6 Applications of H_∞ Control

6.1 A Simple Frequency Domain Design

Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

Assume that the closed loop transfer $\mathcal{F}_l\left(H(s), C(s)\right)$ is SISO.

Frequency Domain Control Problem: Find a controller $C(s)$ such that $\mathcal{F}_l\left(H(s), C(s)\right)$ is asymptotically stable and

$$\left|\mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right| < \phi(\omega), \qquad \text{for all } \omega$$

for some given frequency domain specification $\phi(\omega)$.

Let $W(s) \in RH_\infty$ be such that $W^{-1}(s) \in RH_\infty$ and

$$\left|W^{-1}(j\omega)\right| = \phi(\omega), \qquad \forall\, \omega$$

Then the above problem is solved if

$$\left|\mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right| < \left|W^{-1}(j\omega)\right| = \phi(\omega), \qquad \text{for all } \omega$$

or, equivalently,

$$\left|W(j\omega)\right| \left|\mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right| = \left|W(j\omega)\, \mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right| < 1 \qquad \text{for all } \omega.$$

The function $W(s)$ is called a *scaling* or *shapping* function. This specification can be translated as the following H_∞ control problem.

H_∞ **Control Problem:** Given a scaling function $W(s)$ find a controller $C(s)$ such that be such that $\mathcal{F}_l\left(H(s), C(s)\right)$ is asymptotically stable and

$$\left\|W(s)\, \mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right\| < 1.$$

Note that

$$
\begin{aligned}
&W(s)\mathcal{F}_l\left(H(s), C(s)\right) \\
&\qquad = W(s)\left\{H_{zw}(s) + H_{zu}(s)C(s)\left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s)\right\} \\
&\qquad = W(s)H_{zw}(s) + W(s)H_{zu}(s)C(s)\left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s) \\
&\qquad = \mathcal{F}_l\left(\begin{bmatrix} W(s) & 0 \\ 0 & I \end{bmatrix} H(s), C(s)\right)
\end{aligned}
$$

so that the scaling is applied in the open loop plant $H(s)$.

6.2 Example #1

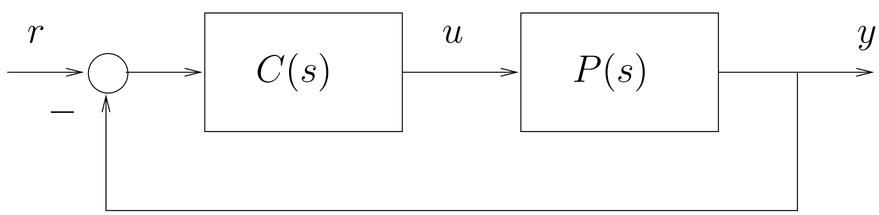


Figure 6: Block diagram

Plant:

$$P(s) = \frac{1}{(s+2)(s-1)}$$

Control objective:
Have a low pass transfer function from r to y. Say, below

$$\phi(\omega) = \frac{1}{\sqrt{(\omega/\omega_b)^2 + 1}}$$

for some $\omega_b > 0$.

Scaling design:
Set

$$W^{-1}(s) = \frac{1}{s/\omega_b + 1}, \qquad W(s) = s/\omega_b + 1.$$

Problem is $W(s) \notin RH_\infty$! General solution is to introduce a pole at 'minus infinity', that is

$$W^{-1}(s) = \frac{\epsilon s + 1}{s/\omega_b + 1}, \qquad\qquad W(s) = \frac{s/\omega_b + 1}{\epsilon s + 1}.$$

Then augment the plant with the scaling.

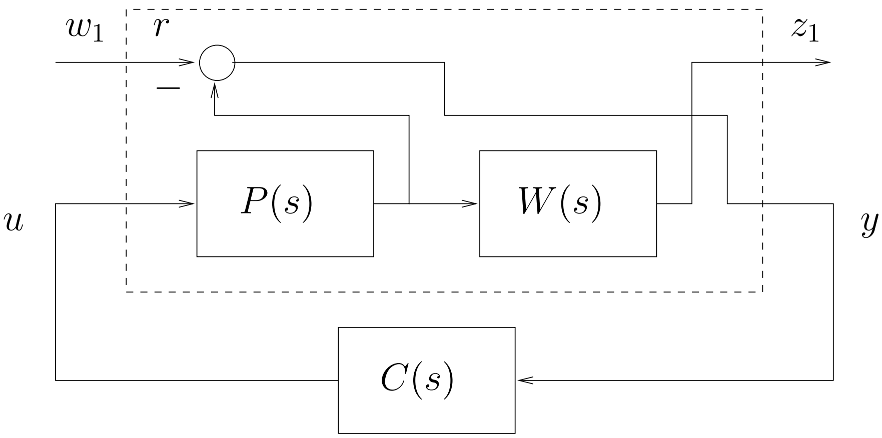


Figure 7: Plant augmented with scaling

The above system does not satisfy the assumptions on the H_∞ control problem for two reasons: $B_w = 0$[3] and $D_{zu} = 0$[4]. This is so because w_1 never gets into the plant before going through the controller and $P(s)$ is strictly proper. In order to overcome that, introduce an extra input w_2 and an extra output $z_2 = \epsilon u$, with $\epsilon > 0$ small, as in the diagram:

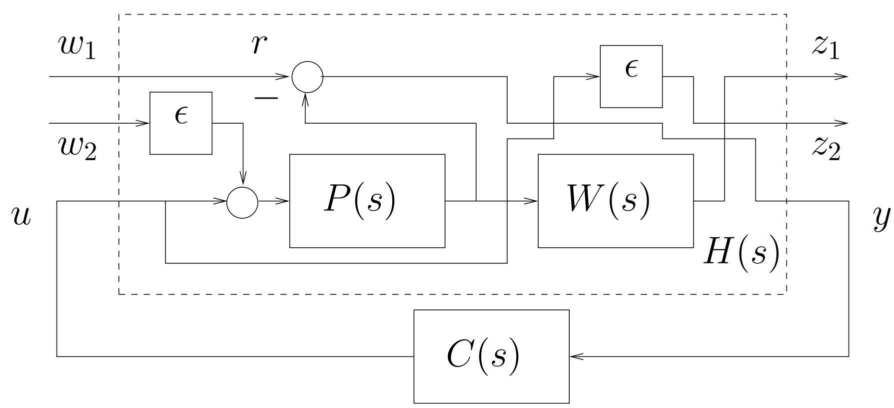


Figure 8: Plant augmented with extra scalings and inputs/outputs

[3] (A, B_w) is not controllable.
[4] $D_{zu}^T D_{zu} \neq 1$. Worse then that, D_{zu} is not full rank.

H_∞ **control problem:** Set $\omega_b = 1$ and find $C(s)$ such that

$$\|\mathcal{F}_l\left(H(s), C(s)\right)\| < 1$$

State space realization:

```
% MAE 284 - Robust and Multivariable Control
% Mauricio de Oliveira
% April 2006
%
% Example #1
% Frequency domain design
% Low pass cutoff frequency
omega_b = 1;

% epsilon
epsilon = 1e-4;

% Plant:
P = tf(1, conv([1 2],[1 -1]))

Transfer function:
     1
-----------
s^2 + s - 2


% Scaling:
S = tf([1/omega_b 1],[epsilon 1])

Transfer function:
   s + 1
------------
0.0001 s + 1


% Convert to state-space
[Ap, Bp, Cp, Dp] = ssdata(P);
[As, Bs, Cs, Ds] = ssdata(S);

% Generalized plant H
A = [As Bs*Cp; zeros(size(Cp'*Bs')) Ap];
Bw1 = zeros(3,1);
Bu = [Bs*Dp; Bp];

Cy = -[zeros(1,1) Cp];
Dyw1 = 1;

Cz1 = [Cs Ds*Cp];
Dzu1 = [Ds*Dp];

% Extra input w2
```

```
Bw2 = epsilon*Bu;
Dyw2 = 0;

% Extra output penalizing control input
Cz2 = zeros(1,3);
Dzu2 = epsilon;

% Augmented plant
Cz = [Cz1; Cz2];
Dzu = [Dzu1; Dzu2];

Bw = [Bw1 epsilon*Bu];
Dyw = [Dyw1 Dyw2];

% Show matrices
A
A =
   1.0e+04 *
   -1.0000         0    0.4096
         0   -0.0001    0.0001
         0    0.0004         0
Bu
Bu =
         0
    0.5000
         0
Bw
Bw =
   1.0e-04 *
         0         0
         0    0.5000
         0         0
Cy
Cy =
         0         0   -0.5000
Dyw
Dyw =
     1     0
Cz
Cz =
   1.0e+04 *
   -1.2206         0    0.5000
         0         0         0
Dzu
Dzu =
   1.0e-04 *
         0
    1.0000

diary off
```

Assumptions:

```
% Checking assumptions
% Stabilizability and detectability
rank(ctrb(A,Bu))
ans =
     3
rank(obsv(A,Cy))  % not observable but detectable
ans =
     2

% Normality
Dzu' * Dzu
ans =
   1.0000e-08
Dyw * Dyw'
ans =
     1

% Enforcing Dyw * Dyw' = 1
Bu = Bu / epsilon
Bu =
            0
         5000
            0
Dzu = Dzu / epsilon
Dzu =
     0
     1

Dzu' * Dzu
ans =
     1
Dyw * Dyw'
ans =
     1

% Auxiliary matrices
Ac = A - Bu * Dzu' * Cz;
Cc = (eye(size(Dzu,1)) - Dzu*Dzu') * Cz;

Af = A - Bw * Dyw' * Cy;
Bf = Bw * (eye(size(Dyw,2)) - Dyw'*Dyw);

% Technical observability and controllability
rank(obsv(Ac,Cc))
ans =
     3
rank(ctrb(Af,Bf))
ans =
     3

diary off
```

Riccati equations:

```
% Set gamma
gamma = 1;

% Solve control ARE
X = are(Ac, Bu*Bu' - Bw*Bw'/gamma^2, Cc'*Cc);

% Solve filtering ARE
Y = are(Af', Cy'*Cy - Cz'*Cz/gamma^2, Bf*Bf');

% Positivity
mX = min(eig(X))
mX =
   2.5140e-09
mY = min(eig(Y))
mY =
   -3.2815

% Spectral condition
mXY = max(abs(eig(X*Y)))/gamma^2
mXY =
   4.6203e-04

% GAMMA = 1 IS NOT FEASIBLE
diary off
```

Conclusion: $\|\mathcal{F}_l\left(H(s), C(s)\right)\| \not< 1$. Raise γ and try again!

```
% Set gamma
gamma = 2.01;

% Solve control ARE
X = are(Ac, Bu*Bu' - Bw*Bw'/gamma^2, Cc'*Cc);

% Solve filtering ARE
Y = are(Af', Cy'*Cy - Cz'*Cz/gamma^2, Bf*Bf');

% Positivity
mX = min(eig(X))
mX =
   2.5180e-09
mY = min(eig(Y))
mY =
  -5.0831e-14

% Spectral condition
mXY = max(abs(eig(X*Y)))/gamma^2
mXY =
    0.0339

if ( mX > -epsilon^2 && mY > -epsilon^2 && mXY < 1 )

  % Constructing central controller
  Zoo = inv(eye(size(A)) - Y*X/gamma^2);
  Loo = -Y*Cy' - Bw*Dyw';
  Koo = -Bu'*X - Dzu'*Cz;
  Aoo = A + Bw*Bw'*X/gamma^2 + Bu*Koo + Zoo*Loo*(Cy + Dyw*Bw'*X/gamma^2);

  % Build controller (do not forget to scale Cc!)
  C = -ss(Aoo, -Zoo*Loo, Koo/epsilon, zeros(size(Koo*Loo))); zpk(C)

Zero/pole/gain:
   -4090077.8764 (s+2) (s+1e04)
-----------------------------------
(s+205.5) (s^2  + 1.732e04s + 1e08)


  % Build closed loop system
  Acl = [Ap, Bp*C.c; C.b*Cp C.a];
  Bcl = [zeros(2,1); C.b*1];
  Ccl = [Cp, zeros(1,3)];
  Dcl = zeros(size(Ccl*Bcl));
  CL = ss(Acl,Bcl,Ccl,Dcl);

  % Check closed loop stability
  max(real(eig(Acl)))
ans =
   -1.0000

  diary off
```

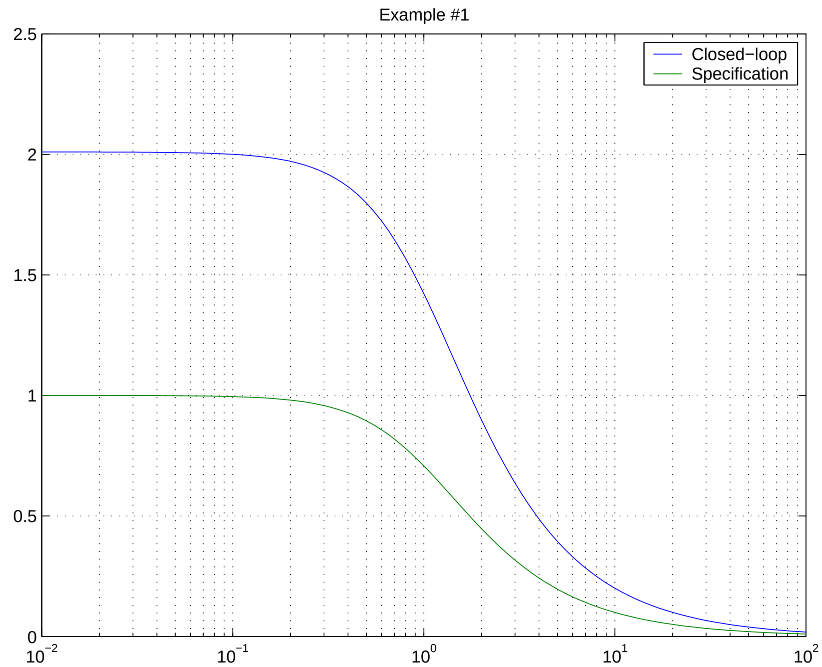


Figure 9: Example #1

6.3 Example #2

Same data as Example #1. Less stringent constraint.

H_∞ **control problem:** Set $\omega_b = 20$ and find $C(s)$ such that

$$\min\{\gamma : \quad \|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma\} = 1.06$$

Solution:

```
% MAE 284 - Robust and Multivariable Control
% Mauricio de Oliveira
% April 2006
%
% Example #2
% Frequency domain design
% Low pass cutoff frequency
omega_b = 20;

% epsilon
epsilon = 1e-4;

% Plant:
P = tf(1, conv([1 2],[1 -1]))
```

```
Transfer function:
     1
-----------
s^2 + s - 2


% Scaling:
S = tf([1/omega_b 1],[epsilon 1])

Transfer function:
 0.05 s + 1
------------
0.0001 s + 1


% Convert to state-space
[Ap, Bp, Cp, Dp] = ssdata(P);
[As, Bs, Cs, Ds] = ssdata(S);

% Generalized plant H
A = [As Bs*Cp; zeros(size(Cp'*Bs')) Ap];
Bw1 = zeros(3,1);
Bu = [Bs*Dp; Bp];

Cy = -[zeros(1,1) Cp];
Dyw1 = 1;

Cz1 = [Cs Ds*Cp];
Dzu1 = [Ds*Dp];

% Extra input w2
Bw2 = epsilon*Bu;
Dyw2 = 0;

% Extra output penalizing control input
Cz2 = zeros(1,3);
Dzu2 = epsilon;

% Augmented plant
Cz = [Cz1; Cz2];
Dzu = [Dzu1; Dzu2];

Bw = [Bw1 epsilon*Bu];
Dyw = [Dyw1 Dyw2];

% Show matrices
A
A =
   1.0e+04 *
   -1.0000         0    0.1024
         0   -0.0001    0.0001
         0    0.0004         0
Bu
Bu =
```

```
         0
    0.5000
         0
Bw
Bw =
   1.0e-04 *
         0         0
         0    0.5000
         0         0
Cy
Cy =
         0         0   -0.5000
Dyw
Dyw =
     1     0
Cz
Cz =
   1.0e+03 *
   -2.4365         0    0.2500
         0         0         0
Dzu
Dzu =
   1.0e-04 *
         0
    1.0000

% Checking assumptions
% Stabilizability and detectability
rank(ctrb(A,Bu))
ans =
     3
rank(obsv(A,Cy))  % not observable but detectable
ans =
     2

% Normality
Dzu' * Dzu
ans =
   1.0000e-08
Dyw * Dyw'
ans =
     1

% Enforcing Dyw * Dyw' = 1
Bu = Bu / epsilon
Bu =
           0
        5000
           0
Dzu = Dzu / epsilon
Dzu =
     0
     1
```

```
Dzu' * Dzu
ans =
     1
Dyw * Dyw'
ans =
     1

% Auxiliary matrices
Ac = A - Bu * Dzu' * Cz;
Cc = (eye(size(Dzu,1)) - Dzu*Dzu') * Cz;

Af = A - Bw * Dyw' * Cy;
Bf = Bw * (eye(size(Dyw,2)) - Dyw'*Dyw);

% Technical observability and controllability
rank(obsv(Ac,Cc))
ans =
     3
rank(ctrb(Af,Bf))
ans =
     3

% Set gamma
gamma = 1.06;

% Solve control ARE
X = are(Ac, Bu*Bu' - Bw*Bw'/gamma^2, Cc'*Cc);

% Solve filtering ARE
Y = are(Af', Cy'*Cy - Cz'*Cz/gamma^2, Bf*Bf');

% Positivity
mX = min(eig(X))
mX =
   3.3507e-07
mY = min(eig(Y))
mY =
  -1.3188e-15

% Spectral condition
mXY = max(abs(eig(X*Y)))/gamma^2
mXY =
    0.2134

if ( mX > -epsilon^2 && mY > -epsilon^2 && mXY < 1 )

  % Constructing central controller
  Zoo = inv(eye(size(A)) - Y*X/gamma^2);
  Loo = -Y*Cy' - Bw*Dyw';
  Koo = -Bu'*X - Dzu'*Cz;
  Aoo = A + Bw*Bw'*X/gamma^2 + Bu*Koo + Zoo*Loo*(Cy + Dyw*Bw'*X/gamma^2);

  % Build controller (do not forget to scale Cc!)
  C = -ss(Aoo, -Zoo*Loo, Koo/epsilon, zeros(size(Koo*Loo))); zpk(C)
```

```
Zero/pole/gain:
-1409708.2925 (s+2) (s+1e04)
----------------------------
(s+9987) (s+491.9) (s+162.3)


  % Build closed loop system
  Acl = [Ap, Bp*C.c; C.b*Cp C.a];
  Bcl = [zeros(2,1); C.b*1];
  Ccl = [Cp, zeros(1,3)];
  Dcl = zeros(size(Ccl*Bcl));
  CL = ss(Acl,Bcl,Ccl,Dcl);

  % Check closed loop stability
  max(real(eig(Acl)))
ans =
   -2.0000

  diary off
```

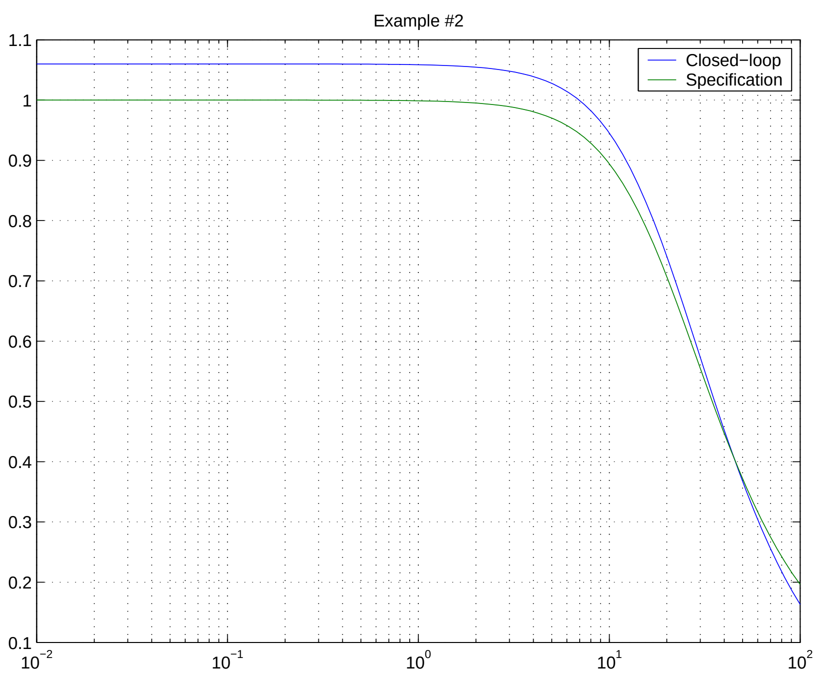


Figure 10: Example #2

6.4 Example #3

Same plant and problem data as Example #1.

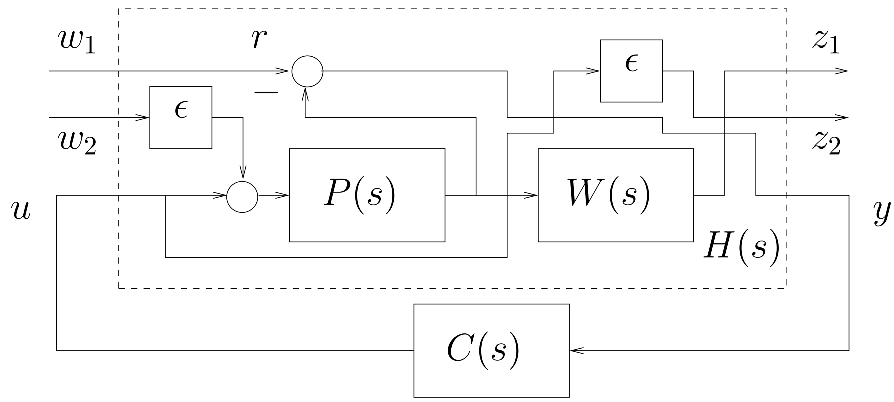


Figure 11: Plant augmented with extra scalings and inputs/outputs

Scaling design:
Set

$$W^{-1}(s) = \frac{1}{s/\omega_b + 1}, \qquad\qquad W(s) = s/\omega_b + 1.$$

$W(s) \notin RH_\infty$ but $P(s)$ is strictly proper! That means $W(s)P(s)$ is realizable!

State space realization:
Let

$$P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array}\right].$$

Then

$$W(s)P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p(A_p/\omega_b + I) & C_pB_p/\omega_b \end{array}\right] = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p(A_p/\omega_b + I) & 0 \end{array}\right]$$

Note that $C_pB_p = 0$ because the original system has relative degree 2!

ADVANTAGE: smaller plant, smaller controller!

```
% MAE 284 - Robust and Multivariable Control
% Mauricio de Oliveira
% April 2006
%
% Example #3
% Frequency domain design
% Low pass cutoff frequency
omega_b = 1;

% epsilon
epsilon = 1e-4;

% Plant:
P = tf(1, conv([1 2],[1 -1]))

Transfer function:
     1
-----------
s^2 + s - 2


% Scaling:
S = tf([1/omega_b 1],1)

Transfer function:
s + 1


% Convert to state-space
[Ap, Bp, Cp, Dp] = ssdata(P);

% Generalized plant H
A = Ap;
Bw1 = zeros(2,1);
Bu = Bp;

Cy = -Cp;
Dyw1 = 1;

Cz1 = Cp*(Ap/omega_b + eye(2));
Dzu1 = Cp*Bu
Dzu1 =
     0

% Extra input w2
Bw2 = epsilon*Bu;
Dyw2 = 0;

% Extra output penalizing control input
Cz2 = zeros(1,2);
Dzu2 = epsilon;

% Augmented plant
```

```
Cz = [Cz1; Cz2];
Dzu = [Dzu1; Dzu2];

Bw = [Bw1 epsilon*Bu];
Dyw = [Dyw1 Dyw2];

% Show matrices
A
A =
   -1.0000    0.5000
    4.0000         0
Bu
Bu =
    0.5000
         0
Bw
Bw =
   1.0e-04 *
         0    0.5000
         0         0
Cy
Cy =
         0   -0.5000
Dyw
Dyw =
     1     0
Cz
Cz =
    2.0000    0.5000
         0         0
Dzu
Dzu =
   1.0e-04 *
         0
    1.0000

diary off
```

H_∞ **control problem:** Set $\omega_b = 1$ and find $C(s)$ such that

$$\min\{\gamma : \quad \|\mathcal{F}_l\,(H(s), C(s))\| < \gamma\} = 2.01$$

Assumptions:

```
% Checking assumptions
% Stabilizability and detectability
rank(ctrb(A,Bu))
ans =
     2
rank(obsv(A,Cy))  % This time it is observable!
ans =
     2

% Normality
Dzu' * Dzu
ans =
   1.0000e-08
Dyw * Dyw'
ans =
     1

% Enforcing Dyw * Dyw' = 1
Bu = Bu / epsilon;
Dzu = Dzu / epsilon;

Dzu' * Dzu
ans =
     1
Dyw * Dyw'
ans =
     1

% Auxiliary matrices
Ac = A - Bu * Dzu' * Cz;
Cc = (eye(size(Dzu,1)) - Dzu*Dzu') * Cz;

Af = A - Bw * Dyw' * Cy;
Bf = Bw * (eye(size(Dyw,2)) - Dyw'*Dyw);

% Technical observability and controllability
rank(obsv(Ac,Cc))
ans =
     2
rank(ctrb(Af,Bf))
ans =
     2

diary off
```

Solution:

```
% Set gamma
gamma = 2.01;

% Solve control ARE
X = are(Ac, Bu*Bu' - Bw*Bw'/gamma^2, Cc'*Cc);

% Solve filtering ARE
Y = are(Af', Cy'*Cy - Cz'*Cz/gamma^2, Bf*Bf');

% Positivity
mX = min(eig(X))
mX =
   4.7048e-09
mY = min(eig(Y))
mY =
   5.8822e-10

% Spectral condition
mXY = max(abs(eig(X*Y)))/gamma^2
mXY =
    0.0200

if ( mX > -epsilon^2 && mY > -epsilon^2 && mXY < 1 )

  % Constructing central controller
  Zoo = inv(eye(size(A)) - Y*X/gamma^2);
  Loo = -Y*Cy' - Bw*Dyw';
  Koo = -Bu'*X - Dzu'*Cz;
  Aoo = A + Bw*Bw'*X/gamma^2 + Bu*Koo + Zoo*Loo*(Cy + Dyw*Bw'*X/gamma^2);

  % Build controller (do not forget to scale Cc!)
  C = -ss(Aoo, -Zoo*Loo, Koo/epsilon, zeros(size(Koo*Loo))); zpk(C)

Zero/pole/gain:
-4112493.4645 (s+2)
-------------------
(s+1e04) (s+206.6)


  % Build closed loop system
  Acl = [Ap, Bp*C.c; C.b*Cp C.a];
  Bcl = [zeros(2,1); C.b*1];
  Ccl = [Cp, zeros(1,2)];
  Dcl = zeros(size(Ccl*Bcl));
  CL = ss(Acl,Bcl,Ccl,Dcl);

  % Check closed loop stability
  max(real(eig(Acl)))
ans =
   -1.0000

  diary off
```

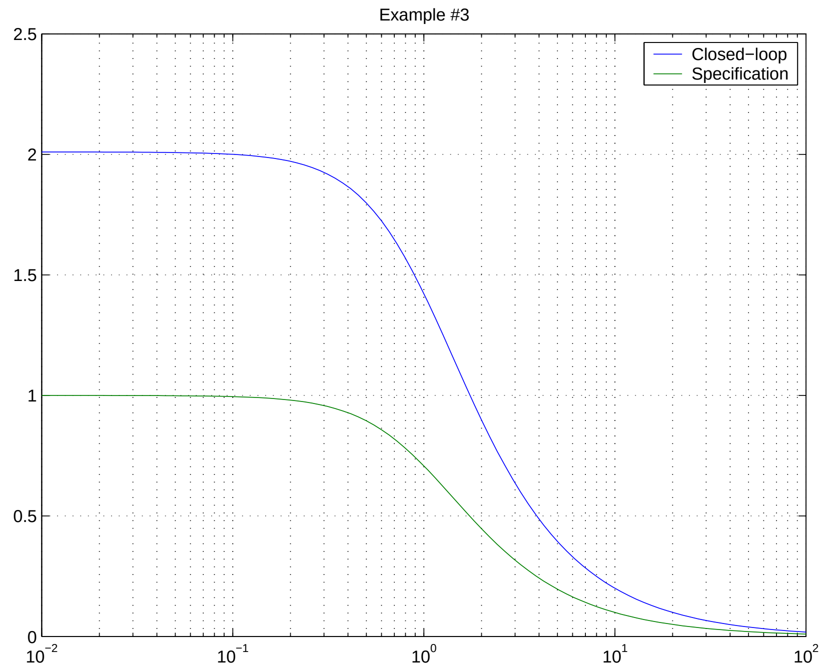


Figure 12: Example #3

6.5 Robust Analysis

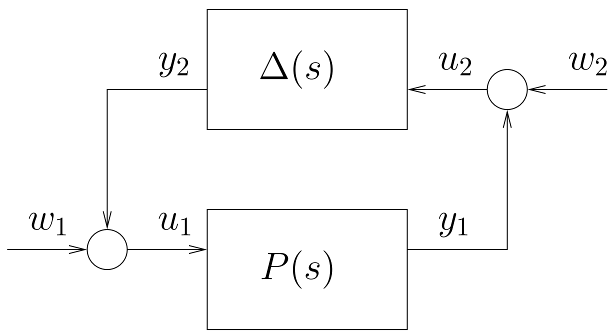


Figure 13: Feedback connection

Theorem [Small Gain]: Let $P(s) \in RH_\infty$. Then the feedback connection if Figure 13 is well posed and internally stable for all $\Delta(s) \in RH_\infty$ such that $\|\Delta(s)\| < 1$ if and only if $\|P(s)\| \leq 1$.

In the proof we will make use of the following technical lemma.

Lemma: $\underline{\sigma}(A + B) \geq \underline{\sigma}(A) - \overline{\sigma}(B)$.

Proof: First note that

$$\frac{\|Ax\| - \|Bx\|}{\|x\|} \geq \frac{\|Ax\|}{\|x\|} - \max_{\|x\| \neq 0} \frac{\|Bx\|}{\|x\|} = \frac{\|Ax\|}{\|x\|} - \overline{\sigma}(B).$$

Consequently

$$\min_{\|x\| \neq 0} \frac{\|Ax\| - \|Bx\|}{\|x\|} \geq \min_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} - \overline{\sigma}(B) = \underline{\sigma}(A) - \overline{\sigma}(B)$$

and

$$\begin{aligned} \underline{\sigma}(A + B) &= \min_{\|x\| \neq 0} \frac{\|Ax + Bx\|}{\|x\|} \\ &\geq \min_{\|x\| \neq 0} \frac{\|Ax\| - \|Bx\|}{\|x\|} \\ &\geq \underline{\sigma}(A) - \overline{\sigma}(B) \end{aligned}$$

Proof of Small Gain Theorem: We prove only sufficiency. Since $P(s) \in RH_\infty$, the feedback connection is asymptotically stable if and only if

$$\Delta(s)[I - P(s)\Delta(s)]^{-1} \in RH_\infty.$$

But as $\Delta(s) \in RH_\infty$ the above condition is equivalent to

$$[I - P(s)\Delta(s)]^{-1} \in RH_\infty,$$

or in other words, $\det[I - P(s)\Delta(s)]$ has no poles in the closed right-half plane. That is

$$\inf_{\mathrm{Re}(s) \geq 0} \underline{\sigma}[I - P(s)\Delta(s)] > 0$$

Using the previous lemma

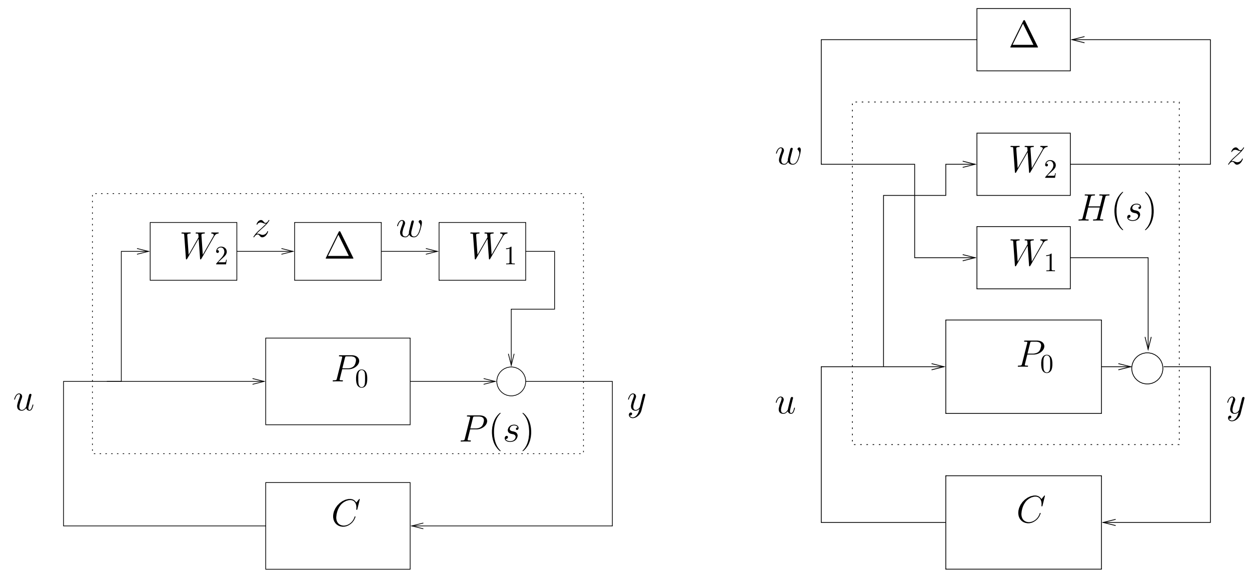
$$\begin{aligned} \inf_{\mathrm{Re}(s) \geq 0} \underline{\sigma}[I - P(s)\Delta(s)] &\geq \inf_{\mathrm{Re}(s) \geq 0} \{\underline{\sigma}(I) - \overline{\sigma}[P(s)\Delta(s)]\} \\ &= 1 + \inf_{\mathrm{Re}(s) \geq 0} \{-\overline{\sigma}[P(s)\Delta(s)]\} \\ &= 1 - \sup_{\mathrm{Re}(s) \geq 0} \overline{\sigma}[P(s)\Delta(s)] \\ &= 1 - \|P(s)\Delta(s)\| \end{aligned}$$

Now if $\|\Delta(s)\| < 1$

$$\begin{aligned} \inf_{\mathrm{Re}(s) \geq 0} \underline{\sigma}[I - P(s)\Delta(s)] &\geq 1 - \|P(s)\Delta(s)\| \\ &\geq 1 - \|P(s)\| \, \|\Delta(s)\| \\ &> 1 - \|P(s)\| \geq 0 \end{aligned}$$

A similar argument can show that the connection is well-posed. For a proof of necessity see Zhou p. 138.

6.5.1 Unstructured Additive Uncertainty



(a) Uncertain System (b) Generalized Plant

Figure 14: Additive Uncertainty

Theorem: The feedback connection in Figure 14(a) where

$$P(s) = P_0(s) + W_1(s)\Delta(s)W_2(s)$$

and the uncertainty $\Delta(s) \in RH_\infty$ is such that $\|\Delta(s)\| < 1$ is robustly stable if and only if

$$\|W_2(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s)\| \leq 1.$$

Proof: Define the input w and output z as in Fig.14(b), where the generalized plant $H(s)$ is given by

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \begin{bmatrix} 0 & W_2(s) \\ W_1(s) & P_0(s) \end{bmatrix}$$

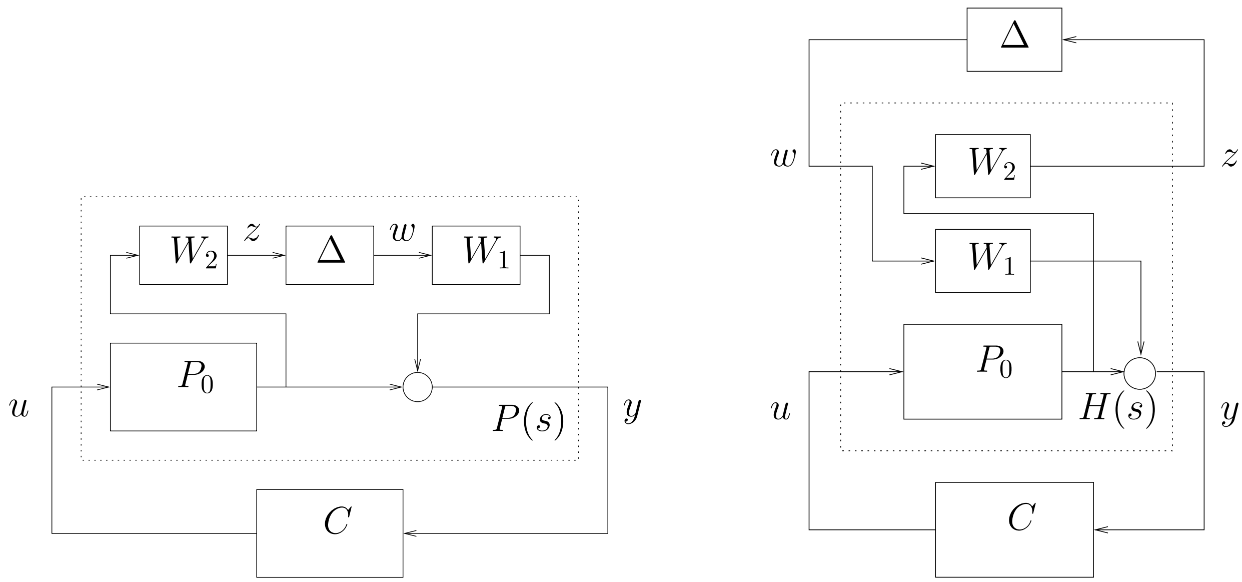
Use the small gain theorem to conclude that the system is robustly stable if and only if

$$\|\mathcal{F}_l\left(H(s), C(s)\right)\| \leq 1$$

where

$$\mathcal{F}_l\left(H(s), C(s)\right) = W_2(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s).$$

6.5.2 Unstructured Multiplicative Uncertainty



(a) Uncertain System

(b) Generalized Plant

Figure 15: Multiplicative Uncertainty

Theorem: The feedback connection in Figure 15(a) where

$$P(s) = [I + W_1(s)\Delta(s)W_2(s)]P_0(s)$$

and the uncertainty $\Delta(s) \in RH_\infty$ is such that $\|\Delta(s)\| < 1$ is robustly stable if and only if

$$\|W_2(s)P_0(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s)\| \leq 1.$$

Proof: Define the input w and output z as in Fig.15(b), where the generalized plant $H(s)$ is given by

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \begin{bmatrix} 0 & W_2(s)P_0(s) \\ W_1(s) & P_0(s) \end{bmatrix}$$

Use the small gain theorem to conclude that the system is robustly stable if and only if

$$\|\mathcal{F}_l\left(H(s), C(s)\right)\| \leq 1$$

where

$$\mathcal{F}_l\left(H(s), C(s)\right) = W_2(s)P_0(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s).$$

6.5.3 Unstructured Fractional Uncertainty

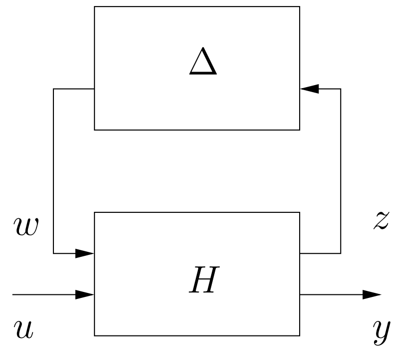


Figure 16: Uncertainty System

Lemma: The uncertain system

$$P(s) = [I + W_1(s)\Delta(s)W_2(s)]^{-1}P_0(s)$$

can be represented as the feedback connection of the plant

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \begin{bmatrix} -W_2(s)W_1(s) & W_2(s)P_0(s) \\ -W_1(s) & P_0(s) \end{bmatrix}$$

as in Figure 16.

Proof: Use the identity

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

to rewrite

$$\begin{aligned} &[I + W_1(s)\Delta(s)W_2(s)]^{-1}P_0(s) \\ &\qquad = P_0(s) - W_1(s)\Delta(s)[I + W_2(s)W_1(s)\Delta(s)]^{-1}W_2(s)P_0(s). \end{aligned}$$

Equivalently

$$\begin{aligned} &\mathcal{F}_l\left(\begin{bmatrix} H_{yu}(s) & H_{yw}(s) \\ H_{zu}(s) & H_{zw}(s) \end{bmatrix}, \Delta(s)\right) \\ &\qquad = \underbrace{P_0(s)}_{H_{yu}(s)} + \underbrace{[-W_1(s)]}_{H_{yw}(s)}\Delta(s)\{I - \underbrace{[-W_2(s)W_1(s)]}_{H_{zw}(s)}\Delta(s)\}^{-1}\underbrace{W_2(s)P_0(s)}_{H_{zu}(s)} \end{aligned}$$

from which the blocks of $H(s)$ are defined.

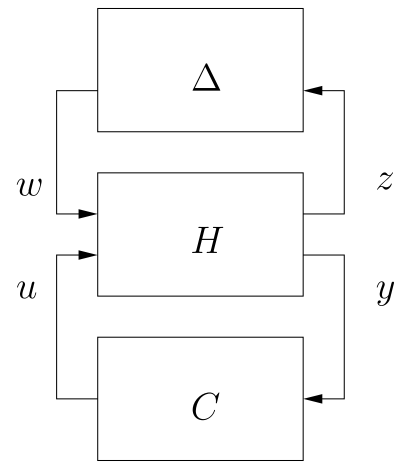


Figure 17: Uncertainty System

Theorem: The feedback connection in Figure 17 where

$$P(s) = [I + W_1(s)\Delta(s)W_2(s)]^{-1}P_0(s)$$

and the uncertainty $\Delta(s) \in RH_\infty$ is such that $\|\Delta(s)\| < 1$ is robustly stable if and only if

$$\|W_2(s)[I - P_0(s)C(s)]^{-1}W_1(s)\| \leq 1.$$

Proof: From the previous lemma and the small gain theorem the system is robustly stable if and only if

$$\|\mathcal{F}_l\left(H(s), C(s)\right)\| \leq 1$$

where

$$\begin{aligned}
\mathcal{F}_l\left(H(s), C(s)\right) &= -W_2(s)W_1(s) - W_2(s)P_0(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s) \\
&= -W_2(s)\left\{I + P_0(s)C(s)[I - P_0(s)C(s)]^{-1}\right\}W_1(s) \\
&= -W_2(s)[I - P_0(s)C(s)]^{-1}W_1(s),
\end{aligned}$$

after using the same identity used in the proof of the previous lemma.

6.5.4 Structured Uncertainty

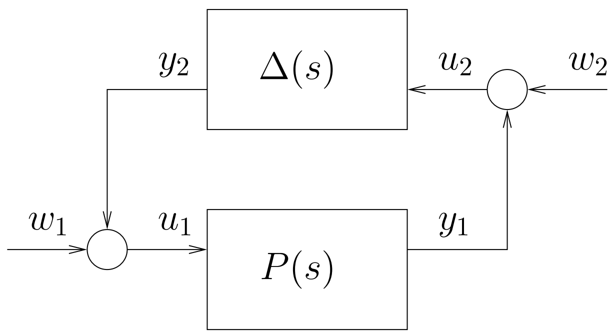


Figure 18: Uncertain System

Structured Uncertainty Robust Analysis: Is the feedback system in Figure 18 robustly stable for all $\Delta(s) \in RH_\infty$ such that $\|\Delta(s)\| < 1$ in the presence of some structural constraint $\Delta(s) \in \mathbf{\Delta}_S(s)$?

Without loss of generality

$$\mathbf{\Delta}_S(s) := \{\Delta(s) : \quad \Delta(s) = \mathrm{diag}[\delta_1(s)I, \ldots, \delta_s(s)I, \quad \Delta_1(s), \ldots, \Delta_F(s)] \in RH_\infty\}$$

or

$$\mathbf{\Delta}_S := \{\Delta : \quad \Delta = \mathrm{diag}[\delta_1 I, \ldots, \delta_s I, \quad \Delta_1, \ldots, \Delta_F], \quad \delta_i \in \mathbb{C}, \quad \Delta_j \in \mathbb{C}^{m_j \times m_j}\}.$$

6.5.5 μ Analysis

In this section we focus on uncertainty in $\mathbf{\Delta}_S$. Results are presented without proof. See Zhou, Chapter 10.

Definition: For $M \in \mathbb{C}^{n\times n}$, $\Delta \in \mathbf{\Delta_S}$, the structured singular value of M is the scalar

$$\mu_{\mathbf{\Delta}_S}(M) := \frac{1}{\min\{\|\Delta\| : \Delta \in \mathbf{\Delta}_S, \quad \det(I - M\Delta) = 0\}}$$

If $\det(I - M\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}_S$ then $\mu_{\mathbf{\Delta}_S}(M) := 0$.

Lemma: $\max_i |\lambda_i(M)| \leq \mu_{\mathbf{\Delta}_S}(M) \leq \overline{\sigma}(M)$.

Scalings:

$$\mathbf{D}_S := \{\Delta : \quad \Delta = \operatorname{diag}[D_1, \ldots, D_s, \quad d_1 I, \ldots, d_F I], \\ D_i \in \mathbb{C}^{r_i \times r_i}, D_i = D_i^* > 0, \quad d_j \in \mathbb{R}, d_j > 0\}.$$

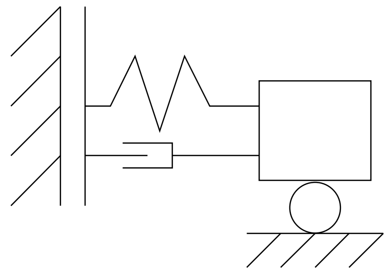
Lemma: $\max_{i, UU^*=I} |\lambda_i(UM)| = \mu_{\boldsymbol{\Delta}_S}(M) \leq \inf_{D \in \mathbf{D}_S} \overline{\sigma}(DMD^{-1})$.

Structured Small Gain Theorem: Let $P(s) \in RH_\infty$. Then the feedback connection if Figure 18 is well posed and internally stable for all $\Delta \in \boldsymbol{\Delta}_S$ such that $\|\Delta\| < 1$ if and only if

$$\sup_{\omega \in \mathbb{R}} \mu_{\boldsymbol{\Delta}_S}[P(j\omega)] \leq 1.$$

6.6 Example #4 [Parametric Uncertainty]

Mass/Spring/Damper system from Zhou, p. 171.



Uncertain System:

$$m\ddot{x} + c\dot{x} + kx = f$$

where $m > 0, \quad c > 0, \quad k > 0.$

Uncertainty:

$$c = \bar{c}(1 + \delta_c) \qquad\qquad k = \bar{k}(1 + \delta_k)$$

where

$$|c - \bar{c}| < \delta_c \bar{c} \qquad\qquad |k - \bar{k}| < \delta_k \bar{k}$$

and $|\delta_c| < \bar{\delta}_c \leq 1, \quad |\delta_k| < \bar{\delta}_k \leq 1.$

Pulling out uncertainty:
Write system equation

$$\begin{aligned}
\ddot{x} &= -m^{-1} c\dot{x} - m^{-1} kx + m^{-1} f \\
&= -m^{-1} \bar{c}(1 + \delta_c)\dot{x} - m^{-1} \bar{k}(1 + \delta_k)x + m^{-1} f \\
&= -m^{-1} \bar{c}\dot{x} - m^{-1} \bar{k}x - m^{-1} \delta_c \bar{c}\dot{x} - m^{-1} \delta_k \bar{k}x + m^{-1} f
\end{aligned}$$

Define auxiliary outputs

$$\begin{aligned}
z_c &= \bar{c}\dot{x}, \\
z_k &= \bar{k}x
\end{aligned}$$

System with auxiliary outputs

$$\begin{aligned}
\ddot{x} &= -m^{-1}\bar{c}\dot{x} - m^{-1}\bar{k}x - m^{-1}\delta_c z_c - m^{-1}\delta_k z_k + m^{-1} f \\
z_c &= \bar{c}\dot{x} \\
z_k &= \bar{k}x
\end{aligned}$$

Define auxiliary inputs in feedback

$$\begin{aligned}
w_c &= \delta_c z_c \\
w_k &= \delta_k z_k
\end{aligned}$$

System with auxiliary inputs and outputs

$$\begin{aligned}
\ddot{x} &= -m^{-1}\bar{c}\dot{x} - m^{-1}\bar{k}x - m^{-1}w_c - m^{-1}w_k + m^{-1} f \\
z_c &= \bar{c}\dot{x} \\
z_k &= \bar{k}x
\end{aligned}$$

Uncertain System in State Space: $(x_2 = \dot{x},\ x_1 = x)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \hline z_c \\ z_k \end{pmatrix} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -m^{-1}\bar{k} & -m^{-1}\bar{c} & -m^{-1} & -m^{-1} & m^{-1} \\ \hline 0 & \bar{c} & 0 & 0 & 0 \\ \bar{k} & 0 & 0 & 0 & 0 \end{array}\right] \begin{pmatrix} x_1 \\ x_2 \\ \hline w_c \\ w_k \\ f \end{pmatrix}$$

$$\begin{pmatrix} w_c \\ w_k \end{pmatrix} = \begin{bmatrix} \delta_c & 0 \\ 0 & \delta_k \end{bmatrix} \begin{pmatrix} z_c \\ z_k \end{pmatrix}$$

6.7 Example #5 [Parametric Uncertainty]

Mass/Spring/Damper system from Zhou, p. 171.

Uncertain System:

$$m\ddot{x} + c\dot{x} + kx = f$$

where $m > 0, \quad c > 0, \quad k > 0.$

Uncertainty:

$$m = \bar{m}(1+\delta_m) \qquad\qquad c = \bar{c}(1+\delta_c) \qquad\qquad k = \bar{k}(1+\delta_k)$$

where

$$|m - \bar{m}| < \delta_m \bar{m} \qquad\qquad |c - \bar{c}| < \delta_c \bar{c} \qquad\qquad |k - \bar{k}| < \delta_k \bar{k}$$

and $|\delta_m| < \bar{\delta}_m \leq 1, \quad |\delta_c| < \bar{\delta}_c \leq 1, \quad |\delta_k| < \bar{\delta}_k \leq 1.$

Auxiliary inputs/outputs:
Define auxiliary outputs

$$\begin{aligned} z_c &= \bar{c}\dot{x}, \\ z_k &= \bar{k}x \end{aligned}$$

Define auxiliary inputs in feedback

$$\begin{aligned} w_c &= \delta_c z_c \\ w_k &= \delta_k z_k \end{aligned}$$

System with auxiliary inputs and outputs

$$\begin{aligned} \ddot{x} &= m^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f\right) \\ z_c &= \bar{c}\dot{x} \\ z_k &= \bar{k}x \end{aligned}$$

The fractional uncertainty:

Use the lemma of the inverse

$$\begin{aligned} m^{-1} &= \bar{m}^{-1}(1+\delta_m)^{-1} \\ &= \bar{m}^{-1}[1-\delta_m(1+\delta_m)^{-1}] \\ &= \bar{m}^{-1}\mathcal{F}_l\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \delta_m\right) \end{aligned}$$

Since for any u

$$y = \mathcal{F}_l\left(M, \delta\right) u$$

is the feedback connection of

$$\begin{pmatrix} y \\ z \end{pmatrix} = M \begin{pmatrix} u \\ w \end{pmatrix}$$

with

$$w = \delta z$$

we have

$$\begin{aligned} \begin{pmatrix} m^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f\right) \\ z_m \end{pmatrix} &= \begin{bmatrix} \bar{m}^{-1} & -\bar{m}^{-1} \\ 1 & -1 \end{bmatrix} \begin{pmatrix} -\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f \\ w_m \end{pmatrix} \\ &= \begin{pmatrix} \bar{m}^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k - w_m + f\right) \\ -\bar{c}\dot{x} - \bar{k}x - w_c - w_k - w_m + f \end{pmatrix} \end{aligned}$$

System with auxiliary inputs and outputs

$$\begin{aligned} \ddot{x} &= \bar{m}^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k - w_m + f\right) \\ z_c &= \bar{c}\dot{x} \\ z_k &= \bar{k}x \\ z_m &= -\bar{c}\dot{x} - \bar{k}x - w_c - w_k - w_m + f \end{aligned}$$

Uncertain System in State Space: $(x_2 = \dot{x},\, x_1 = x)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \hline z_c \\ z_k \\ z_m \end{pmatrix} = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\bar{m}^{-1}\bar{k} & -\bar{m}^{-1}\bar{c} & -\bar{m}^{-1} & -\bar{m}^{-1} & -\bar{m}^{-1} & \bar{m}^{-1} \\ \hline 0 & \bar{c} & 0 & 0 & 0 & 0 \\ \bar{k} & 0 & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & -1 & -1 & -1 & 1 \end{array}\right] \begin{pmatrix} x_1 \\ x_2 \\ \hline w_c \\ w_k \\ w_m \\ f \end{pmatrix}$$

$$\begin{pmatrix} w_c \\ w_k \\ w_m \end{pmatrix} = \begin{bmatrix} \delta_c & 0 & 0 \\ 0 & \delta_k & 0 \\ 0 & 0 & \delta_m \end{bmatrix} \begin{pmatrix} z_c \\ z_k \\ z_m \end{pmatrix}$$

6.8 Example #6 [Parametric Uncertainty]

Mass/Spring/Damper system from Zhou, p. 171.

Uncertain System:

$$\ddot{x} + m^{-1} c\dot{x} + m^{-1} k x = m^{-1} f$$

where $m > 0, \quad c > 0, \quad k > 0.$

Uncertainty:

$$m^{-1} = \bar{m}^{-1}(1 + \delta_m) \qquad c = \bar{c}(1 + \delta_c) \qquad k = \bar{k}(1 + \delta_k)$$

where

$$|m^{-1} - \bar{m}^{-1}| < \delta_m \bar{m}^{-1} \qquad |c - \bar{c}| < \delta_c \bar{c} \qquad |k - \bar{k}| < \delta_k \bar{k}$$

and $|\delta_m| < \bar{\delta}_m \leq 1, \quad |\delta_c| < \bar{\delta}_c \leq 1, \quad |\delta_k| < \bar{\delta}_k \leq 1.$

Auxiliary inputs/outputs:
Define auxiliary outputs

$$\begin{aligned} z_c &= \bar{c}\dot{x}, \\ z_k &= \bar{k}x \end{aligned}$$

Define auxiliary inputs in feedback

$$\begin{aligned} w_c &= \delta_c z_c \\ w_k &= \delta_k z_k \end{aligned}$$

System with auxiliary inputs and outputs

$$\begin{aligned} \ddot{x} &= m^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f\right) \\ z_c &= \bar{c}\dot{x} \\ z_k &= \bar{k}x \end{aligned}$$

The multiplicative uncertainty:
Since

$$m^{-1} = \bar{m}^{-1}(1 + \delta_m)$$

we have

$$m^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f\right) = \bar{m}^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + w_m + f\right)$$

where

$$\begin{aligned} w_m &= \delta_m z_m \\ z_m &= -\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f \end{aligned}$$

System with auxiliary inputs and outputs

$$\begin{aligned} \ddot{x} &= \bar{m}^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + w_m + f\right) \\ z_c &= \bar{c}\dot{x} \\ z_k &= \bar{k}x \\ z_m &= -\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f \end{aligned}$$

Uncertain System in State Space: $(x_2 = \dot{x},\, x_1 = x)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \hline z_c \\ z_k \\ z_m \end{pmatrix} = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\bar{m}^{-1}\bar{k} & -\bar{m}^{-1}\bar{c} & -\bar{m}^{-1} & -\bar{m}^{-1} & \bar{m}^{-1} & \bar{m}^{-1} \\ \hline 0 & \bar{c} & 0 & 0 & 0 & 0 \\ \bar{k} & 0 & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & -1 & -1 & 0 & 1 \end{array}\right] \begin{pmatrix} x_1 \\ x_2 \\ \hline w_c \\ w_k \\ w_m \\ f \end{pmatrix}$$

$$\begin{pmatrix} w_c \\ w_k \\ w_m \end{pmatrix} = \begin{bmatrix} \delta_c & 0 & 0 \\ 0 & \delta_k & 0 \\ 0 & 0 & \delta_m \end{bmatrix} \begin{pmatrix} z_c \\ z_k \\ z_m \end{pmatrix}$$