5 The H_∞ Optimal Control Problem

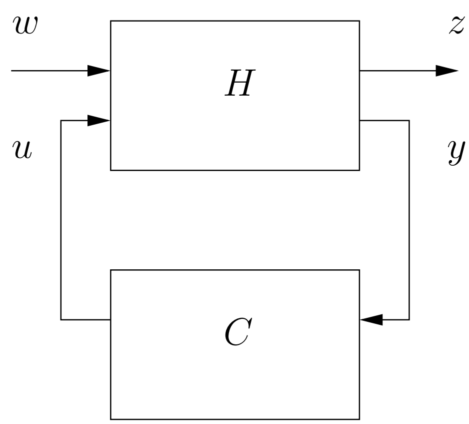

Figure 5: Feedback connection

5.1 Problem Statement

Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

where (A, B_u) is stabilizable and (A, C_y) is detectable.

H_∞ **Control Problem:** Find $C(s)$ such that

$$C(s) = \arg \min_{C(s)} \|\mathcal{F}_l\left(H(s), C(s)\right)\|_\infty$$

From the parametrization of all stabilizing controllers

$$\mathcal{F}_l\left(H(s), C(s)\right) = T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)$$

so that

$$\min_{C(s)} \|\mathcal{F}_l\left(H(s), C(s)\right)\|_\infty = \min_{Q(s) \in RH_\infty} \|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty$$

5.2 Equivalent Formulation as a Hankel Approximation Problem

Suppose there exist $T_{12}^{\perp}$ and $T_{21}^{\perp}$ such that

$$\begin{bmatrix} T_{12}(s) & T_{12}^{\perp}(s) \end{bmatrix}^{\sim} \begin{bmatrix} T_{12}(s) & T_{12}^{\perp}(s) \end{bmatrix} = I, \qquad \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix} \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix}^{\sim} = I$$

Note that

$$\begin{bmatrix} T_{12}(s) & T_{12}^{\perp}(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix}$$

are inner and co-inner, respectively, and square. Once can chose K and L such that $T_{12}^{\perp}$ and $T_{21}^{\perp}$ with such properties always exist (not shown here!).

Lemma:
Let $G(s) \in RH_\infty$ be inner (i.e. $G^{\sim}(s)G(s) = I$), then for any $F(s) \in RH_\infty$

$$\|G(s)F(s)\|_\infty = \|F(s)\|_\infty$$

Proof:

$$\begin{aligned} \|F(s)G(s)\|_\infty &= \sup_\omega \overline{\sigma}(G(j\omega)F(j\omega)) \\ &= \sup_\omega \overline{\lambda}(F^{\sim}(j\omega)G^{\sim}(j\omega)G(j\omega)F(j\omega)) \\ &= \sup_\omega \overline{\lambda}(F^{\sim}(j\omega)F(j\omega)) \\ &= \|F(s)\|_\infty \end{aligned}$$

Lemma:
Let $G(s) \in RH_\infty$ be co-inner (i.e. $G(s)G^{\sim}(s) = I$), then for any $F(s) \in RH_\infty$

$$\|F(s)G(s)\|_\infty = \|F(s)\|_\infty$$

Proof:

$$\begin{aligned} \|F(s)G(s)\|_\infty &= \sup_\omega \overline{\sigma}(F(j\omega)G(j\omega)) \\ &= \sup_\omega \overline{\lambda}(F(j\omega)G(j\omega)G^{\sim}(j\omega)F^{\sim}(j\omega)) \\ &= \sup_\omega \overline{\lambda}(F(j\omega)F^{\sim}(j\omega)) \\ &= \|F(s)\|_\infty \end{aligned}$$

Therefore

$$
\begin{aligned}
&\|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty \\
&\qquad = \left\| T_{11}(s) + \begin{bmatrix} T_{12}(s) & T_{12}^{\perp}(s) \end{bmatrix} \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix} \right\|_\infty \\
&\qquad = \left\| \begin{bmatrix} T_{12}(s) & T_{12}^{\perp}(s) \end{bmatrix} \left(\begin{bmatrix} T_{12}(s) & T_{12}(s)^{\perp} \end{bmatrix}^\sim T_{11}(s) \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix}^\sim \right.\right. \\
&\qquad\qquad \left.\left. + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix} \right\|_\infty \\
&\qquad = \left\| \begin{bmatrix} T_{12}(s) & T_{12}^{\perp}(s) \end{bmatrix}^\sim T_{11}(s) \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix}^\sim + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty \\
&\qquad = \left\| \begin{bmatrix} R_{11}(s) + Q(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{bmatrix} \right\|_\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
R_{11}(s) &= T_{12}^\sim(s)T_{11}(s)T_{21}^\sim(s), & R_{21}(s) &= T_{12}^\sim(s)T_{11}(s)T_{21}^{\perp\sim}(s), \\
R_{21}(s) &= T_{12}^{\perp\sim}(s)T_{11}(s)T_{21}^\sim(s), & R_{22}(s) &= T_{12}^{\perp\sim}(s)T_{11}(s)T_{21}^{\perp\sim}(s).
\end{aligned}
$$

5.2.1 The 1-Block Problem

If $T_{12}(s)$ and $T_{21}(s)$ are square and

$$T_{12}^{\sim}(s)T_{12}(s) = I, \qquad T_{21}(s)T_{21}^{\sim}(s) = I$$

then

$$\begin{aligned}
\|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty &= \|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty \\
&= \|T_{12}(s)\left(T_{12}^{\sim}(s)T_{11}(s)T_{21}^{\sim}(s) + Q(s)\right)T_{21}(s)\|_\infty \\
&= \|T_{12}^{\sim}(s)T_{11}(s)T_{21}^{\sim}(s) + Q(s)\|_\infty
\end{aligned}$$

Furthermore

$$\|T_{12}^{\sim}(s)T_{11}(s)T_{21}^{\sim}(s) + Q(s)\|_\infty = \|T_{21}(s)T_{11}^{\sim}(s)T_{12}(s) + Q^{\sim}(s)\|_\infty$$

where $T_{21}(s)T_{11}^{\sim}(s)T_{12}(s)$ can be shown to be in RH_∞.

Hankel Approximation Problem: Find $Q(s) \in RH_\infty$ such that

$$\min_{Q \in RH_\infty} \|T_{21}(s)T_{11}^{\sim}(s)T_{12}(s) + Q^{\sim}(s)\|_\infty$$

AKA as **Nehari Extension Problem**.

Solution is known (Glover, 1984)

5.3 Equivalent Formulation as a Model Matching Problem

Model Matching Problem: Find $Q(s) \in RH_\infty$ such that

$$Q(s) = \arg \min_{Q(s) \in RH_\infty} \|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty$$

Model Matching Problem (SISO): Find $Q(s) \in RH_\infty$ such that

$$Q^*(s) = \arg \min_{Q(s) \in RH_\infty} \|T_1(s) + T_2(s)Q(s)\|_\infty$$

Trivial case: If $T_1(s)T_2^{-1}(s)$ is stable then

$$Q^*(s) = -T_1(s)T_2^{-1}(s), \qquad \|T_1(s) + T_2(s)Q^*(s)\|_\infty = 0$$

Simplest next case: If $T_2(s)$ has one zero s_0 such that $s_0 + s_0^* > 0$. Then

$$\begin{aligned} \|T_1(s) + T_2(s)Q(s)\|_\infty &= \sup_{\mathrm{Re}(s)>0} |T_1(s) + T_2(s)Q(s)| \\ &\geq |T_1(s_0) + T_2(s_0)Q(s_0)| \\ &\geq |T_1(s_0)| \end{aligned}$$

But for $Q^*(s) = [T_1(s) - T_1(s_0)]T_2^{-1}(s)$

$$\|T_1(s) + T_2(s)Q^*(s)\|_\infty = \|T_1(s_0)\|_\infty$$

Note that $Q^*(s)$ is stable because $T_1(s) - T_1(s_0)$ has a zero on s_0!

General case: **Nevanlina-Pick Problem**.

5.4 Super Simplified H_∞ Control Problem Solution

Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

Assumptions:

1. (A, B_u) is stabilizable and (A, C_y) is detectable
2. $D_{zw} = 0$
3. $D_{yu} = 0$
4. $D_{zu}^T C_z = 0$ and $D_{zu}^T D_{zu} = I$
5. $B_w D_{yw}^T = 0$ and $D_{yw} D_{yw}^T = I$
6. (A, B_w) is controllable and (A, C_z) is observable

Theorem: There exists a controller $C(s)$ such that $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ under the above assumptions if and only if all the following conditions are satisfied

1. There exists X symmetric and positive semidefinite stabilizing solution of the Riccati equation
$$A^T X + XA - X(B_u B_u^T - \gamma^{-2} B_w B_w^T)X + C_z^T C_z = 0$$
2. There exists Y symmetric and positive semidefinite stabilizing solution of the Riccati equation
$$AY + YA^T - Y(C_y^T C_y - \gamma^{-2} C_z^T C_z)Y + B_w B_w^T = 0$$
3. $\max_i |\lambda_i(XY)| < \gamma^2$

If so, all controllers such that $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ are parametrized as the feedback connection of

$$R(s) = \left[\begin{array}{c|cc} A_\infty & -Z_\infty L_\infty & Z_\infty B_u \\ \hline K_\infty & 0 & I \\ -C_y & I & 0 \end{array}\right],$$

where

$$\begin{aligned}
A_\infty &:= A + \gamma^{-2} B_w B_w^T X + B_u K_\infty + Z_\infty L_\infty C_y \\
Z_\infty &:= (I - \gamma^{-2} Y X)^{-1} \\
L_\infty &:= -Y C_y^T \\
K_\infty &:= -B_u^T X
\end{aligned}$$

with $Q(s) \in RH_\infty$ such that $\|Q(s)\| < \gamma$.

5.5 Removing the Assumptions

AKA as *Loop Shifting*.

5.5.1 $D_{yu} = 0$

Define

$$\tilde{y} = y - D_{yw} u$$

so that

$$\tilde{H}(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

5.5.2 Normality of D_{zu} and D_{yw}

This assumption can be replaced by D_{zu} full column-rank and D_{yw} full row-rank. Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

If D_{zu} is full column rank, compute

$$D_{zu} = U \Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} 0 \\ \Sigma V^T \end{bmatrix} =: U_z \begin{bmatrix} 0 \\ S_u \end{bmatrix}, \qquad U_z^T U_z = I.$$

If D_{yw} is full row rank, compute

$$D_{yw} = U \Sigma V^T = \begin{bmatrix} 0 & U\Sigma \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} =: \begin{bmatrix} 0 & S_y \end{bmatrix} V_w^T, \qquad V_w^T V_w = I.$$

Now define

$$\tilde{w} = V_w^T w \qquad\qquad \tilde{u} = S_u u$$
$$\tilde{z} = U_z^T z \qquad\qquad \tilde{y} = S_y^{-1} y$$

so that

$$w = V_w \tilde{w} \qquad\qquad u = S_u^{-1} \tilde{u}$$

and

$$\tilde{H}(s) = \left[\begin{array}{c|cc} A & B_w V_w & B_u S_u^{-1} \\ \hline U_z^T C_z & U_z^T D_{zw} V_w & U_z^T D_{zu} S_u^{-1} \\ S_y^{-1} C_y & S_y^{-1} D_{yw} V_w & 0 \end{array}\right].$$

Note that

$$U_z^T D_{zu} S_u^{-1} = U_z^T U_z \begin{bmatrix} 0 \\ S_u \end{bmatrix} S_u^{-1} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

and

$$S_y^{-1} D_{yw} V_w = S_y^{-1} \begin{bmatrix} 0 & S_y \end{bmatrix} V_w^T V_w = \begin{bmatrix} 0 & I \end{bmatrix}$$

Also

$$\begin{aligned} \mathcal{F}_l\left(\tilde{H}(s), \tilde{C}(s)\right) &= \tilde{H}_{zw}(s) + \tilde{H}_{zu}(s)\tilde{C}(s)\left[I - \tilde{H}_{yu}(s)\tilde{C}(s)\right]^{-1} \tilde{H}_{yw}(s) \\ &= U_z^T H_{zw}(s) V_w + U_z^T H_{zu}(s) C(s) \left[I - H_{yu}(s) C(s)\right]^{-1} H_{yw}(s) V_w \\ &= U_z^T \mathcal{F}_l\left(H(s), C(s)\right) V_w \end{aligned}$$

and

$$\left\|\mathcal{F}_l\left(\tilde{H}(s), \tilde{C}(s)\right)\right\|_\infty = \left\|U_z^T \mathcal{F}_l\left(H(s), C(s)\right) V_w\right\|_\infty = \left\|\mathcal{F}_l\left(H(s), C(s)\right)\right\|_\infty$$

5.5.3 Orthogonality of C_z and D_{zu}

Define

$$\tilde{u} = u + D_{zu}^T C_z x, \qquad\qquad u = \tilde{u} - D_{zu}^T C_z x$$

so that

$$\tilde{H}(s) = \left[\begin{array}{c|cc} A - B_u D_{zu}^T C_z & B_w & B_u \\ \hline (I - D_{zu} D_{zu}^T) C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

Because $D_{zu}^T D_{zu} = I$ then

$$D_{zu}^T (I - D_{zu} D_{zu}^T) C_z = [D_{zu}^T - (D_{zu}^T D_{zu}) D_{zu}^T] C_z = 0$$

5.5.4 Orthogonality of B_w and D_{yw}

Note that

$$\begin{aligned}
\mathcal{F}_l\left(H(s), C(s)\right)^T &= \left[H_{zw}(s) + H_{zu}(s)C(s)\left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s)\right]^T \\
&= H_{zw}^T(s) + H_{yw}^T(s)\left[I - C^T(s)H_{yu}^T(s)\right]^{-1} C^T(s)H_{zu}^T(s) \\
&= \mathcal{F}_u\left(H^T(s), C^T(s)\right)
\end{aligned}$$

Note that

$$H^T(s) = \left[\begin{array}{c|cc} A^T & C_z^T & C_y^T \\ \hline B_w^T & 0 & D_{yw}^T \\ B_u^T & D_{zu}^T & 0 \end{array}\right]$$

and the definitions

$$\tilde{u} = u + D_{yw}B_w^T x, \qquad u = \tilde{u} - D_{yw}B_w^T x$$

are so that

$$\tilde{H}^T(s) = \left[\begin{array}{c|cc} A^T - C_y^T D_{yw} B_w^T & C_z^T & C_y^T \\ \hline (I - D_{yw}^T D_{yw})B_w^T & 0 & D_{yw}^T \\ B_u^T & D_{zu}^T & 0 \end{array}\right]$$

and because $D_{yw}D_{yw}^T = I$ then

$$D_{yw}(I - D_{yw}^T D_{yw})B_w^T = [D_{yw} - (D_{yw}D_{yw}^T)D_{yw}]B_w^T = 0$$

Of course

$$\begin{aligned}
\mathcal{F}_u\left(H^T(s), C^T(s)\right)^T &= \mathcal{F}_u\left(\tilde{H}^T(s), \tilde{C}^T(s)\right)^T \\
&= \mathcal{F}_l\left(\tilde{H}(s), \tilde{C}(s)\right)
\end{aligned}$$

where

$$H(s) = \left[\begin{array}{c|cc} A & B_w(I - D_{yw}^T D_{yw}) & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

and $B_w(I - D_{yw}^T D_{yw})$ and D_{yw} are orthogonal.

5.5.5 $D_{zw} = 0$ can be relaxed, but it is not so simple!

5.6 Not So Simplified H_∞ Control Problem Solution

Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

Definitions:

1. $A_c := A - B_u D_{zu}^T C_z$
2. $C_c := (I - D_{zu} D_{zu}^T) C_z$
3. $A_f := A - B_w D_{yw}^T C_y$
4. $B_f := B_w (I - D_{yw}^T D_{yw})$

Assumptions:

1. (A, B_u) is stabilizable and (A, C_y) is detectable
2. $D_{zw} = 0$
3. $D_{yu} = 0$
4. $D_{zu}^T D_{zu} = I$
5. $D_{yw} D_{yw}^T = I$
6. (A_c, C_c) is observable and (A_f, B_f) is controllable.

Theorem: There exists a controller $C(s)$ such that $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ under the above assumptions if and only if all the following conditions are satisfied

1. There exists X symmetric and positive semidefinite stabilizing solution of the Riccati equation
$$A_c^T X + X A_c - X(B_u B_u^T - \gamma^{-2} B_w B_w^T) X + C_c^T C_c = 0$$
2. There exists Y symmetric and positive semidefinite stabilizing solution of the Riccati equation
$$A_f Y + Y A_f^T - Y(C_y^T C_y - \gamma^{-2} C_z^T C_z) Y + B_f B_f^T = 0$$
3. $\max_i |\lambda_i(XY)| < \gamma^2$

If so, all controllers such that $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ are parametrized as the feedback connection of

$$R(s) = \left[\begin{array}{c|cc} A_\infty & -Z_\infty L_\infty & Z_\infty(B_u + \gamma^{-2} Y C_z^T D_{zu}) \\ \hline K_\infty & 0 & I \\ -C_y - \gamma D_{yw} B_w^T X & I & 0 \end{array}\right],$$

where

$$\begin{aligned} A_\infty &:= A + \gamma^{-2} B_w B_w^T X + B_u K_\infty + Z_\infty L_\infty (C_y + \gamma^{-2} D_{yw} B_w^T X) \\ Z_\infty &:= (I - \gamma^{-2} Y X)^{-1} \\ L_\infty &:= -Y C_y^T - B_w D_{yw}^T \\ K_\infty &:= -B_u^T X - D_{zu}^T C_z \end{aligned}$$

with $Q(s) \in RH_\infty$ such that $\|Q(s)\| < \gamma$.

5.7 A Proof Using Linear Matrix Inequalities

5.7.1 Preliminaries

Lemma [Schur Complement]: The following statements are equivalent:

1. $Z \succ 0, \quad X - Y^T Z^{-1} Y \succeq 0.$
2. $Z \succ 0, \quad \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succeq 0.$

Lemma [Schur Complement (strict version)]: The following statements are equivalent:

1. $Z \succ 0, \quad X - Y^T Z^{-1} Y \succ 0.$
2. $X \succ 0, \quad Z - Y X^{-1} Y^T \succ 0.$
3. $\begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succ 0.$

Proof: Assume $Z \succ 0$ and

$$\begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succeq 0 \quad (\succ 0).$$

The nonsingular matrix

$$T = \begin{bmatrix} I & 0 \\ -Z^{-1} Y & I \end{bmatrix}$$

establishes the congruence transformation

$$T^T \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} T = \begin{bmatrix} X - Y^T Z^{-1} Y & 0 \\ 0 & Z \end{bmatrix} \succeq 0 \quad (\succ 0).$$

A similar argument proves 2.

Lemma [Projection]: The following are equivalent statements:

1. There exists X such that
$$Q + B^T X C + C^T X^T B \prec 0$$
2. The following two conditions hold
$$B_\perp^T Q B_\perp \prec 0 \quad \text{or} \quad B^T B \succ 0$$
$$C_\perp^T Q C_\perp \prec 0 \quad \text{or} \quad C^T C \succ 0$$

Lemma [Bounded Real]: Let $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a realization of $H(s) \in RH_\infty$. Define

$$R_\gamma := \gamma^2 I - D^T D, \qquad F_\gamma := A + B R_\gamma^{-1} D^T C$$

The following statements are equivalent:

1. $\|H\| < \gamma$
2. $\overline{\sigma}(D) < \gamma$ and the Hamiltonean matrix
$$Z_\gamma := \begin{bmatrix} -F_\gamma^T & -C^T(I + D R_\gamma^{-1} D^T)C \\ B R_\gamma^{-1} B^T & F_\gamma \end{bmatrix}$$
has no eigenvalue on the imaginary axis
3. $\overline{\sigma}(D) < \gamma$ and there exists $X \succeq 0$ stabilizing solution to the ARE
$$F_\gamma^T X + X F_\gamma + X B R_\gamma^{-1} B^T X + C^T(I + D R_\gamma^{-1} D^T)C = 0$$
4. $\overline{\sigma}(D) < \gamma$ and there exists $X \succ 0$ solution to the inequality
$$F_\gamma^T X + X F_\gamma + X B R_\gamma^{-1} B^T X + C^T(I + D R_\gamma^{-1} D^T)C \prec 0$$
5. There exists a symmetric positive definite solution X to the Linear Matrix Inequality
$$\begin{bmatrix} A^T X + X A & X B & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \prec 0$$

Proof: Equivalence of 1. to 3. has been shown before. For the equivalence of 3. to 4. define

$$H_\epsilon(s) = \begin{bmatrix} H(s) \\ \epsilon(sI - A)^{-1}B \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ \epsilon I & 0 \end{array}\right].$$

such that form some small enough $\epsilon > 0$

$$\|H(s)\| < \|H_\epsilon(s)\| \le \|H(s)\| + \epsilon\|(sI - A)^{-1}B\| < \gamma.$$

Note that

$$\begin{bmatrix} C^T & \epsilon I \end{bmatrix} \left(I + \begin{bmatrix} D \\ 0 \end{bmatrix} R_\gamma^{-1} \begin{bmatrix} D^T & 0 \end{bmatrix} \right) \begin{bmatrix} C \\ \epsilon I \end{bmatrix} = C^T(I + DR_\gamma^{-1}D^T)C + \epsilon I.$$

That X is strictly positive is a consequence of the fact that

$$\left(A, \begin{bmatrix} C \\ \epsilon 0 \end{bmatrix} \right)$$

is observable for $\epsilon > 0$.

The equivalence of 4. and 5. comes from using Schur Complement. Indeed

$$\begin{aligned} 0 &\succ F_\gamma^T X + XF_\gamma + XBR_\gamma^{-1}B^TX + C^T(I + DR_\gamma^{-1}D^T)C \\ &= (A + BR_\gamma^{-1}D^TC)^TX + X(A + BR_\gamma^{-1}D^TC) \\ &\qquad + XBR_\gamma^{-1}B^TX + C^T(I + DR_\gamma^{-1}D^T)C \\ &= \underbrace{A^TX + XA + C^TC}_{X} + \underbrace{(XB + C^TD)}_{Y^T}\underbrace{R_\gamma^{-1}}_{Z^{-1}}\underbrace{(B^TX + D^TC)}_{Y} \end{aligned}$$

is equivalent to

$$\begin{aligned} 0 &\succ \begin{bmatrix} A^TX + XA + C^TC & XB + C^TD \\ B^TX + D^TC & -\gamma^2 I + D^TD \end{bmatrix} \\ &= \gamma \left(\underbrace{\begin{bmatrix} A^T(X/\gamma) + (X/\gamma)A & (X/\gamma)B \\ B^T(X/\gamma) & -\gamma I \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} C^T \\ D^T \end{bmatrix}}_{Y^T} \underbrace{\gamma^{-1}I}_{Z^{-1}} \underbrace{\begin{bmatrix} C & D \end{bmatrix}}_{Y} \right). \end{aligned}$$

Using Schur Complement once again after recalling that $\gamma > 0$ and defining $X_\gamma := X/\gamma$ we obtain

$$\begin{bmatrix} A^T X_\gamma + X_\gamma A & X_\gamma B & C^T \\ B^T X_\gamma & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \prec 0$$

which is the expression in 5.

5.7.2 Equivalence with Static Output Feedback

Lemma: Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

There exists a *dynamic output feedback controller*

$$C(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

that stabilizes $H(s)$ and $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exits a *static output feedback controller*

$$\mathcal{K} = \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix}$$

that stabilizes the system $\mathcal{H}(s)$ so that $\|\mathcal{F}_l\left(\mathcal{H}(s), \mathcal{K}\right)\| < \gamma$, where

$$\mathcal{H}(s) = \begin{bmatrix} \mathcal{H}_{zw}(s) & \mathcal{H}_{zu}(s) \\ \mathcal{H}_{yw}(s) & \mathcal{H}_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_w & \mathcal{B}_u \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} & \mathcal{D}_{zu} \\ \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{array}\right]$$

with

$$\left[\begin{array}{c|c|c} \mathcal{A} & \mathcal{B}_w & \mathcal{B}_u \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} & \mathcal{D}_{zu} \\ \hline \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{array}\right] := \left[\begin{array}{cc|c|cc} A & 0 & B_w & B_u & 0 \\ 0 & 0 & 0 & 0 & I \\ \hline C_z & 0 & D_{zw} & D_{zu} & 0 \\ \hline C_y & 0 & D_{yw} & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{array}\right]. \tag{2}$$

Proof: Follows from

$$\begin{aligned} \mathcal{F}_l\left(H(s), C(s)\right) &= \left[\begin{array}{cc|c} A + B_u\hat{D}C_y & B_u\hat{C} & B_w + B_u\hat{D}D_{yw} \\ \hat{B}C_y & \hat{A} & \hat{B}D_{yw} \\ \hline C_z + D_{zu}\hat{D}C_y & D_{zu}\hat{C} & D_{zw} + D_{zu}\hat{D}D_{yw} \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathcal{A} + \mathcal{B}_u\mathcal{K}\mathcal{C}_y & \mathcal{B}_w + \mathcal{B}_u\mathcal{K}\mathcal{D}_{yw} \\ \hline \mathcal{C}_z + \mathcal{D}_{zu}\mathcal{K}\mathcal{C}_y & \mathcal{D}_{zw} + \mathcal{D}_{zu}\mathcal{K}\mathcal{D}_{yw} \end{array}\right] \\ &= \mathcal{F}_l\left(\mathcal{H}(s), \mathcal{K}\right). \end{aligned}$$

Theorem: Let

$$\mathcal{H}(s) = \begin{bmatrix} \mathcal{H}_{zw}(s) & \mathcal{H}_{zu}(s) \\ \mathcal{H}_{yw}(s) & \mathcal{H}_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_w & \mathcal{B}_u \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} & \mathcal{D}_{zu} \\ \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{array}\right].$$

There exists a static output feedback controller $\mathcal{K}$ that stabilizes $\mathcal{H}(s)$ and keeps $\|\mathcal{F}_l(\mathcal{H}(s), \mathcal{K})\| < \gamma$ if and only if there exists $\mathcal{X} \succ 0$ such that

$$\begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T\mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp \prec 0$$

$$\begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{Y}\mathcal{A}^T + \mathcal{A}\mathcal{Y} & \mathcal{B}_w & \mathcal{Y}\mathcal{C}_z^T \\ \mathcal{B}_w^T & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z\mathcal{Y} & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp \prec 0$$

with $\mathcal{Y} = \mathcal{X}^{-1}$.

Proof: Define

$$\left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right] := \left[\begin{array}{c|c} \mathcal{A} + \mathcal{B}_u\mathcal{K}\mathcal{C}_y & \mathcal{B}_w + \mathcal{B}_u\mathcal{K}\mathcal{D}_{yw} \\ \hline \mathcal{C}_z + \mathcal{D}_{zu}\mathcal{K}\mathcal{C}_y & \mathcal{D}_{zw} + \mathcal{D}_{zu}\mathcal{K}\mathcal{D}_{yw} \end{array}\right]$$

Now use item 4. of the Bounded Real Lemma to establish that

$$\|\mathcal{F}_l(\mathcal{H}(s), \mathcal{K})\| < \gamma$$

if and only if there exits $\mathcal{X} \succ 0$ such that

$$\begin{aligned} 0 \succ & \begin{bmatrix} \mathsf{A}^T\mathcal{X} + \mathcal{X}\mathsf{A} & \mathcal{X}\mathsf{B} & \mathsf{C}^T \\ \mathsf{B}^T\mathcal{X} & -\gamma I & \mathsf{D}^T \\ \mathsf{C} & \mathsf{D} & -\gamma I \end{bmatrix} \\ = & \underbrace{\begin{bmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T\mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix}}_{Q} \\ & + \underbrace{\begin{bmatrix} \mathcal{X}\mathcal{B}_u \\ 0 \\ \mathcal{D}_{zu} \end{bmatrix}}_{B^T} \underbrace{\mathcal{K}}_{X} \underbrace{\begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}}_{C} + \underbrace{\begin{bmatrix} \mathcal{C}_y^T \\ \mathcal{D}_{yw}^T \\ 0 \end{bmatrix}}_{C^T} \underbrace{\mathcal{K}^T}_{X^T} \underbrace{\begin{bmatrix} \mathcal{B}_u^T\mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix}}_{B} \end{aligned}$$

The first inequality in the Theorem comes from direct application of the Projection Lemma to the above inequality. For the second inequality observe that

$$\begin{bmatrix} \mathcal{B}_u^T \mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix} = \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix} \begin{bmatrix} \mathcal{X} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

so that

$$\begin{bmatrix} \mathcal{B}_u^T \mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp = \begin{bmatrix} \mathcal{X}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp$$

The second inequality of the projection lemma yields

$$\begin{aligned}
0 \succ & \begin{bmatrix} \mathcal{B}_u^T \mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{A}^T \mathcal{X} + \mathcal{X} \mathcal{A} & \mathcal{X} \mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T \mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T \mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp \\
= & \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \\
& \times \begin{bmatrix} \mathcal{X}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{A}^T \mathcal{X} + \mathcal{X} \mathcal{A} & \mathcal{X} \mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T \mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{X}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp \\
= & \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{X}^{-1} \mathcal{A}^T + \mathcal{A} \mathcal{X}^{-1} & \mathcal{B}_w & \mathcal{X}^{-1} \mathcal{C}_z^T \\ \mathcal{B}_w^T & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z \mathcal{X}^{-1} & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp
\end{aligned}$$

which is the second inequality listed in the theorem after defining

$$\mathcal{Y} := \mathcal{X}^{-1}.$$

Lemma [Matrix Completion]: Let $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n \times n}$ be given such that $X \succ 0$ and $Y \succ 0$ and $X \succeq Y^{-1}$ with $\operatorname{rank}(X - Y^{-1}) = m \leq n$. There exists matrices $U, V \in \mathbb{R}^{n \times m}$ and $\hat{X}, \hat{Y} \in \mathbb{R}^{m \times m}$ such that

$$\begin{bmatrix} X & U \\ U^T & \hat{X} \end{bmatrix}^{-1} = \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix} \succ 0.$$

Proof: The proof is constructive. From Schur Complement, for positiveness we need

$$\hat{X} \succ U^T X^{-1} U$$

since $X \succ 0$. One such $\hat{X}$ is

$$\hat{X} = U^T X^{-1} U + I.$$

Using the formula for the inverse of a 2-by-2 block matrix we have

$$\begin{bmatrix} X & U \\ U^T & \hat{X} \end{bmatrix}^{-1} = \begin{bmatrix} X^{-1} + X^{-1} U E^{-1} U^T X^{-1} & -X^{-1} U E^{-1} \\ -E^{-1} U^T X^{-1} & E^{-1} \end{bmatrix}$$

where

$$E = \hat{X} - U^T X^{-1} U = I.$$

That is,

$$\begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix} = \begin{bmatrix} X^{-1} + X^{-1} U U^T X^{-1} & -X^{-1} U \\ -U^T X^{-1} & I \end{bmatrix}$$

which implies

$$\hat{Y} = I, \qquad V = -X^{-1} U,$$

for any U satisfying

$$Y = X^{-1} + X^{-1} U U^T X^{-1}.$$

Note that this equation is always solvable since

$$U U^T = X(Y - X^{-1})X$$

and the right hand side has rank m.

Theorem: Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

There exists a dynamic output feedback controller $C(s)$ of order n that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exists symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$ satisfying the linear matrix inequalities

$$\begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_w & C_z^T \\ B_w^T X & -\gamma I & D_{zw}^T \\ C_z & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_z^T & B_w \\ C_z Y & -\gamma I & D_{zw} \\ B_w^T & D_{zw}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0.$$

Proof: Start by substituting $\mathcal{A}$, $\mathcal{B}_w$, $\mathcal{B}_u$, $\mathcal{C}_z$, $\mathcal{D}_{zw}$, $\mathcal{D}_{zu}$, $\mathcal{C}_y$, $\mathcal{D}_{yw}$ by the augmented system matrices from (2). Partition

$$\mathcal{X} = \begin{bmatrix} X & U \\ U^T & \hat{X} \end{bmatrix}, \qquad \mathcal{Y} = \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix},$$

and compute

$$\left[\begin{array}{c|c|c} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \hline \mathcal{B}_w^T\mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{array}\right] = \left[\begin{array}{cc|c|c} A^T X + XA & A^T U & XB_w & C_z^T \\ U^T A & 0 & U^T B_w & 0 \\ \hline B_w^T X & B_w U & -\gamma I & D_{zw}^T \\ \hline C_z & 0 & D_{zw} & -\gamma I \end{array}\right],$$

$$\left[\begin{array}{c|c|c} \mathcal{Y}\mathcal{A}^T + \mathcal{A}\mathcal{Y} & \mathcal{B}_w & \mathcal{Y}\mathcal{C}_z^T \\ \hline \mathcal{B}_w^T & -\gamma I & \mathcal{D}_{zw}^T \\ \hline \mathcal{C}_z\mathcal{Y} & \mathcal{D}_{zw} & -\gamma I \end{array}\right] = \left[\begin{array}{cc|c|c} AY + YA^T & AV & B_w & YC_z^T \\ V^T A & 0 & 0 & V^T C_z \\ \hline B_w^T & 0 & -\gamma I & D_{zw}^T \\ \hline C_z Y & C_z V & D_{zw} & -\gamma I \end{array}\right].$$

Note that

$$\begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix} = \begin{bmatrix} C_y & 0 & D_{yw} & 0 \\ 0 & I & 0 & 0 \end{bmatrix} = \begin{bmatrix} C_y & D_{yw} & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

so that

$$\begin{aligned} \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} C_y & D_{yw} & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}_\perp \\ &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

Using the above expressions, the first inequality of the previous theorem become

$$
\begin{aligned}
0 &\succ \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T\mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp \\
&= \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}^T \\
&\qquad \times \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} A^TX + XA & A^TU & XB_w & C_z^T \\ U^TA & 0 & U^TB_w & 0 \\ B_w^TX & B_wU & -\gamma I & D_{zw}^T \\ C_z & 0 & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \\
&\qquad\qquad \times \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^TX + XA & XB_w & A^TU & C_z^T \\ B_w^TX & -\gamma I & B_wU & D_{zw}^T \\ U^TA & U^TB_w & 0 & 0 \\ C_z & D_{zw} & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^TX + XA & XB_w & C_z^T \\ B_w^TX & -\gamma I & D_{zw}^T \\ C_z & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}
\end{aligned}
$$

which is the first inequality in this theorem.
Similarly

$$
\begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp = \begin{bmatrix} B_u^T & 0 & 0 & D_{zu}^T \\ 0 & I & 0 & 0 \end{bmatrix}_\perp = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}
$$

and

$$0 \succ \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{A}\mathcal{Y} + \mathcal{Y}\mathcal{A}^T & \mathcal{B}_w & \mathcal{Y}\mathcal{C}_z^T \\ \mathcal{B}_w^T & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z\mathcal{Y} & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp$$

$$= \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_z^T & B_w \\ C_zY & -\gamma I & D_{zw} \\ B_w^T & D_{zw}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}$$

which is the second inequality in this theorem.
In order to complete the proof we have to establish that $\mathcal{Y}$ is the inverse of $\mathcal{X} \succ 0$. First note that from the formula of the inverse of a 2-by-2 block matrix we have

$$Y = (X - U\hat{X}^{-1}U^T)^{-1}$$

where $X \succ 0$, $\hat{X} \succ 0$ and $Y \succ 0$ since $\mathcal{X} \succ 0$ and $\mathcal{Y} \succ 0$. Therefore

$$X = Y^{-1} + U\hat{X}^{-1}U^T \succeq Y^{-1}$$

It is possible to show that if the controller has order equal to n then U can be assumed to be nonsingular so that the above inequality is strict since $\hat{X} \succ 0$.
The Schur Complement of the above strict inequality is

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0$$

which is the third inequality in this theorem, and for which we have established necessity.
Now for sufficiency, note that only the sub-blocks X and Y appear on the inequalities and that for some feasible $X \succ 0$ and $Y \succ 0$ we are free to choose U, V, $\hat{X}$ and $\hat{Y}$. We can therefore use the Matrix Completion Lemma with $\operatorname{rank}(X - Y^{-1}) = n$ to conclude on the existence of such matrices.

Corollary: Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

There exists a dynamic output feedback controller $C(s)$ of order $n_c \leq n$ that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exists symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$ satisfying the linear matrix inequalities

$$\begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_w & C_z^T \\ B_w^T X & -\gamma I & D_{zw}^T \\ C_z & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_z^T & B_w \\ C_z Y & -\gamma I & D_{zw} \\ B_w^T & D_{zw}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0, \qquad X \succ 0, \qquad Y \succ 0.$$

and $\mathrm{rank}(X - Y^{-1}) = n_c$.

Proof: Necessity is as in the proof of the previous theorem, and sufficiency through the Matrix Completion Lemma with $\mathrm{rank}(X - Y^{-1}) = n_c$.

Theorem: Assume

1. $D_{zw} = 0$,
2. $D_{yu} = 0$,
3. $D_{zu}^T C_z = 0$ and $D_{zu}^T D_{zu} = I$,
4. $B_w D_{yw}^T = 0$ and $D_{yw} D_{yw}^T = I$.

There exists a controller $C(s)$ of order n that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exists symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$ satisfying the linear matrix inequalities

$$\begin{bmatrix} A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z & XB_w (D_{yw})_\perp \\ (D_{yw})_\perp^T B_w^T X & -\gamma (D_{yw})_\perp^T (D_{yw})_\perp \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} AY + YA^T - \gamma C_y^T C_y + \gamma^{-1} B_w B_w^T & Y C_z^T (D_{zu}^T)_\perp \\ (D_{zu}^T)_\perp^T C_z Y & -\gamma (D_{zu}^T)_\perp^T (D_{zu}^T)_\perp \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0.$$

Proof: If $D_{yw} D_{yw}^T = I$ and $D_{zu}^T D_{zu} = I$ then

$$\begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp = \begin{bmatrix} I & 0 \\ -D_{yw}^T C_y & (D_{yw})_\perp \end{bmatrix}, \quad \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp = \begin{bmatrix} I & 0 \\ -D_{zu} B_u^T & (D_{zu}^T)_\perp \end{bmatrix}.$$

Consequently

$$\begin{aligned} 0 &\succ \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_w & C_z^T \\ B_w^T X & -\gamma I & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & 0 \\ -D_{yw}^T C_y & (D_{yw})_\perp & 0 \\ 0 & 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_w & C_z^T \\ B_w^T X & -\gamma I & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -D_{yw}^T C_y & (D_{yw})_\perp & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A^T X + XA - \gamma C_y^T C_y & XB_w (D_{yw})_\perp & C_z^T \\ (D_{yw})_\perp^T B_w^T X & -\gamma (D_{yw})_\perp^T (D_{yw})_\perp & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} \end{aligned}$$

where we have used the fact that $D_{zw} = 0$, $B_w D_{yw}^T = 0$, and $D_{yw} D_{yw}^T = I$. Applying Schur Complement on the last inequality we obtain the equivalent inequality

$$\begin{bmatrix} A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z & X B_w (D_{yw})_\perp \\ (D_{yw})_\perp^T B_w^T X & -\gamma (D_{yw})_\perp^T (D_{yw})_\perp \end{bmatrix} \prec 0$$

which is the first inequality in the theorem.
Similarly for the second inequality

$$\begin{aligned} 0 &\prec \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_z^T & B_w \\ C_z Y & -\gamma I & 0 \\ B_w^T & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} AY + YA^T - \gamma C_y^T C_y & Y C_z^T (D_{zu}^T)_\perp & B_w \\ (D_{zu}^T)_\perp^T C_z Y & -\gamma (D_{zu}^T)_\perp^T (D_{zu}^T)_\perp & 0 \\ B_w^T & 0 & -\gamma I \end{bmatrix} \end{aligned}$$

Using Schur Complement

$$\begin{bmatrix} AY + YA^T - \gamma C_y^T C_y + \gamma^{-1} B_w B_w^T & Y C_z^T (D_{zu}^T)_\perp \\ (D_{zu}^T)_\perp^T C_z Y & -\gamma (D_{zu}^T)_\perp^T (D_{zu}^T)_\perp \end{bmatrix} \prec 0$$

which is the second inequality in the theorem.

Corollary: Assume

1. $D_{zw} = 0$,
2. $D_{yu} = 0$,
3. $D_{zu}^T C_z = 0$ and $D_{zu}^T D_{zu} = I$,
4. $B_w D_{yw}^T = 0$ and $D_{yw} D_{yw}^T = I$.

There exists a controller $C(s)$ of order n that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exists symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$, $X \succ 0$, $Y \succ 0$ satisfying the Riccati inequalities

$$\begin{aligned} A^T X + XA - X\left(C_y^T C_y - \gamma^{-2} B_w B_w^T\right) X + C_z^T C_z \prec 0, \\ AY + YA^T - Y\left(C_y^T C_y - \gamma^{-2} C_z^T C_z\right) Y + B_w B_w^T \prec 0, \end{aligned}$$

with $\max_i |\lambda_i(XY)| < \gamma^2$.

Proof: We can always choose $(D_{yw})_\perp$ such that

$$(D_{yw})_\perp^T (D_{yw})_\perp = I$$

in which case the first inequality in the previous theorem simplifies to

$$\begin{bmatrix} A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z & XB_w(D_{yw})_\perp \\ (D_{yw})_\perp^T B_w^T X & -\gamma I \end{bmatrix} \prec 0$$

Applying Schur Complement and the fact that $B_w D_{yw}^T = 0$ we obtain the equivalent inequality

$$\begin{aligned} 0 &\succ A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z + \gamma^{-1} X B_w (D_{yw})_\perp (D_{yw})_\perp^T B_w^T X \\ &= A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z + \gamma^{-1} X B_w \begin{bmatrix} D_{yw}^T & (D_{yw})_\perp \end{bmatrix} \begin{bmatrix} D_{yw} \\ (D_{yw})_\perp^T \end{bmatrix} B_w^T X \\ &= A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z + \gamma^{-1} X B_w B_w^T X \end{aligned}$$

The last inequality is a consequence of

$$\begin{bmatrix} D_{yw}^T & (D_{yw})_\perp \end{bmatrix} \begin{bmatrix} D_{yw} \\ (D_{yw})_\perp^T \end{bmatrix} = \begin{bmatrix} D_{yw} \\ (D_{yw})_\perp^T \end{bmatrix} \begin{bmatrix} D_{yw}^T & (D_{yw})_\perp \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Now defining $X_\gamma := \gamma^{-1} X$ we have

$$0 \succ \gamma \left(A^T X_\gamma + X_\gamma A - C_y^T C_y + \gamma^{-2} C_z^T C_z + X_\gamma B_w B_w^T X_\gamma \right).$$

Dividing by $\gamma > 0$ and multiplying on the left and on the right by $\tilde{Y} := X_\gamma^{-1}$ we obtain

$$A\tilde{Y} + \tilde{Y} A^T - \tilde{Y} \left(C_y^T C_y - \gamma^{-2} C_z^T C_z \right) \tilde{Y} + B_w B_w^T \prec 0$$

which is the second inequality in the corollary.
Similarly for the first inequality in the previous theorem we can choose

$$(D_{zu}^T)_\perp^T (D_{zu}^T)_\perp = I$$

so that it becomes equivalent to

$$AY + YA^T - \gamma C_y^T C_y + \gamma^{-1} B_w B_w^T + \gamma^{-1} Y C_z^T (D_{zu}^T)_\perp (D_{zu}^T)_\perp^T C_z Y \prec 0$$

Using the fact that $D_{zu}^T C_z = 0$ and defining $Y_\gamma := \gamma^{-1} Y$ we obtain

$$\begin{aligned} 0 &\succ AY + YA^T - \gamma C_y^T C_y + \gamma^{-1} B_w B_w^T + \gamma^{-1} Y C_z^T C_z Y, \\ &= \gamma \left(AY_\gamma + Y_\gamma A^T - C_y^T C_y + \gamma^{-2} B_w B_w^T + Y_\gamma C_z^T C_z Y_\gamma \right). \end{aligned}$$

Dividing by $\gamma > 0$ and multiplying on the left and on the right by $\tilde{X} := Y_\gamma^{-1}$ we obtain

$$A^T \tilde{X} + \tilde{X} A - \tilde{X} \left(C_y^T C_y - \gamma^{-2} B_w B_w^T \right) \tilde{X} + C_z^T C_z \prec 0$$

which is the first inequality in the corollary.
The third inequality in the previous theorem is

$$\begin{bmatrix} \gamma \tilde{Y}^{-1} & I \\ I & \gamma \tilde{X}^{-1} \end{bmatrix} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0$$

which, using Schur Complement, is equivalent to

$$\gamma^2 \tilde{Y}^{-1} \succ \tilde{X}$$

or after multiplication by $\tilde{Y}^{1/2}$ on the left or on the right

$$\tilde{Y}^{1/2} \tilde{X} \tilde{Y}^{1/2} \prec \gamma^2 I$$

which means

$$\max_i \lambda_i(\tilde{Y}^{1/2}\tilde{X}\tilde{Y}^{1/2}) < \gamma^2.$$

After noting that

$$\tilde{X}\tilde{Y}^{1/2} = \tilde{Y}^{-1/2}\left(\tilde{Y}^{1/2}\tilde{X}\tilde{Y}^{1/2}\right)\tilde{Y}^{1/2} \qquad \sim \qquad \tilde{Y}^{1/2}\tilde{X}\tilde{Y}^{1/2}$$

we have

$$\max_i |\lambda_i(\tilde{X}\tilde{Y})| = \max_i \lambda_i(\tilde{Y}^{1/2}\tilde{X}\tilde{Y}^{1/2}) < \gamma^2.$$

Corollary: Assume

1. (A, B_u) is stabilizable and (A, C_y) is detectable.
2. $D_{zw} = 0$,
3. $D_{yu} = 0$,
4. $D_{zu}^T C_z = 0$ and $D_{zu}^T D_{zu} = I$,
5. $B_w D_{yw}^T = 0$ and $D_{yw} D_{yw}^T = I$.
6. (A, B_w) is controllable and (A, C_z) is observable.

There exists a controller $C(s)$ of order n that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exists $X \succeq 0$, $Y \succeq 0$ stabilizing solutions of the Riccati equations

$$\begin{aligned} A^T X + XA - X\left(C_y^T C_y - \gamma^{-2} B_w B_w^T\right) X + C_z^T C_z = 0, \\ AY + YA^T - Y\left(C_y^T C_y - \gamma^{-2} C_z^T C_z\right) Y + B_w B_w^T = 0, \end{aligned}$$

with $\max_i |\lambda_i(XY)| < \gamma^2$.

Proof: The extra assumptions are sufficient conditions for the existence of solutions to the Ricatti equations, which, through an argument similar to the one used in the proof of the Bounded Real Lemma, are equivalent to the Riccati inequalities of the previous corollary.