamma$
2. $\overline{\sigma}(D) < \gamma$ and the Hamiltonean matrix
$$Z_\gamma := \begin{bmatrix} -F_\gamma^T & -C^T(I + D R_\gamma^{-1} D^T)C \\ B R_\gamma^{-1} B^T & F_\gamma \end{bmatrix}$$
has no eigenvalue on the imaginary axis
3. $\overline{\sigma}(D) < \gamma$ and there exists a symmetric positive semidefinite stabilizing solution to the ARE (Algebraic Riccati Equation)
$$F_\gamma^T X + X F_\gamma + X B R_\gamma^{-1} B^T X + C^T(I + D R_\gamma^{-1} D^T)C = 0$$

Proof: Let $\Phi(s) := \gamma^2 I - H^{\sim}(s)H(s)$. Note that

$$\overline{\sigma}[H(j\omega)] < \gamma \quad \iff \quad \Phi(j\omega) \succ 0 \quad \iff \quad \Phi(j\omega) \text{ is nonsingular}$$

That positivity of $\Phi(j\omega)$ implies nonsingularity is immediate. The converse is also true. To see that, assume $\gamma > \|H\|$ and that $\Phi(j\omega)$ is singular. Then there exists $x \neq 0$ such that $\Phi(j\omega)x = 0$. Without loss of generality one can assume $\|x\| = 1$ such that

$$0 = x^*\Phi(s)x = \gamma^2 - x^*H^{\sim}(j\omega)H(j\omega)x \geq \gamma^2 - \overline{\sigma}^2[H(j\omega)] \geq \gamma^2 - \|H\|^2$$

which implies $\gamma \leq \|H\|$, establishing a contradiction.
In other words, $\Phi(s)$ must have no zeros on the imaginary axis. Equivalently $\Phi^{-1}(s)$ has no poles on the imaginary axis. Let us build a realization for $\Phi^{-1}(s)$. First construct

$$\begin{aligned}
\Phi(s) &= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \gamma^2 I \end{array}\right] - \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\
&= \left[\begin{array}{cc|c} -A^T & -C^T C & C^T D \\ 0 & A & -B \\ \hline B^T & D^T C & \gamma^2 I - D^T D \end{array}\right]
\end{aligned}$$

and then

$$\begin{aligned}
\Phi^{-1}(s) &= \left[\begin{array}{cc|c} -A^T & -C^T C & C^T D \\ 0 & A & -B \\ \hline B^T & D^T C & R_\gamma \end{array}\right]^{-1} \\
&= \left[\begin{array}{c|c} \begin{bmatrix} -A^T & -C^T C \\ 0 & A \end{bmatrix} - \begin{bmatrix} C^T D \\ -B \end{bmatrix} R_\gamma^{-1} \begin{bmatrix} B^T & D^T C \end{bmatrix} & -\begin{bmatrix} C^T D \\ -B \end{bmatrix} R_\gamma^{-1} \\ \hline R_\gamma^{-1} \begin{bmatrix} B^T & D^T C \end{bmatrix} & R_\gamma^{-1} \end{array}\right] \\
&= \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{Z_γ}} & -C^T D R_\gamma^{-1} \\ & & B R_\gamma^{-1} \\ \hline R_\gamma^{-1} B^T & R_\gamma^{-1} D^T C & R_\gamma^{-1} \end{array}\right]
\end{aligned}$$

Note that $\|H\| < \gamma$ implies $\lim_{\omega\to\infty} \overline{\sigma}[H(j\omega)] = \overline{\sigma}(D) < \gamma$, which implies $R_\gamma \succ 0$, therefore R_γ is nonsingular.

To prove that $2. \Rightarrow 1.$ note that if the matrix Z_γ has no pole on the imaginary axis then $\Phi^{-1}(s)$ has no pole on the imaginary axis. To prove that $1. \Rightarrow 2.$ note that if $\Phi^{-1}(s)$ has no pole on the imaginary axis then matrix Z_γ has no pole on the imaginary axis if $\Phi^{-1}(s)$ is controllable and observable. To prove that $\Phi^{-1}(s)$ is observable assume it is not and that $j\bar{\omega}$ is an unobservable pole of $\Phi^{-1}(s)$. Then there exists $x \neq 0$ such that

$$Z_\gamma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = j\bar{\omega} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad R_\gamma^{-1} \begin{bmatrix} B^T & D^T C \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Expand

$$\begin{aligned} Z_\gamma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{bmatrix} -(A + BR_\gamma^{-1} D^T C)^T & -C^T (I + DR_\gamma^{-1} D^T) C \\ BR_\gamma^{-1} B^T & A + BR_\gamma^{-1} D^T C \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} -A^T x_1 - C^T C x_2 - C^T D R_\gamma^{-1} (B^T x_1 + D^T C x_2) \\ Ax_2 + BR_\gamma^{-1} (B^T x_1 + D^T C x_2) \end{pmatrix} \\ &= \begin{pmatrix} -A^T x_1 - C^T C x_1 \\ Ax_2 \end{pmatrix} \end{aligned}$$

so that

$$(j\bar{\omega} + A^T) x_1 = -C^T C x_2 \qquad (j\bar{\omega} - A) x_2 = 0$$

Since A is Hurwitz it has no eigenvalue on the imaginary axis which implies $x_2 = x_1 = 0$ which is a contradiction, and $\Phi^{-1}(s)$ has no unobservable poles on the imaginary axis. The proof for uncontrollable modes on the imaginary axis follow the same pattern.

The proof of equivalence of $2.$ and $3.$ comes with the next result. Note that the simmilarity transformation

$$\begin{aligned} &\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^{-1} \begin{bmatrix} -F_\gamma^T & -C^T (I + DR_\gamma^{-1} D^T) C \\ BR_\gamma^{-1} B^T & F_\gamma \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} F_\gamma & BR_\gamma^{-1} B^T \\ -C^T (I + DR_\gamma^{-1} D^T) C & -F_\gamma^T \end{bmatrix} \end{aligned}$$

puts Z_γ in the form in which the next results are presented.

3.5 Algebraic Riccati Equations

Lemma 1: Consider the *Hamiltonean matrix*

$$Z := \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}.$$

where A, $R = R^T$ and $Q = Q^T \in \mathbb{R}^{n\times n}$.

1. λ is an eigenvalue of Z if and only if $-\lambda$ is an eigenvalue of Z.
2. If Z has no eigenvalues on the imaginary axis then there exists a matrix $W \in \mathbb{R}^{n\times n}$ such that

$$ZV_1 = V_1 W \tag{1}$$

   where W is Hurwitz.

Proof:
Item 1. Z has eigenvalues pairs which are symmetric w.r.t the imaginary axis because

$$J^{-1}ZJ = -JZJ = -Z^T, \qquad J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad J^{-1} = -J$$

Item 2. Let Z_J be the Jordan form of matrix Z so that

$$ZV = VZ_J$$

where $V \in \mathbb{R}^{2n\times 2n}$ is a matrix whose columns are the (generalized) eigenvectors of Z. Since the eigenvalues of Z are symmetric with respect to the imaginary axis and there are no eigenvalues on the imaginary axis, there exists at least two distinct Jordan blocks

$$Z \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} Z_{J_-} & 0 \\ 0 & Z_{J_+} \end{bmatrix}$$

where all n eigenvalues of Z_{J_-} have negative real part, i.e., Z_{J_-} is Hurwitz. The first columns of the above equation are in the form (1) with $W = Z_{J_-}$ Hurwitz.

Lemma 2: Consider the Algebraic Riccati Equation (ARE)

$$A^T X + XA + XRX + Q = 0$$

where A, $R = R^T$ and $Q = Q^T \in \mathbb{R}^{n \times n}$ and the associated *Hamiltonean matrix*

$$Z := \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}.$$

which is assumed to have no eigenvalue on the imaginary axis.

1. Let
$$V_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{C}^{2n \times n}$$
be (generalized) eigenvectors of Z associated with all n eigenvalues with negative real part. If X_1 is nonsingular then $X = X_2 X_1^{-1}$ solves the ARE.
2. The solution obtained in item 1. is
   (a) real,
   (b) symmetric,
   (c) unique stabilizing.
3. If $R \succeq 0$ (or $R \preceq 0$) then X_1 is invertible if and only if (A, R) is stabilizable.

Proof:
Item 1. From Item 2. of Lemma 1 there exists a Hurwitz matrix W such that

$$ZV_1 = V_1 W$$

Then, multiplying the above by X_1^{-1} on the right and by $\begin{bmatrix} X & -I \end{bmatrix}$ on the left we get

$$\begin{bmatrix} X & -I \end{bmatrix} Z \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} X_1 W X_1^{-1} = 0$$

Note that

$$\begin{bmatrix} X & -I \end{bmatrix} Z \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = A^T X + XA + XRX + Q$$

Item 2. (a) Since Z is real, the columns of V_1 can be chosen complex conjugates in pairs so that

$$\bar{V}_1 = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} P = \begin{bmatrix} X_1 P \\ X_2 P \end{bmatrix}$$

where P is some permutation matrix and

$$\bar{X} = \bar{X}_2 \bar{X}_1^{-1} = X_2 P P^{-1} X_1^{-1} = X_2 X_1^{-1} = X$$

that is X is real.

Item 2. (b) X is symmetric if

$$X = X_2 X_1^{-1} = X_1^{-*} X_2^* = X^*$$

or in other words

$$T = X_2^* X_1 - X_1^* X_2 = 0.$$

Now note that

$$T = V_1^* J V_1 = \begin{bmatrix} X_1^* & X_2^* \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Since $JZJ = Z^T$, Z and J satisfy the Lyapunov equation

$$JZ + Z^T J = 0$$

Multiplying the above equation by V_1^* on the left and V_1 on the right and using (1)

$$\begin{aligned} 0 &= V_1^* J Z V_1 + V_1^* Z^T J V_1 \\ &= V_1^* J V_1 W + W^* V_1^* J V_1 \\ &= TW + W^* T. \end{aligned}$$

Because W is Hurwitz $T = 0$.

Item 2. (c) Multiply (1) by X_1^{-1} on the right and by $\begin{bmatrix} I & 0 \end{bmatrix}$ on the left to obtain

$$\begin{bmatrix} I & 0 \end{bmatrix} Z \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = A + RX = X_1 W X_1^{-1}.$$

Therefore, $A + RX$ is stable because it is similar to a stable matrix. For unicity assume $\tilde{X}$ is also a stabilizing solution to the ARE. Therefore subtracting the two AREs

$$\begin{aligned} 0 &= (A^T X + XA + XRX + Q) - (A^T \tilde{X} + \tilde{X} A + \tilde{X} R \tilde{X} + Q) \\ &= A^T (X - \tilde{X}) + (X - \tilde{X}) A + XRX - \tilde{X} R \tilde{X} \\ &= A^T (X - \tilde{X}) + (X - \tilde{X}) A + XRX - \tilde{X} R \tilde{X} - XR\tilde{X} + XR\tilde{X} \\ &= A^T (X - \tilde{X}) + (X - \tilde{X}) A + XR(X - \tilde{X}) + (X - \tilde{X}) R \tilde{X} \\ &= (A + RX)^T (X - \tilde{X}) + (X - \tilde{X})(A + R\tilde{X}) \end{aligned}$$

The last equation can be seen as a Sylvester equation in $(X - \tilde{X})$ and since $A + RX$ and $A + R\tilde{X}$ are both Hurwitz $\lambda_i(A + RX) + \lambda_j(A + R\tilde{X}) < 0$ so that it admits only the trivial solution, that is, $X - \tilde{X} = 0$.

Item 3. To proves necessity assume that $R \succeq 0$ (or $R \preceq 0$), (A, R) is stabilizable and that X_1 is singular, such that there exists $x \neq 0$ such that $X_1 x = 0$. Multiply (1) by $\begin{bmatrix} I & 0 \end{bmatrix}$ on the left to obtain

$$\begin{bmatrix} I & 0 \end{bmatrix} Z \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = AX_1 + RX_2 = X_1 W$$

Multiply on the left by $x^* X_2^*$ and on the right by x

$$x^* X_2^* A X_1 x + x^* X_2^* R X_2 x = x^* X_2^* X_1 W x = x^* X_1^* X_2 W x$$

and use the fact that $X_1 x = 0$ to obtain

$$x^* X_2^* R X_2 x = 0$$

which implies $RX_2 x = 0$ because $R \succeq 0$ (or $R \preceq 0$). Note that this also implies

$$X_1 W x = 0.$$

If X_1 is singular then it can be shown that there exists $\bar{x} \neq 0$ such that

$$X_1 \bar{x} = 0, \quad W \bar{x} = \bar{\lambda} \bar{x}, \quad \bar{\lambda} + \bar{\lambda}^* < 0$$

Now multiply (1) by $\begin{bmatrix} 0 & I \end{bmatrix}$ on the left to obtain

$$\begin{bmatrix} 0 & I \end{bmatrix} Z \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -QX_1 - A^* X_2 = X_2 W$$

and multiply by $\bar{x}$ such that $X_1 \bar{x} = 0$ on the right

$$\begin{aligned} 0 &= (A^* X_2 - X_2 W)\bar{x} \\ &= (A^* - \bar{\lambda} I) X_2 \bar{x} \end{aligned}$$

Since $RX_2\bar{x} = 0$ (A, R) is not stabilizable, which is a contradiction.
To prove sufficiency note that if X_1 is invertible than $A + RX$ is stable such that (A, R) is stabilizable.