Robust and Multivariable Control — MAE 284

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Contents

1 Linear Multivariable Systems

1.1 Nomenclature

Inputs (u), outputs (y), and state (x).

1.2 SISO

Single-Input-Single-Output
Several special results
"Classic" control theory (until the 60's)

1.3 MIMO

Multiple-Input-Multiple-Output
"Modern" control theory (from the 60's on)

1.4 Linear Time-Invariant (LTI)

Linear and ... $u(t) \to y(t) \quad \Rightarrow \quad u(t-\tau) \to y(t-\tau)$ for any τ.

1.5 State Space

LTI

$$\dot{x} = Ax + Bu, \quad x(0) = x_0,$$
$$y = Cx + Du.$$

Dimensions:

$$x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m, \qquad y \in \mathbb{R}^p.$$

Notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Transfer function:

$$H(s) = C(sI - A)^{-1}B + D$$

Note that $H(s)$ is always proper!

Similarity transformation:

$$\left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Similarity does not change transfer function

$$H(s) = CT(sI - T^{-1}AT)^{-1}T^{-1}B + D = C(sI - A)^{-1}B + D$$

System response:

$$Y(s) = \underbrace{H(s)U(s)}_{\text{Input response}} + \underbrace{C(sI - A)^{-1}x(0)}_{\text{Initial conditions response}}$$

MIMO comes for free!

1.6 Controllability

Controllability Matrix:

$$\mathcal{C}(A, B) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

Controllability Gramian:

$$X(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau,$$

Theorem: The following are equivalent:

1. The pair (A, B) is controllable
2. The controllability Gramian $X(t) \succ 0$ for any $0 < t < \infty$.
3. The controllability matrix $\mathcal{C}(A, B)$ has full-row rank.
4. There exists no $\lambda \in \mathbb{C}$ and $z \neq 0$ such that $z^* A = \lambda z^*, \quad z^* B = 0.$
5. There exists $K \in \mathbb{R}^{m \times n}$ so that the eigenvalues of the real matrix $A + BK$ can be placed anywhere in the complex plane. (recall that complex poles must appear as complex conjugates!)
6. The pair (A^T, B^T) is observable.

Lemma: If $X = \lim_{t \to \infty} X(t)$ exists then

$$AX + XA^T + BB^T = 0.$$

1.7 Observability

Observability Matrix:

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Controllability Gramian:

$$Y(t) = \int_0^t e^{A^T \tau} C^T C e^{A\tau} d\tau.$$

Theorem: The following are equivalent:

1. The pair (A, C) is observable.
2. The observability Gramian $Y(t) \succ 0$ for any $0 < t < \infty$.
3. The observability matrix $\mathcal{O}(A, C)$ has full-column rank.
4. There exists no $\lambda \in \mathbb{C}$ and $x \neq 0$ such that $Ax = \lambda x, \quad Cx = 0$.
5. There exists $L \in \mathbb{R}^{m \times n}$ so that the eigenvalues of the real matrix $A + LC$ can be placed anywhere in the complex plane.
6. The pair (A^T, C^T) is controllable.

Lemma: If $Y = \lim_{t \to \infty} Y(t)$ exists then

$$A^T Y + Y A + C^T C = 0.$$

1.8 Stabilizability

Theorem: The following are equivalent:

1. The pair (A, B) is stabilizable.
2. If there exists λ and $z \neq 0$ such that $z^* A = \lambda z^*$ with $\lambda + \lambda^* \geq 0$ then $z^* B \neq 0$.
3. There exists $K \in \mathbb{R}^{m \times n}$ so that the real matrix $A + BK$ is Hurwitz.
4. The pair (A^T, B^T) is detectable.

1.9 Detectability

Theorem: The following are equivalent:

1. The pair (A, C) is detectable.
2. If there exists λ and $x \neq 0$ such that $Ax = \lambda x$ with $\lambda + \lambda^* \geq 0$ then $Cx \neq 0$.
3. There exists $L \in \mathbb{R}^{m \times n}$ so that the real matrix $A + LC$ is Hurwitz.
4. The pair (A^T, C^T) is stabilizable.

1.10 Operations

1. Transpose: $H^T(s)$
2. Conjugate: $H^\sim(s) = H^T(-s)$
3. Inverse: $H^{-1}(s)H(s) = H(s)H^{-1}(s) = I$
4. Series: $H(s) = H_1(s)H_2(s)$
5. Parallel: $H(s) = H_1(s) + H_2(s)$

WARNING: MIMO systems do not commute $H_1(s)H_2(s) \neq H_2(s)H_1(s)$!

State space formulas make it easy...

1. Transpose: $H^T(s) = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$
2. Conjugate: $H^\sim(s) = H^T(-s) = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right]$
3. Inverse: $H^{-1}(s) = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$
4. Series: $H(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$
5. Parallel: $H(s) = H_1(s) + H_2(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$

but usually lead to non minimal realizations.

1.11 Polynomial and Rational Matrices

Definition: Let $Q(s)$ be a matrix with polynomial entries on $s \in \mathbb{C}$.

$$\text{normal rank}(Q(s)) = \max_{s \in \mathbb{C}} \operatorname{rank} Q(s)$$

Example:

$$Q(s) = \begin{bmatrix} s & s+1 \\ 0 & 1 \end{bmatrix}$$

Normal rank $Q(s) = 2$ since rank $Q(1) = 2$. Note that rank $Q(0) = 1$.

Definition: A polynomial square matrix $U(s)$ is called *unimodular* if its inverse is also a polynomial matrix.

Theorem: A polynomial matrix $U(s)$ is *unimodular* if and only if $\det U(s)$ is independent of s, i.e., a constant (not zero).

Proof:

$$U^{-1}(s) = \frac{1}{\det U(s)} \operatorname{Adj} U(s)$$

and $\operatorname{Adj} U(s)$ is always a polynomial matrix.

Example:

$$U(s) = \begin{bmatrix} s+1 & s \\ s & s-1 \end{bmatrix}$$

is unimodular since

$$|U(s)| = (s+1)(s-1) - s^2 = s^2 - 1 - s^2 = -1.$$

Definition: The square polynomial matrix $R(s)$ is a *right divisor* of the polynimial matrix $N(s)$ if

$$N(s) = \bar{N}(s)R(s)$$

for some polynomial matrix $\bar{N}(s)$.

Definition: The square polynomial matrix $R(s)$ is a *greatest common right divisor* of the polynimial matrices $D(s)$ and $N(s)$ if $R(s)$ is a right divisor of $D(s)$ and $N(s)$ and $R(s) = W(s)\bar{R}(s)$ with $W(s)$ polynomial for any other right divisor $\bar{R}(s)$ of $D(s)$ and $N(s)$.

Example:

$$D(s) = \begin{bmatrix} s^2 - s & s^2 \\ 2s^2 + 9s + 5 & 2s^2 + 5s + 5 \end{bmatrix}, \qquad N(s) = \begin{bmatrix} s^2 + 1 & s^2 + 2s + 1 \end{bmatrix}$$

Verify that

$$\begin{aligned} \bar{D}(s) &= \begin{bmatrix} s^2 - s & \frac{1}{6}(17s^3 - 23s^2 + 6s + 6) \\ 2s^2 + 9s + 5 & \frac{1}{6}(34s^3 + 141s^2 + 31s - 54) \end{bmatrix}, \\ \bar{N}(s) &= \begin{bmatrix} s^2 + 1 & \frac{1}{6}(17s^3 - 6s^2 + 17s + 6) \end{bmatrix}, \\ R(s) &= \begin{bmatrix} 1 & \frac{1}{6}(-17s^2 + 6s + 6) \\ 0 & s \end{bmatrix} \end{aligned}$$

are such that

$$N(s) = \bar{N}(s)R(s), \qquad D(s) = \bar{D}(s)R(s)$$

In this case $R(s)$ is also a greatest common right divisor.

Matrix divisors can be computed using a combination of the Gauss elimination algorithm with the Euclidian division of scalar polynomials. Usually messy!

Definition: Two polynomial matrices $N(s)$ and $D(s)$ are *right coprime* if the equations

$$N(s) = \bar{N}(s)U(s), \qquad D(s) = \bar{D}(s)U(s)$$

can be solved only with $U(s)$ unimodular. In other words, if all greatest common divisors are unimodulars.

Theorem [Bezout identity]: Two polynomial matrices $N(s)$ and $D(s)$ are *right coprime* if and only if there exists polynomial matrices $X(s)$ and $Y(s)$ such that

$$X(s)N(s) + Y(s)D(s) = I.$$

Proof: Sufficiency: Find the greatest common right divisor $R(s)$

$$N(s) = \bar{N}(s)R(s), \qquad D(s) = \bar{D}(s)R(s).$$

If the Bezout identity is satisfied for some $X(s)$ and $Y(s)$ then

$$\begin{aligned} I &= X(s)N(s) + Y(s)D(s) \\ &= X(s)\bar{N}(s)R(s) + Y(s)\bar{D}(s)R(s) \\ &= \left[X(s)\bar{N}(s) + Y(s)\bar{D}(s)\right] R(s) \end{aligned}$$

so that

$$R^{-1}(s) = X(s)\bar{N}(s) + Y(s)\bar{D}(s)$$

Since the right hand side of the above equation is a polynomial $R(s)$ must be unimodular, i.e., $N(s)$ and $D(s)$ are coprime.
Necessity: By construction. It is possible to show that any greatest common right divisor of $N(s)$ and $D(s)$ can be written as

$$R(s) = \hat{X}(s)N(s) + \hat{Y}(s)D(s)$$

for some polynomial $\hat{X}(s)$ and $\hat{Y}(s)$. Therefore

$$\begin{aligned} I &= R^{-1}(s)R(s) \\ &= R^{-1}(s)\hat{X}(s)N(s) + R^{-1}(s)\hat{Y}(s)D(s) \\ &= X(s)N(s) + Y(s)D(s) \end{aligned}$$

for $X(s) = R^{-1}(s)\hat{X}(s)$, $Y(s) = R^{-1}(s)\hat{Y}(s)$.

Theorem [Smith form]: Let $N(s)$ be a polynomial matrix on s with normal rank r. There exists two polynomial and unimodular matrices $L(s)$ and $R(s)$ such that $N(s) = L(s)S(s)R(s)$ and

$$S(s) = \begin{bmatrix} \sigma_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \sigma_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

where

1. $\sigma_i(s)$ is monic,
2. $\sigma_i(s)$ divides $\sigma_{i+1}(s)$,

for all $i = 1, \cdots, r$.

Corresponding definitions and properties hold for *left* divisor, coprime, etc.

Similar definitions and properties hold for *left* divisor, coprime, etc, for *rational* matrices. We will see some of that next!

Theorem [Bezout identity]: Two rational matrices $F(s)$ and $G(s)$ are *right coprime* if and only if there exists rational matrices $X(s)$ and $Y(s)$ such that

$$X(s)F(s) + Y(s)G(s) = I.$$

Theorem [Smith-McMillian form]: Let $H(s)$ be a proper rational matrix function of s with normal rank r. There exists two polynomial and unimodular matrices $L(s)$ and $R(s)$ such that $H(s) = L(s)M(s)R(s)$ and

$$M(s) = \begin{bmatrix} \epsilon_1(s)/\psi_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \epsilon_r(s)/\psi_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

where

1. $\epsilon_i(s), \psi_i(s)$ are monic,
2. $\epsilon_i(s), \psi_i(s)$ are coprime,

for all $i = 1, \cdots, r$ and

3. $\epsilon_i(s)$ divides $\epsilon_{i+1}(s)$,
4. $\psi_{i+1}(s)$ divides $\psi_i(s)$,

for all $i = 1, \cdots, r-1$.

1.12 Poles and Zeros

Definition: Let $H(s)$ be a rational transfer matrix and $S(s)$ its associated Smith-McMillian form. The poles of $H(s)$ are the roots of $\prod_{i=1}^{r} \psi_i(s)$ and the zeros (also known as transmission zeros) are the roots of $\prod_{i=1}^{r} \epsilon_i(s)$.

Example: Let

$$H(s) = \begin{bmatrix} \frac{(s-3)(s+1)}{(s-5)(s-2)} \\ \frac{(s-3)}{(s-5)} \end{bmatrix} = \begin{bmatrix} s+1 & 1 \\ s-2 & 1 \end{bmatrix} \begin{bmatrix} \frac{(s-3)}{(s-5)(s-2)} \\ 0 \end{bmatrix}$$

$H(s)$ has two poles $\{2, 5\}$ and one zero $\{-3\}$. Note that $\{-1\}$ is not a zero!

Theorem: If $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a state space realization of $H(s)$ then the eigenvalues of A are the poles of $H(s)$.

Theorem: If $s_0 \in \mathbb{C}$ is such that

$$\operatorname{rank} H(s_0) < \text{normal rank}\, H(s)$$

then s_0 is a *transmission zero* of $H(s)$.

Example: Note that normal rank $H(s) = 1$, rank $H(3) = 0$ but rank $H(-1) = 1$!

Theorem: If $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a state space realization of $H(s)$ and $s_0 \in \mathbb{C}$ is such that

$$\operatorname{rank} \begin{bmatrix} A - s_0 I & B \\ C & D \end{bmatrix} < \text{normal rank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$$

then s_0 is an *invariant zero* of $H(s)$.

Example: A state space realization for $H(s)$ on the previous example is

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -10 & 7 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 & 2 \\ 1 & -1 & 1 & 6 & 0 \\ \hline -13 & 5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Therefore

$$P(s) = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} -s & 1 & 0 & 0 & 0 \\ -10 & 7-s & 0 & 0 & 1 \\ 0 & 0 & 5-s & 0 & 2 \\ 1 & -1 & 1 & 6-s & 0 \\ -13 & 5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

so that normal rank $P(s) = 5$, e.g. rank $P(0) = 5$, while rank $P(3) = 4$ and rank $P(6) = 4$. Invariant zeros are then $\{3, 6\}$. Note that 6 is not a transmission zero!

Theorem: A transmission zero is also an invariant zero.

Lemma: Invariant zeros are not affected by state feedback or similarity transformations.
Proof: Note that

$$\begin{bmatrix} A + BK - s_0 I & B \\ C + DK & D \end{bmatrix} = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ K & I \end{bmatrix}$$

and $\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}$ has full rank.
For the same reason

$$\begin{bmatrix} T^{-1}AT - sI & T^{-1}B \\ CT & D \end{bmatrix} = \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}.$$

1.13 Minimal Realizations

Theorem: A state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is minimal if and only if (A, B) is controllable and (A, C) is observable.

Check out material from MAE 280A on how to construct minimal realizations.

WARNING: Realizations for MIMO transfer functions are not trivial! Check out Kailath, Chapter 6 and beyond.

Theorem: A state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is minimal if and only if $n = r$, where r is the degree of the Smith-McMillan form associated with $H(s) = C(sI - A)^{-1}B + D$.

Theorem: If $H(s)$ is minimal, then all invariant zeros are transmission zeros.

2 H_2 and H_∞ Spaces

2.1 Complex-valued Functions

Let $S \subset \mathbb{C}$ and $f(s) : S \to \mathbb{C}$.

f is *analytic at* $s_0 \in S$ if it is differentiable at s_0

If f is analytic at $s_0 \in S$ then it has continuous derivatives of all orders at s_0

If f is analytic at $s_0 \in S$ then it addmits a *power series expansion* at s_0; The converse is also true

f is *analytic in* S if it is analytic at all $s \in S$

Maximum Modulus Theorem: Let S be a closed and bounded set and f be analytic in the interior of S ($\operatorname{int} S$). The function $|f(s)|$ attain its maximum in $\operatorname{int} S$ if and only if $f(s)$ is constant.

Cauchy Integral Theorem: Let S be a simply connected region and f be analytic in $\operatorname{int} S$. The integral of f on any closed curve in S is zero.

Examples

1. e^s is analytic everywhere in $\mathbb{C}$
2. $e^s = \sum_{i=0}^{\infty} \frac{s^i}{i!}$ for all $s \in \mathbb{C}$
3. $\frac{1}{s-a}$ is analytic in $\mathbb{C} \backslash \{a\}$

2.2 Linear Vector Spaces

Things we should know:

1. What is a linear vector space?
2. What is a subspace?
3. What is the dimension of a linear space (subspace)?
4. Let X be a vector space over $\mathbb{C}$
   (a) $\|x\| : X \to \mathbb{R}$ is a *norm* if
      i. $\|x\| \geq 0 \quad \forall x \in X, \quad \|x\| = 0 \Leftrightarrow x = 0$
      ii. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$ (triangle inequality)
      iii. $\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in \mathbb{C}, x \in X$
   (b) $\langle x, y \rangle : X \times X \to \mathbb{C}$ is an *inner product* if
      i. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
      ii. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
      iii. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \forall \alpha \in \mathbb{C}$
      iv. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
5. Normed Spaces: vector space + norm

Examples (Linear Vector Spaces):

1. $\mathbb{C}^n$ and $\mathbb{C}^{m \times n}$
2. $x = p(t) \in \mathbb{C}^{m \times n} \quad \forall\, t \in [a, b] \quad p(\cdot)$ is a function in $[a, b]$
3. $x = p(s) \in \mathbb{C}^{m \times n} \quad \forall\, s \in \mathbb{C} \quad p(\cdot)$ is a polynomial

Examples (Norms)

1. $\mathbb{C}^n : \quad \|x\| = \sum_i |x_i|, \quad \|x\| = \max_i |x_i|$
2. $\mathbb{C}^{m \times n} : \quad \|x\| = \max_i \sum_j |x_{ij}|, \quad \|x\| = \max_j \sum_i |x_{ij}|$
3. $\|x\| = \sup_{t \in [a,b]} \|x(t)\|$

2.3 $L_\infty[a,b]$ Space

Linear Vector Space: real-valued functions

Norm: $\|f\| = \sup_{t\in[a,b]} \overline{\sigma}[f(t)]$

$\overline{\sigma}(X)$ is the maximum singular value of the complex matrix X

L_∞ Space: all f such that $\|f\| < \infty$

Examples:

1. $f(t) = t^2 - t + 3 \in L_\infty[0,1]$
2. $f(t) = t^{-1} \notin L_\infty(-\infty,\infty)$, $f \in L_\infty[1,\infty)$
3. $f(t) = t^{-1/2} \in L_\infty[1,\infty)$
4. $f(t) = e^{-t} \in L_\infty[0,\infty)$
5. $f(t) = e^{t} \notin L_\infty[0,\infty)$
6. $f(t) = e^{-|t|} \in L_\infty(-\infty,\infty)$

2.4 $L_\infty(j\mathbb{R})$ Space

Linear Vector Space: complex-valued functions

Norm: $\|F\| = \sup_{\omega\in\mathbb{R}} \overline{\sigma}[F(j\omega)]$

$\overline{\sigma}(X)$ is the maximum singular value of the complex matrix X

$L_\infty(j\mathbb{R})$ Space: all F such that $\|F\| < \infty$

$RL_\infty(j\mathbb{R})$ Space: F is rational and $F \in L_\infty(j\mathbb{R})$; $RL_\infty(j\mathbb{R}) \subset L_\infty(j\mathbb{R})$

SISO $RL_\infty(j\mathbb{R})$ = rational, proper, no poles on the imaginary axis

Examples:

1. $F(s) = \frac{1}{s+2} \in RL_\infty(j\mathbb{R})$
2. $F(s) = \frac{1}{s-2} \in RL_\infty(j\mathbb{R})$
3. $F(s) = \frac{s+1}{s} \notin L_\infty(j\mathbb{R})$
4. $F(s) = \frac{e^{-s}}{s+2} \in L_\infty(j\mathbb{R}), \quad F \notin RL_\infty(j\mathbb{R})$
5. $F(s) = \begin{bmatrix} \frac{(s-3)(s+1)}{(s-5)(s-2)} \\ \frac{(s-3)}{(s-5)} \end{bmatrix} \in RL_\infty(j\mathbb{R})$

2.5 H_∞ Space

Linear Vector Space: complex-valued functions, analytic in $\mathrm{Re}(s) > 0$

Norm:

$$\begin{aligned}\|F\| &= \sup_{\mathrm{Re}(s)>0} \overline{\sigma}[F(s)] \\ &= \sup_{\omega\in\mathbb{R}} \overline{\sigma}[F(j\omega)]\end{aligned}$$

H_∞ Space: F analytic in $\mathrm{Re}(s) > 0$ and $\|F\| < \infty$; $H_\infty \subset L_\infty(j\mathbb{R})$

RH_∞ Space: F is rational and $F \in H_\infty$; $RH_\infty \subset H_\infty$

SISO RH_∞ : rational, proper and asymptotically stable

Examples:

1. $F(s) = \frac{1}{s+2} \in RH_\infty$
2. $F(s) = \frac{1}{s-2} \notin H_\infty$
3. $F(s) = \frac{e^{-s}}{s+2} \in H_\infty, \quad F \notin RH_\infty$
4. $F(s) = \begin{bmatrix} \frac{(s-3)(s+1)}{(s+5)(s+2)} \\ \frac{(s-3)}{(s+5)} \end{bmatrix} \in RH_\infty$

2.6 Pre-Hilbert and Hilbert Spaces

Pre-Hilbert Spaces: vector space + inner product

The function $\|x\| := \sqrt{\langle x, x\rangle}$ is a norm

(Pre-)Hilbert Spaces are normed spaces

Pre-Hilbert Space + completeness = Hilbert Space

Some facts:

1. x, y are *orthogonal* if $\langle x, y\rangle = 0$. $(x \perp y)$
2. $|\langle x, y\rangle| \leq \|x\|\|y\|$; (Cauchy-Schwarz Inequality)
   $|\langle x, y\rangle| = \|x\|\|y\|$ iff $x = \alpha y$, $\alpha \in \mathbb{C}$, or $x(y) = 0$
3. $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (Parallelogram Law)
4. $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if $x \perp y$ (Pythagorean Theorem)

Projection Theorem: Let S be a closed subspace of a Hilbert space X. For a given $x \in X$ there is a *unique* vector $\bar{s} \in S$ such that $\|x - \bar{s}\| \leq \|x - s\|, \forall s \in S$, or, in other words, $\bar{s} = \arg\inf_{s \in S} \|x - s\|$. This vector is such that $(x - \bar{s}) \perp S$.

Examples (Inner Products)

1. $\mathbb{C}^n : \quad \langle x, y\rangle = x^* y$
2. $\mathbb{C}^{m\times n} : \quad \langle x, y\rangle = \operatorname{trace}(x^* y)$
3. $L_2[a, b] : \quad \langle x, y\rangle = \int_a^b \operatorname{trace}[x^*(t) y(t)]\, dt$

2.7 $L_2[a,b]$ Space

Linear Vector Space: real-valued functions in $[a,b]$

Inner product:

$$\langle f, g\rangle = \int_a^b \operatorname{trace}[f^T(t)g(t)]\,dt$$

Norm: induced by the inner product (Hilbert space)

$L_2[a,b]$ Space: f such that $\|f\| < \infty$

Examples:

1. $f(t) = t^2 - t + 3 \in L_2[0,1]$
2. $f(t) = t^{-1} \notin L_2[0,\infty)$, $f \in L_2[1,\infty)$, $(\int_1^\infty t^{-2}\,dt = -t^{-1}\big|_1^\infty = 1)$
3. $f(t) = t^{-1/2} \notin L_2[1,\infty)$ $(\int_1^\infty t^{-1}\,dt = \log t|_1^\infty = \infty)$
4. $f(t) = e^{-t} \in L_2(0,\infty)$
5. $f(t) = e^{t} \notin L_2(0,\infty)$
6. $f(t) = e^{-|t|} \in L_2(-\infty,\infty)$

2.8 $L_2(j\mathbb{R})$ Space

Linear Vector Space: complex-valued functions

Inner product:

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{trace}[F^{\sim}(j\omega) G(j\omega)] \, d\omega$$

Norm: induced by the inner product (Hilbert space)

$L_2(j\mathbb{R})$ Space: F such that $\|F\| < \infty$

$RL_2(j\mathbb{R})$ Space: F is rational and $F \in L_2(j\mathbb{R})$; $RL_2(j\mathbb{R}) \subset L_2(j\mathbb{R})$

For SISO $RL_2(j\mathbb{R})$ = strictly proper, no poles on the imaginary axis

Parseval's Theorem: Let $f \in L_2(-\infty, \infty)$ and $F(s)$ be its Laplace Transform. Then $F \in L_2(j\mathbb{R})$ and $\|f\| = \|F\|$.

Examples:

1. $F(s) = \frac{1}{s+2} \in RL_2(j\mathbb{R})$
2. $F(s) = \frac{1}{s-2} \in RL_2(j\mathbb{R})$
3. $F(s) = \frac{1}{s(s+1)} \notin RL_2(j\mathbb{R})$
4. $F(s) = \frac{s+1}{s+2} \notin L_2(j\mathbb{R})$ $\quad (F(j\infty) \to 1!)$
5. $F(s) = e^{-s} \notin L_2(j\mathbb{R})$ $\quad (F^{\sim}(j\omega)F(j\omega) = 1 \quad \forall\omega!)$
6. $F(s) = \frac{e^{-s}}{s-2} \in L_2(j\mathbb{R})$

2.9 H_2 Space

Linear Vector Space: complex-valued functions, analytic in $\mathrm{Re}(s) > 0$

Inner product:

$$\begin{aligned}\langle F, G\rangle &= \frac{1}{2\pi}\sup_{\sigma>0}\left\{\int_{-\infty}^{\infty}\mathrm{trace}[F^{\sim}(\sigma+j\omega)G(\sigma+j\omega)]\,d\omega\right\}\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{trace}[F^{\sim}(j\omega)G(j\omega)]\,d\omega\end{aligned}$$

Norm: induced by the inner product (Hilbert space)

H_2 Space: F analytic in $\mathrm{Re}(s) > 0$ and $\|F\| < \infty$; $H_2 \subset L_2(j\mathbb{R})$

RH_2 Space: F is rational and $F \in H_2$; $RH_2 \subset H_2$

For SISO $RH_2 =$ rational, strictly proper and asymptotically stable

Parseval's Theorem: Let $f \in L_2[0,\infty)$ and $F(s)$ be its Laplace Transform. Then $F \in H_2$ and $\|f\| = \|F\|$.

Examples:

1. $F(s) = \frac{1}{s+2} \in RH_2$
2. $F(s) = \frac{1}{s-2} \notin RH_2$ (not stable!)
3. $F(s) = \frac{s+1}{s+2} \notin H_2$ $(F(j\infty) \to 1!)$
4. $F(s) = e^{-s} \notin H_2$ $(F^{\sim}(j\omega)F(j\omega) = 1 \quad \forall\omega!)$
5. $F(s) = \frac{e^{-s}}{s+2} \in H_2$
6. $F(s) = \begin{bmatrix}\frac{(s+1)}{(s+2)}\\ 1\end{bmatrix} \notin RH_2,$ $\left(F^{\sim}(j\omega)F(j\omega) = 1 + \frac{|j\omega+1|^2}{|j\omega+2|^2} \geq 1 \quad \forall\omega!\right)$

2.10 $H_2^\perp$ Space

Linear Vector Space: complex-valued functions, analytic in $\mathrm{Re}(s) < 0$

Inner product:

$$\langle F, G \rangle = \frac{1}{2\pi} \sup_{\sigma<0} \left\{ \int_{-\infty}^{\infty} \mathrm{trace}[F^\sim(\sigma + j\omega) G(\sigma + j\omega)]\, d\omega \right\}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{trace}[F^\sim(j\omega) G(j\omega)]\, d\omega$$

Norm: induced by the inner product (Hilbert space)

$H_2^\perp$ Space: F analytic in $\mathrm{Re}(s) < 0$ and $\|F\| < \infty$; $H_2^\perp \subset L_2(j\mathbb{R})$

$RH_2^\perp$ Space: F is rational and $F \in H_2^\perp$; $RH_2^\perp \subset H_2^\perp$

For SISO $RH_2^\perp =$ rational, strictly proper and asymptotically anti-stable

Parseval's Theorem: Let $f \in L_2(-\infty, 0]$ and $F(s)$ be its Laplace Transform. Then $F \in H_2^\perp$ and $\|f\| = \|F\|$.

Examples:

1. $F(s) = \frac{1}{s+2} \notin RH_2^\perp$ (stable!)
2. $F(s) = \frac{1}{s-2} \in RH_2^\perp$
3. $F(s) = \frac{s+1}{s-2} \notin H_2^\perp$ $(F(j\infty) \to 1!)$
4. $F(s) = e^{-s} \notin H_2^\perp$ $(F^\sim(j\omega)F(j\omega) = 1 \quad \forall\omega!)$
5. $F(s) = \frac{e^{-s}}{s+2} \in H_2^\perp$
6. $F(s) = \begin{bmatrix} \frac{(s+1)}{(s-2)} \\ 1 \end{bmatrix} \notin RH_2,$ $\left(F^\sim(j\omega)F(j\omega) = 1 + \frac{|j\omega+1|^2}{|j\omega-2|^2} \geq 1 \quad \forall\omega!\right)$

2.11 Relations between H_2, $H_2^\perp$ and $L_2(j\mathbb{R})$ spaces

Lemma: Let $G(s) \in RH_2^\perp$. Then $G^\sim(s) \in RH^2$.
Proof: Because $G^\sim(s) = G^T(-s)$, if $\bar{s}$ is a pole of $G(s)$ then $-\bar{s}$ is a pole of $G^\sim(s)$.

Lemma: Let $F(s) \in RH_2$ and $G(s) \in RH_2^\perp$. Then $F(s) \perp G(s)$.
Proof: Consider the semicircle C_ρ of radius ρ centered at the origin and entirely contained in the right hand side of the complex plane. Because $G^\sim(s) \in RH_2$, the rational scalar function

$$T(s) = \mathrm{trace}[F(s)G^\sim(s)] \in RH_2.$$

Therefore $T(s)$ is analytic in C_ρ for any value of $\rho > 0$, and the Cauchy Integral Theorem states that

$$\oint_{C_\rho} T(s)\,ds = \int_{-\rho}^{\rho} T(j\omega)\,d\omega - \int_{-\pi/2}^{\pi/2} T(\rho e^{j\theta})\,d\theta = 0$$

Since $T(s)$ is in RH_2 it is strictly proper, hence

$$\lim_{\rho\to\infty} \int_{-\pi/2}^{\pi/2} T(\rho e^{j\theta})\,d\theta = 0.$$

Consequently

$$\langle F(s), G(s)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} T(j\omega)\,d\omega = \oint_{C_\infty} T(s)\,ds = 0$$

Corollary: $RL_2(j\mathbb{R}) = RH_2 \oplus RH_2^{\perp}$
Proof: Using *partial fractions expansion*, any $H(s) \in RL_2(j\mathbb{R})$ can be written

$$H(s) = H_{\mathrm{s}}(s) + H_{\mathrm{a}}(s)$$

where $H_{\mathrm{s}}(s) \in RH_2$ and $H_{\mathrm{a}}(s) \in RH_2^{\perp}$, so that $H_{\mathrm{s}}(s) \perp H_{\mathrm{a}}(s)$.

Corollary: Let $H(s) = H_{\mathrm{s}}(s) + H_{\mathrm{a}}(s) \in RL_2(j\mathbb{R})$ where $H_{\mathrm{s}}(s) \in RH_2$ and $H_{\mathrm{a}}(s) \in RH_2^{\perp}$. Then $\|H\|^2 = \|H_{\mathrm{s}}(s)\|^2 + \|H_{\mathrm{a}}(s)\|^2$

Example:

$$H(s) = \frac{2s}{(s+1)(s-1)} = \underbrace{\frac{1}{s+1}}_{H_{\mathrm{s}}(s)} \quad + \quad \underbrace{\frac{1}{s-1}}_{H_{\mathrm{a}}(s)}$$

Note that

$$\begin{aligned}\|H_{\mathrm{s}}\|^2 = \left\|(s+1)^{-1}\right\|^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2+1}\,d\omega \\ &= \frac{1}{2\pi}\arctan\omega\big|_{-\infty}^{\infty} = \frac{1}{2\pi}(\frac{\pi}{2} - \frac{-\pi}{2}) = \frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\|H\|^2 &= \|H_{\mathrm{s}}\|^2 + \|H_{\mathrm{a}}\|^2 \\ &= \|H_{\mathrm{s}}\|^2 + \|H_{\mathrm{a}}^{\sim}\|^2 \\ &= \left\|(s+1)^{-1}\right\|^2 + \left\|-(s+1)^{-1}\right\|^2 \\ &= 2\left\|(s+1)^{-1}\right\|^2 = 1\end{aligned}$$

2.12 Linear Operators

Definition: A linear map between two linear spaces. $T : X \to Y$

Notation: $T : X \to Y$, $y = Tx$

Example: A linear time-invariant system H! Takes a signal $u \in L_2[0, \infty)$ and produces a $y = Hu \in L_2[0, \infty)$.

Operator norm: When X and Y are normed spaces

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|$$

Submultiplicative property: $\|AB\| \leq \|A\|\|B\|$
Proof: Note that for any $x \in X$

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \geq \frac{\|Tx\|}{\|x\|} \quad \Rightarrow \quad \|Tx\| \leq \|T\|\|x\|$$

Therefore

$$\begin{aligned} \|AB\| &= \sup_{x \neq 0} \frac{\|ABx\|}{\|x\|} \\ &\leq \sup_{x \neq 0} \frac{\|A\|\|Bx\|}{\|x\|} \\ &\leq \|A\| \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} \\ &\leq \|A\|\|B\| \end{aligned}$$

Example: $T : \mathbb{C}^n \to \mathbb{C}^n$, $y = Tx$
Since

$$\|Tx\|^2 = x^* T^* T x \leq \max_i \lambda_i(T^*T)\, x^* x = \overline{\sigma}^2(T)\, \|x\|^2$$

with equality holding when x is an eigenvector of T

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \overline{\sigma}(T)$$

2.13 H_∞ norm as an operator norm

Theorem: Let $H(s)$ be the transfer function of a linear time-invariant system. Let the input $u \in L_2[0,\infty)$ and assume the output $y \in L_2[0,\infty)$. The norm of the linear operator $H : L_2[0,\infty) \to L_2[0,\infty)$ is equal to the H_∞ norm of the transfer function $H(s)$.

Half proof: First note that

$$\|H\| = \sup_{\|u\|=1} \|y\|, \quad u \in L_2[0,\infty), \quad y = Hu \in L_2[0,\infty)$$

As u and y are in $L_2[0,\infty)$ compute the corresponding pair of unilateral Laplace transform U, $Y \in H_2$. Then use Parseval's theorem to compute

$$\|H\| = \sup_{\|U\|=1} \|Y\|, \quad U(s) \in H_2, \quad Y(s) = H(s)U(s) \in H_2$$

Now note that for any U

$$\begin{aligned}
\|Y\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}[U^\sim(j\omega)H^\sim(j\omega)H(j\omega)U(j\omega)]\, d\omega, \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}[U^\sim(j\omega)\sup_{\omega\in\mathbb{R}}[H^\sim(j\omega)H(j\omega)]\,U(j\omega)]\, d\omega, \\
&\leq \sup_{\omega\in\mathbb{R}} \overline{\sigma}^2[H(j\omega)]\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}[U^\sim(j\omega)U(j\omega)]\, d\omega, \\
&\leq \|H\|_\infty^2 \|U\|_2^2
\end{aligned}$$

which implies that

$$\|H\| = \sup_{U\neq 0} \frac{\|Y\|_2}{\|U\|_2} \leq \|H\|_\infty.$$

The second part is to prove that $\|H\| \geq \|H\|_\infty$. See Zhou p. 52 or Colaneri p. 53.

2.14 Inner Functions

Definition: A transfer function $F(s) \in RL_\infty$ is called *all-pass* if F is square and $F^\sim(s)F(s) = I$.

Definition: A transfer function $F(s) \in RH_\infty$ is called *inner* if $F^\sim(s)F(s) = I$.

Definition: A transfer function $F(s) \in RH_\infty$ is called *co-inner* if $F(s)F^\sim(s) = I$.

Definition: A transfer function $F(s) \in RH_\infty$ is called *outer* if there exists $X(s)$ analytic in $\mathrm{Re}(s) > 0$ such that $F(s)X(s) = I$.

Lemma: Let $F(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RH_\infty$ and let $X = X^T$ solve the Lyapunov equation

$$A^*X + XA + C^*C = 0$$

If $D^TC + B^TX = 0$ then $F^\sim(s)F(s) = D^TD$. If (A, B) is controlable and $F^\sim(s)F(s) = D^TD$ then $D^TC + B^TX = 0$.

Proof: Note that

$$\begin{aligned} F^\sim(s)F(s) &= \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\ &= \left[\begin{array}{cc|c} -A^T & -C^TC & -C^TD \\ 0 & A & B \\ \hline B^T & D^TC & D^TD \end{array}\right] \end{aligned}$$

Applying the similarity transformation defined by

$$T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

we obtain

$$F^{\sim}(s)F(s) = \left[\begin{array}{cc|c} -A^T & -A^TX - XA - C^TC & -C^TD - BX \\ 0 & A & B \\ \hline B^T & B^TX + D^TC & D^TD \end{array}\right]$$
$$= \left[\begin{array}{cc|c} -A^T & 0 & -C^TD - BX \\ 0 & A & B \\ \hline B^T & B^TX + D^TC & D^TD \end{array}\right]$$

Corollary: Let $F(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RH_\infty$ be a minimal realization and $X = X^T$ solve the Lyapunov equation

$$A^*X + XA + C^*C = 0$$

$F(s)$ is inner if and only if $D^TC + B^TX = 0$ and $D^TD = I$.

Lemma: Let $F(s) \in RH_\infty$ be a scalar transfer function. There exists an inner function $F_i(s)$ and an outer function $F_o(s)$ such that $F(s) = F_i(s)F_o(s)$. If $F(j\omega) \neq 0$ then $F_o^{-1}(s) \in RH_\infty$.
Proof: Assume $F(s)$ has $m_r + 2m_c$ non-minimum phase zeros

$$\{\alpha_1, \ldots, \alpha_{m_r}, \quad \alpha_{m_r+1} \pm j\beta_{m_r+1}, \ldots, \alpha_{m_r+m_c} \pm j\beta_{m_r+m_c}\}$$

where $\alpha_i > 0$, $i = 1, \ldots, m_r + m_c$.
Define

$$F_i(s) = \prod_{i=1}^{m_r} \frac{s - \alpha_i}{s + \alpha_i} \prod_{i=m_r+1}^{m_r+m_c} \frac{s - \alpha_i \mp j\beta_i}{s + \alpha_i \pm j\beta_i}$$

Note that $F_i(s)$ is inner, i.e.,

$$F_i^{\sim}(s)F_i(s) = \prod_{i=1}^{m_r} \frac{s + \alpha_i}{s - \alpha_i}\frac{s - \alpha_i}{s + \alpha_i} \prod_{i=m_r+1}^{m_r+m_c} \frac{s + \alpha_i \pm j\beta_i}{s - \alpha_i \mp j\beta_i}\frac{s - \alpha_i \mp j\beta_i}{s + \alpha_i \pm j\beta_i} = 1$$

Now define

$$F_o(s) = F(s)F_i^{-1}(s).$$

All zeros of $F_o(s)$ have nonpositive real part. Therefore $F_o^{-1}(s)$ is analytic in $\mathrm{Re}(s) > 0$. If $F(j\omega) \neq 0$ then $F_o(s)$ has no zeros on the imaginary axis so that $F_o(s)^{-1} \in RH_\infty$.

3 Computing RH_2 and RH_∞ Norms

3.1 Sylvester and Lyapunov Equations

Lemma 1: Consider the Sylvester Equation

$$AX + XB = C$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times m}$ and $C \in \mathbb{C}^{n\times m}$. A solution $X \in \mathbb{C}^{n\times m}$ exists and is unique if and only if $\lambda_i(A) + \lambda_j(B) \neq 0$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Proof: The Sylvester Equation is a linear equation and it has a unique solution if and only if the homogeneous equation associated with the Sylvester equation admits only the trivial solution. Assume it does not, that is, there $\bar{X} \neq 0$ such that

$$A\bar{X} + \bar{X}B = 0$$

Then, multiplication of the above on the left by $v_i^* \neq 0$, the ith left eigenvector of A, and on the right by $u_j \neq 0$, the jth right eigenvector of B, yields

$$0 = v_i^* A\bar{X}u_j + v_i^*\bar{X}Bu_j = [\lambda_i(A) + \lambda_j(B)]\, v_i^*\bar{X}u_j.$$

Since $\lambda_i(A) + \lambda_j(B) \neq 0$ by hypothesis we must have $v_i^*\bar{X}u_j$ for all i, j. One can show that this indeed implies $\bar{X} = 0$, establishing a contradiction.

Lemma 2: Consider the Lyapunov Equation

$$A^T X + XA + C^T C = 0$$

where $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$.

1. A solution $X \in \mathbb{C}^{n\times n}$ exists and is unique if and only if $\lambda_i(A) + \lambda_j^*(A) \neq 0$ for all $i, j = 1, \ldots, n$. Furthermore X is symmetric.
2. If A is Hurwitz then X is positive semidefinite.
3. If (A, C) is detectable and X is positive semidefinite then A is Hurwitz.
4. If (A, C) is observable and A is Hurwitz then X is positive definite.

Proof:
Item 1. is a corollary of Lemma 1. That X is symmetric follows from unicity since

$$\begin{aligned} 0 &= (A^T X + XA + C^T C)^T - (A^T X + XA + C^T C) \\ &= A^T (X^T - X) + (X^T - X)A \end{aligned}$$

so that $X^T - X = 0$.

Item 2. If A is Hurwitz then

$$X = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt \succeq 0$$

and

$$A^T X + XA = \lim_{t\to\infty} \int_0^\infty \frac{d}{dt} e^{A^T t} C^T C e^{At}\, dt = e^{A^T t} C^T C e^{At}\Big|_0^\infty = -C^T C$$

since $\lim_{t\to\infty} e^A t = 0$.

Item 3. Assume (A, C) is detectable, $X \succeq 0$ and that A is not Hurwitz. Then there exists λ and $x \neq 0$ such that $Ax = \lambda x$ with $\lambda + \lambda^* \geq 0$. But if X solves the Lyapunov equation

$$-x^* C^T C x = x^* A^T X x + x^* X A x = (\lambda + \lambda^*)\, x^* X x \geq 0$$

which implies $Cx = 0$, hence (A, C) not detectable.

Item 4. Assume that (A, C) is observable and A is Hurwitz. From Item 2. $X \succeq 0$. Assume X is not positive definite, that is, there exists $\bar{x} \neq 0$ such that $X\bar{x} = 0$. It follows that

$$0 = \bar{x}^* X \bar{x} = \int_0^\infty \bar{x}^* e^{A^T t} C^T C e^{At} \bar{x}\, dt = \int_0^\infty y^*(t) y(t)\, dt$$

which implies that response $y(t) = Ce^{At}\bar{x} = 0$ to a non null initial condition $x(0) = \bar{x}$ is null, which contradicts the hypothesis that (A, C) is observable.

3.2 Computing the RH_2 norm

Let $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a realization of $H(s) \in RH_2$. It follows that A can be assumed to be Hurwitz and $D = 0$. From Parseval's Theorem

$$\|H\| = \|h\|$$

where $h = h(t) \in L_2[0, \infty)$ is the impulse response

$$h(t) = \begin{cases} 0, & t < 0 \\ Ce^{At}B, & t \geq 0 \end{cases}$$

Then

$$\begin{aligned} \|H\|^2 &= \|h\|^2 \\ &= \int_0^\infty \mathrm{trace}[h^T(t)h(t)]\,dt \\ &= \int_0^\infty \mathrm{trace}[B^T e^{A^T t} C^T C e^{At} B]\,dt, \\ &= \mathrm{trace}\left[B^T \left(\int_0^\infty e^{A^T t} C^T C e^{At}\,dt\right) B\right], \\ &= \mathrm{trace}\left[B^T X B\right], \qquad X = \int_0^\infty e^{A^T t} C^T C e^{At}\,dt \succeq 0 \end{aligned}$$

Since A is Hurwitz X can be obtained by computing the unique[1] solution to the Lyapunov equation

$$A^T X + XA + C^T C = 0$$

[1] Because A is Hurwitz, $\mathrm{Re}[\lambda_i(A)] < 0$ so that $\lambda_i(A) + \lambda_j^*(A) < 0$ for any $i, j = 1, \ldots, n$.

Alternatively

$$
\begin{aligned}
\|H\|^2 &= \|h\|^2 \\
&= \int_0^\infty \mathrm{trace}[h(t)h^T(t)]\,dt \\
&= \mathrm{trace}\left[C\left(\int_0^\infty e^{At}BB^Te^{A^Tt}\,dt\right)C^T\right], \\
&= \mathrm{trace}\left[CYC^T\right], \qquad Y = \left(\int_0^\infty e^{At}BB^Te^{A^Tt}\,dt\right) \succeq 0
\end{aligned}
$$

and

$$AY + YA^T + BB^T = 0$$

3.3 Computing the RL_2 and $RH_2^\perp$ norms

If $H(s) \in RH_2^\perp$ then $H^\sim(s) \in RH_2$ and $\|H\| = \|H^\sim\|$

If $H(s) \in RL_2$ then find

$$H(s) = H_\mathrm{s}(s) + H_\mathrm{a}(s)$$

where $H_\mathrm{s}(s) \in RH_2$ and $H_\mathrm{a}(s) \in RH_2^\perp$.

$$\|H\| = \sqrt{\|H_\mathrm{s}\|^2 + \|H_\mathrm{a}\|^2} = \sqrt{\|H_\mathrm{s}\|^2 + \|H_\mathrm{a}^\sim\|^2}$$

3.4 Computing the RL_∞ norm

Theorem: Let $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a realization of $H(s)$. Assume A is Hurwitz and define

$$R_\gamma := \gamma^2 I - D^T D, \qquad\qquad F_\gamma := A + B R_\gamma^{-1} D^T C$$

The following statements are equivalent:

1. $\|H\| < \gamma$
2. $\overline{\sigma}(D) < \gamma$ and the Hamiltonean matrix

   $$Z_\gamma := \begin{bmatrix} -F_\gamma^T & -C^T(I + D R_\gamma^{-1} D^T)C \\ B R_\gamma^{-1} B^T & F_\gamma \end{bmatrix}$$

   has no eigenvalue on the imaginary axis
3. $\overline{\sigma}(D) < \gamma$ and there exists a symmetric positive semidefinite stabilizing solution to the ARE (Algebraic Riccati Equation)

   $$F_\gamma^T X + X F_\gamma + X B R_\gamma^{-1} B^T X + C^T(I + D R_\gamma^{-1} D^T)C = 0$$

Proof: Let $\Phi(s) := \gamma^2 I - H^{\sim}(s)H(s)$. Note that

$$\overline{\sigma}[H(j\omega)] < \gamma \quad \iff \quad \Phi(j\omega) \succ 0 \quad \iff \quad \Phi(j\omega) \text{ is nonsingular}$$

That positivity of $\Phi(j\omega)$ implies nonsingularity is immediate. The converse is also true. To see that, assume $\gamma > \|H\|$ and that $\Phi(j\omega)$ is singular. Then there exists $x \neq 0$ such that $\Phi(j\omega)x = 0$. Without loss of generality one can assume $\|x\| = 1$ such that

$$0 = x^*\Phi(s)x = \gamma^2 - x^* H^{\sim}(j\omega)H(j\omega)x \geq \gamma^2 - \overline{\sigma}^2[H(j\omega)] \geq \gamma^2 - \|H\|^2$$

which implies $\gamma \leq \|H\|$, establishing a contradiction.
In other words, $\Phi(s)$ must have no zeros on the imaginary axis. Equivalently $\Phi^{-1}(s)$ has no poles on the imaginary axis. Let us build a realization for $\Phi^{-1}(s)$. First construct

$$\begin{aligned}
\Phi(s) &= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \gamma^2 I \end{array}\right] - \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\
&= \left[\begin{array}{cc|c} -A^T & -C^T C & C^T D \\ 0 & A & -B \\ \hline B^T & D^T C & \gamma^2 I - D^T D \end{array}\right]
\end{aligned}$$

and then

$$\begin{aligned}
\Phi^{-1}(s) &= \left[\begin{array}{cc|c} -A^T & -C^T C & C^T D \\ 0 & A & -B \\ \hline B^T & D^T C & R_\gamma \end{array}\right]^{-1} \\
&= \left[\begin{array}{c|c} \begin{bmatrix} -A^T & -C^T C \\ 0 & A \end{bmatrix} - \begin{bmatrix} C^T D \\ -B \end{bmatrix} R_\gamma^{-1} \begin{bmatrix} B^T & D^T C \end{bmatrix} & -\begin{bmatrix} C^T D \\ -B \end{bmatrix} R_\gamma^{-1} \\ \hline R_\gamma^{-1} \begin{bmatrix} B^T & D^T C \end{bmatrix} & R_\gamma^{-1} \end{array}\right] \\
&= \left[\begin{array}{cc|c} \multicolumn{2}{c|}{Z_\gamma} & \begin{matrix} -C^T D R_\gamma^{-1} \\ B R_\gamma^{-1} \end{matrix} \\ \hline R_\gamma^{-1} B^T & R_\gamma^{-1} D^T C & R_\gamma^{-1} \end{array}\right]
\end{aligned}$$

Note that $\|H\| < \gamma$ implies $\lim_{\omega \to \infty} \overline{\sigma}[H(j\omega)] = \overline{\sigma}(D) < \gamma$, which implies $R_\gamma \succ 0$, therefore R_γ is nonsingular.

To prove that $2. \Rightarrow 1.$ note that if the matrix Z_γ has no pole on the imaginary axis then $\Phi^{-1}(s)$ has no pole on the imaginary axis. To prove that $1. \Rightarrow 2.$ note that if $\Phi^{-1}(s)$ has no pole on the imaginary axis then matrix Z_γ has no pole on the imaginary axis if $\Phi^{-1}(s)$ is controllable and observable. To prove that $\Phi^{-1}(s)$ is observable assume it is not and that $j\bar{\omega}$ is an unobservable pole of $\Phi^{-1}(s)$. Then there exists $x \neq 0$ such that

$$Z_\gamma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = j\bar{\omega} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad R_\gamma^{-1} \begin{bmatrix} B^T & D^T C \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Expand

$$\begin{aligned} Z_\gamma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{bmatrix} -(A + BR_\gamma^{-1} D^T C)^T & -C^T (I + DR_\gamma^{-1} D^T) C \\ BR_\gamma^{-1} B^T & A + BR_\gamma^{-1} D^T C \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} -A^T x_1 - C^T C x_2 - C^T D R_\gamma^{-1} (B^T x_1 + D^T C x_2) \\ A x_2 + B R_\gamma^{-1} (B^T x_1 + D^T C x_2) \end{pmatrix} \\ &= \begin{pmatrix} -A^T x_1 - C^T C x_1 \\ A x_2 \end{pmatrix} \end{aligned}$$

so that

$$(j\bar{\omega} + A^T) x_1 = -C^T C x_2 \qquad (j\bar{\omega} - A) x_2 = 0$$

Since A is Hurwitz it has no eigenvalue on the imaginary axis which implies $x_2 = x_1 = 0$ which is a contradiction, and $\Phi^{-1}(s)$ has no unobservable poles on the imaginary axis. The proof for uncontrollable modes on the imaginary axis follow the same pattern.

The proof of equivalence of 2. and 3. comes with the next result. Note that the simmilarity transformation

$$\begin{aligned} &\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^{-1} \begin{bmatrix} -F_\gamma^T & -C^T (I + DR_\gamma^{-1} D^T) C \\ BR_\gamma^{-1} B^T & F_\gamma \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} F_\gamma & BR_\gamma^{-1} B^T \\ -C^T (I + DR_\gamma^{-1} D^T) C & -F_\gamma^T \end{bmatrix} \end{aligned}$$

puts Z_γ in the form in which the next results are presented.

3.5 Algebraic Riccati Equations

Lemma 1: Consider the *Hamiltonean matrix*

$$Z := \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}.$$

where A, $R = R^T$ and $Q = Q^T \in \mathbb{R}^{n\times n}$.

1. λ is an eigenvalue of Z if and only if $-\lambda$ is an eigenvalue of Z.
2. If Z has no eigenvalues on the imaginary axis then there exists a matrix $W \in \mathbb{R}^{n\times n}$ such that

$$ZV_1 = V_1 W \tag{1}$$

   where W is Hurwitz.

Proof:
Item 1. Z has eigenvalues pairs which are symmetric w.r.t the imaginary axis because

$$J^{-1}ZJ = -JZJ = -Z^T, \qquad J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad J^{-1} = -J$$

Item 2. Let Z_J be the Jordan form of matrix Z so that

$$ZV = VZ_J$$

where $V \in \mathbb{R}^{2n\times 2n}$ is a matrix whose columns are the (generalized) eigenvectors of Z. Since the eigenvalues of Z are symmetric with respect to the imaginary axis and there are no eigenvalues on the imaginary axis, there exists at least two distinct Jordan blocks

$$Z \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} Z_{J_-} & 0 \\ 0 & Z_{J_+} \end{bmatrix}$$

where all n eigenvalues of Z_{J_-} have negative real part, i.e., Z_{J_-} is Hurwitz. The first columns of the above equation are in the form (1) with $W = Z_{J_-}$ Hurwitz.

Lemma 2: Consider the Algebraic Riccati Equation (ARE)

$$A^T X + XA + XRX + Q = 0$$

where A, $R = R^T$ and $Q = Q^T \in \mathbb{R}^{n \times n}$ and the associated *Hamiltonean matrix*

$$Z := \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}.$$

which is assumed to have no eigenvalue on the imaginary axis.

1. Let
$$V_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{C}^{2n \times n}$$
be (generalized) eigenvectors of Z associated with all n eigenvalues with negative real part. If X_1 is nonsingular then $X = X_2 X_1^{-1}$ solves the ARE.

2. The solution obtained in item 1. is
   (a) real,
   (b) symmetric,
   (c) unique stabilizing.

3. If $R \succeq 0$ (or $R \preceq 0$) then X_1 is invertible if and only if (A, R) is stabilizable.

Proof:
Item 1. From Item 2. of Lemma 1 there exists a Hurwitz matrix W such that

$$Z V_1 = V_1 W$$

Then, multiplying the above by X_1^{-1} on the right and by $\begin{bmatrix} X & -I \end{bmatrix}$ on the left we get

$$\begin{bmatrix} X & -I \end{bmatrix} Z \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} X_1 W X_1^{-1} = 0$$

Note that

$$\begin{bmatrix} X & -I \end{bmatrix} Z \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = A^T X + XA + XRX + Q$$

Item 2. (a) Since Z is real, the columns of V_1 can be chosen complex conjugates in pairs so that

$$\bar{V}_1 = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} P = \begin{bmatrix} X_1 P \\ X_2 P \end{bmatrix}$$

where P is some permutation matrix and

$$\bar{X} = \bar{X}_2 \bar{X}_1^{-1} = X_2 P P^{-1} X_1^{-1} = X_2 X_1^{-1} = X$$

that is X is real.

Item 2. (b) X is symmetric if

$$X = X_2 X_1^{-1} = X_1^{-*} X_2^* = X^*$$

or in other words

$$T = X_2^* X_1 - X_1^* X_2 = 0.$$

Now note that

$$T = V_1^* J V_1 = \begin{bmatrix} X_1^* & X_2^* \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Since $JZJ = Z^T$, Z and J satisfy the Lyapunov equation

$$JZ + Z^T J = 0$$

Multiplying the above equation by V_1^* on the left and V_1 on the right and using (1)

$$\begin{aligned} 0 &= V_1^* J Z V_1 + V_1^* Z^T J V_1 \\ &= V_1^* J V_1 W + W^* V_1^* J V_1 \\ &= TW + W^* T. \end{aligned}$$

Because W is Hurwitz $T = 0$.

Item 2. (c) Multiply (1) by X_1^{-1} on the right and by $\begin{bmatrix} I & 0 \end{bmatrix}$ on the left to obtain

$$\begin{bmatrix} I & 0 \end{bmatrix} Z \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = A + RX = X_1 W X_1^{-1}.$$

Therefore, $A + RX$ is stable because it is similar to a stable matrix. For unicity assume $\tilde{X}$ is also a stabilizing solution to the ARE. Therefore subtracting the two AREs

$$\begin{aligned}
0 &= (A^T X + XA + XRX + Q) - (A^T \tilde{X} + \tilde{X}A + \tilde{X}R\tilde{X} + Q) \\
&= A^T(X - \tilde{X}) + (X - \tilde{X})A + XRX - \tilde{X}R\tilde{X} \\
&= A^T(X - \tilde{X}) + (X - \tilde{X})A + XRX - \tilde{X}R\tilde{X} - XR\tilde{X} + XR\tilde{X} \\
&= A^T(X - \tilde{X}) + (X - \tilde{X})A + XR(X - \tilde{X}) + (X - \tilde{X})R\tilde{X} \\
&= (A + RX)^T(X - \tilde{X}) + (X - \tilde{X})(A + R\tilde{X})
\end{aligned}$$

The last equation can be seen as a Sylvester equation in $(X - \tilde{X})$ and since $A + RX$ and $A + R\tilde{X}$ are both Hurwitz $\lambda_i(A + RX) + \lambda_j(A + R\tilde{X}) < 0$ so that it admits only the trivial solution, that is, $X - \tilde{X} = 0$.

Item 3. To proves necessity assume that $R \succeq 0$ (or $R \preceq 0$), (A, R) is stabilizable and that X_1 is singular, such that there exists $x \neq 0$ such that $X_1 x = 0$. Multiply (1) by $\begin{bmatrix} I & 0 \end{bmatrix}$ on the left to obtain

$$\begin{bmatrix} I & 0 \end{bmatrix} Z \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = AX_1 + RX_2 = X_1 W$$

Multiply on the left by $x^* X_2^*$ and on the right by x

$$x^* X_2^* A X_1 x + x^* X_2^* R X_2 x = x^* X_2^* X_1 W x = x^* X_1^* X_2 W x$$

and use the fact that $X_1 x = 0$ to obtain

$$x^* X_2^* R X_2 x = 0$$

which implies $RX_2 x = 0$ because $R \succeq 0$ (or $R \preceq 0$). Note that this also implies

$$X_1 W x = 0.$$

If X_1 is singular then it can be shown that there exists $\bar{x} \neq 0$ such that

$$X_1 \bar{x} = 0, \quad W\bar{x} = \bar{\lambda}\bar{x}, \quad \bar{\lambda} + \bar{\lambda}^* < 0$$

Now multiply (1) by $\begin{bmatrix} 0 & I \end{bmatrix}$ on the left to obtain

$$\begin{bmatrix} 0 & I \end{bmatrix} Z \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -QX_1 - A^* X_2 = X_2 W$$

and multiply by $\bar{x}$ such that $X_1 \bar{x} = 0$ on the right

$$\begin{aligned} 0 &= (A^* X_2 - X_2 W)\bar{x} \\ &= (A^* - \bar{\lambda} I) X_2 \bar{x} \end{aligned}$$

Since $R X_2 \bar{x} = 0$ (A, R) is not stabilizable, which is a contradiction.
To prove sufficiency note that if X_1 is invertible than $A + RX$ is stable such that (A, R) is stabilizable.

4 Stability

4.1 Well-posedeness

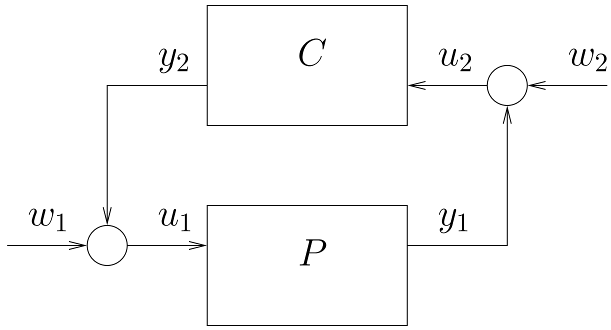


Figure 1: Feedback connection

Definition: A feedback system is said to be *well-posed* if all closed-loop matrices are well-defined and proper.

Lemma: The feedback system in Figure 1 is well-posed if and only if

$$I - C(\infty)P(\infty)$$

is invertible.
Proof: The system equations are

$$u_1 = w_1 + y_2 \qquad\qquad y_1 = P(s)u_1$$
$$u_2 = w_2 + y_1 \qquad\qquad y_2 = C(s)u_2$$

so that

$$u_1 = w_1 + C(s)[w_2 + P(s)u_1]$$

and

$$[I - C(s)P(s)]u_1 = w_1 + C(s)w_2$$

For u_1 to be well defined $[I - C(s)P(s)]^{-1}$ must exist and be proper. This is the same as having $[I - C(\infty)P(\infty)]$ nonsingular. Note that if u_1 is well defined then u_2 will also be.

4.2 Internal Stability

Definition: The feedback system in Figure 1 is said to be *internally stable* if the transfer function from $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ to $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is in RH_∞.

Lemma: The feedback system in Figure 1 is said to be *internally stable* if and only if the transfer function

$$T_{wu}(s) = \begin{bmatrix} I & -C(s) \\ -P(s) & I \end{bmatrix}^{-1}$$

is in RH_∞.

Proof: Just determine the transfer function from $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ to $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. From the diagram

$$\begin{aligned} u_1 &= w_1 + C(s)u_2 \\ u_2 &= w_2 + P(s)u_1 \end{aligned}$$

or

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{bmatrix} I & -C(s) \\ -P(s) & I \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Lemma: The transfer function

$$T_{wu}(s) := \begin{bmatrix} I & -C(s) \\ -P(s) & I \end{bmatrix}^{-1}$$

is in RH_∞ if and only if

$$\begin{bmatrix} I + C(s)[I - P(s)C(s)]^{-1}P(s) & C(s)[I - P(s)C(s)]^{-1} \\ [I - P(s)C(s)]^{-1}P(s) & [I - P(s)C(s)]^{-1} \end{bmatrix}$$

is in RH_∞.

Proof: Use the fact that if A is nonsingular

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}DE^{-1}CA^{-1} & -A^{-1}DE^{-1} \\ -E^{-1}CA^{-1} & E^{-1} \end{bmatrix}$$

where

$$E = B - CA^{-1}D$$

is known as *Schur Complement* of the matrix.
In the case of the matrix in the lemma it produces

$$T_{wu}(s) = \begin{bmatrix} I + C(s)E(s)^{-1}P(s) & C(s)E(s)^{-1} \\ -E^{-1}P(s) & E(s)^{-1} \end{bmatrix}, \quad E(s) = I - P(s)C(s).$$

Example: (From Zhou, p. 69) Let

$$P(s) = \frac{s-1}{s+1}, \qquad\qquad C(s) = -\frac{1}{s-1}.$$

Then

$$\begin{aligned}
[I - P(s)C(s)]^{-1} &= \left(1 + \frac{1}{s+1}\right)^{-1} = \frac{s+1}{s+2}, \\
1 + C(s)[I - P(s)C(s)]^{-1}P(s) &= 1 - \frac{1}{s+2} = \frac{s+1}{s+2}
\end{aligned}$$

and

$$\begin{bmatrix} I + C(s)[I - P(s)C(s)]^{-1}P(s) & C(s)[I - P(s)C(s)]^{-1} \\ [I - P(s)C(s)]^{-1}P(s) & [I - P(s)C(s)]^{-1} \end{bmatrix} \\
= \begin{bmatrix} \frac{s+1}{s+2} & -\frac{s+1}{(s-1)(s+2)} \\ \frac{s-1}{s+2} & \frac{s+1}{s+2} \end{bmatrix}$$

Not internally stable!

Corollary: If $C(s) \in RH_\infty$ then the feedback system in Figure 1 is *internally stable* if and only if $[I - P(s)C(s)]^{-1}P(s) \in RH_\infty$.

Proof: Necessity is immediate since $[I - P(s)C(s)]^{-1}P(s)$ is a block of $T_{wu}(s)$.
Sufficiency commes from

$$\begin{aligned}
[I - P(s)C(s)]^{-1} &= [I - P(s)C(s)]^{-1}[I - P(s)C(s) + P(s)C(s)] \\
&= [I - P(s)C(s)]^{-1}[I - P(s)C(s)] + [I - P(s)C(s)]^{-1}P(s)C(s) \\
&= I + [I - P(s)C(s)]^{-1}P(s)C(s) \in RH_\infty
\end{aligned}$$

since $C(s)$ and $[I - P(s)C(s)]^{-1}P(s)$ are in RH_∞.

Corollary: If $P(s) \in RH_\infty$ then the feedback system in Figure 1 is *internally stable* if and only if $C(s)[I - P(s)C(s)]^{-1} \in RH_\infty$.

Proof: Necessity is immediate again.
Sufficiency commes from

$$\begin{aligned}
[I - P(s)C(s)]^{-1} &= [I - P(s)C(s) + P(s)C(s)][I - P(s)C(s)]^{-1} \\
&= [I - P(s)C(s)][I - P(s)C(s)]^{-1} + P(s)C(s)[I - P(s)C(s)]^{-1} \\
&= I + P(s)C(s)[I - P(s)C(s)]^{-1} \in RH_\infty
\end{aligned}$$

since $C(s)$ and $[I - P(s)C(s)]^{-1}P(s)$ are in RH_∞.

Lemma: Let $P(s)$ and $C(s)$ have state space realizations

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \qquad\qquad C(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right].$$

Assume that (A, B) and $(\hat{A}, \hat{B})$ are stabilizable and that (A, C) and $(\hat{A}, \hat{C})$ are detectable. Then the feedback system in Figure 1 is said to be *internally stable* if and only if the realization

$$T_{wu}(s) = \left[\begin{array}{cc|cc} A + B\Delta^{-1}\hat{D}C & B\Delta^{-1}\hat{C} & B\Delta^{-1} & B\Delta^{-1}\hat{D} \\ \hat{B}\hat{\Delta}^{-1}C & \hat{A} + \hat{B}\hat{\Delta}^{-1}D\hat{C} & \hat{B}\hat{\Delta}^{-1}D & \hat{B}\hat{\Delta}^{-1} \\ \hline \Delta^{-1}\hat{D}C & \Delta^{-1}\hat{C} & \Delta^{-1} & \Delta^{-1}\hat{D} \\ \hat{\Delta}^{-1}C & \hat{\Delta}^{-1}D\hat{C} & \hat{\Delta}^{-1}D & \hat{\Delta}^{-1} \end{array}\right]$$

where

$$\Delta = (I - \hat{D}D)^{-1}, \qquad\qquad \hat{\Delta} = (I - D\hat{D})^{-1}$$

is Hurwitz.
Proof: First show that the above is indeed a realization for $T_{wu}(s)$. Then all we have to prove that this realization is stabilizable and detectable. Before we prove this note that

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} = \begin{bmatrix} I + \hat{D}\hat{\Delta}^{-1}D & \hat{D}\hat{\Delta}^{-1} \\ \hat{\Delta}^{-1}D & \hat{\Delta}^{-1} \end{bmatrix}$$

Using the identity

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$$

we have

$$I + \hat{D}\hat{\Delta}^{-1}D = I + \hat{D}(I - D\hat{D})^{-1}D = (I - \hat{D}D)^{-1} = \Delta^{-1}.$$

Also

$$\hat{D} - \hat{D}D\hat{D} = \hat{D}(I - D\hat{D}) = (I - \hat{D}D)\hat{D}$$

so that

$$\hat{D}\hat{\Delta}^{-1} = \hat{D}(I - D\hat{D})^{-1} = (I - \hat{D}D)^{-1}\hat{D} = \Delta^{-1}\hat{D}.$$

We have established that

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} = \begin{bmatrix} \Delta^{-1} & \Delta^{-1}\hat{D} \\ \hat{\Delta}^{-1}D & \hat{\Delta}^{-1} \end{bmatrix}$$

Define

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] := \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & \hat{A} & 0 & \hat{B} \\ \hline 0 & \hat{C} & I & -\hat{D} \\ C & 0 & -D & I \end{array}\right]$$

such that

$$\begin{bmatrix} A + B\Delta^{-1}\hat{D}C & B\Delta^{-1}\hat{C} \\ \hat{B}\hat{\Delta}^{-1}C & \hat{A} + \hat{B}\hat{\Delta}^{-1}D\hat{C} \end{bmatrix} = \tilde{A} + \tilde{B}\tilde{D}^{-1}\tilde{C},$$

$$\begin{bmatrix} \Delta^{-1}\hat{D}C & \Delta^{-1}\hat{C} \\ \hat{\Delta}^{-1}C & \hat{\Delta}^{-1}D\hat{C} \end{bmatrix} = \tilde{D}^{-1}\tilde{C}$$

Now assume that

$$\left(\begin{bmatrix} A + B\Delta^{-1}\hat{D}C & B\Delta^{-1}\hat{C} \\ \hat{B}\hat{\Delta}^{-1}C & \hat{A} + \hat{B}\hat{\Delta}^{-1}D\hat{C} \end{bmatrix}, \quad \begin{bmatrix} \Delta^{-1}\hat{D}C & \Delta^{-1}\hat{C} \\ \hat{\Delta}^{-1}C & \hat{\Delta}^{-1}D\hat{C} \end{bmatrix} \right)$$
$$= \left(\tilde{A} + \tilde{B}\tilde{D}^{-1}\tilde{C}, \quad \tilde{D}^{-1}\tilde{C} \right)$$

is not stabilizable, that is, there exists $x \neq 0$ and $\lambda + \lambda^* > 0$ such that

$$\left(\tilde{A} + \tilde{B}\tilde{D}^{-1}\tilde{C} \right) x = \lambda x, \qquad \tilde{D}^{-1}\tilde{C}x = 0$$

which implies, because $\tilde{D}$ is nonsingular,

$$\tilde{A}x = \lambda x, \qquad \tilde{C}x = 0.$$

Expanding we conclude that

$$Ax_1 = \lambda x_1, \qquad Cx_1 = 0, \qquad \hat{A}x_2 = \lambda x_2, \qquad Cx_2 = 0$$

which implies that (A, C) and $(\hat{A}, \hat{C})$ are not detectable, establishing a contradiction.

4.3 Coprime Factorization over RH_∞

Lemma: Two rational matrices $M(s)$ and $N(s)$ in RH_∞ are *right coprime* if and only if there exists rational matrices $\hat{X}(s)$ and $\hat{Y}(s)$ in RH_∞ such that

$$\hat{X}(s)M(s) + \hat{Y}(s)N(s) = \begin{bmatrix} \hat{X}(s) & \hat{Y}(s) \end{bmatrix} \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = I.$$

Lemma: Two rational matrices $\hat{M}(s)$ and $\hat{N}(s)$ in RH_∞ are *left coprime* if and only if there exists rational matrices $X(s)$ and $Y(s)$ in RH_∞ such that

$$\hat{M}(s)X(s) + \hat{N}(s)Y(s) = \begin{bmatrix} \hat{M}(s) & \hat{N}(s) \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = I.$$

Theorem: Let $H(s)$ be a proper rational matrix. There exists matrices $M(s)$, $N(s)$, $\hat{M}(s)$, $\hat{N}(s)$, $X(s)$, $Y(s)$, $\hat{X}(s)$, and $\hat{Y}(s)$ all in RH_∞ such that

$$H(s) = N(s)M^{-1}(s) = \hat{M}^{-1}(s)\hat{N}(s)$$

and

$$\begin{bmatrix} \hat{X}(s) & \hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & -Y(s) \\ N(s) & X(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Proof: Let $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization[2] for $H(s)$ and define

$$\begin{bmatrix} \hat{X}(s) & \hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & -(B + LD) & L \\ \hline K & I & 0 \\ C & -D & I \end{array}\right],$$

$$\begin{bmatrix} M(s) & -Y(s) \\ N(s) & X(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + BK & B & -L \\ \hline K & I & 0 \\ C + DK & D & I \end{array}\right].$$

[2] We just need (A, B) stabilizable and (A, C) detectable.

Then

$$
\begin{aligned}
&\begin{bmatrix} \hat{X}(s) & \hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & -Y(s) \\ N(s) & X(s) \end{bmatrix} \\
&\quad = \left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline K & I & 0 \\ C & -D & I \end{array}\right] \left[\begin{array}{c|cc} A+BK & B & -L \\ \hline K & I & 0 \\ C+DK & D & I \end{array}\right] \\
&\quad = \left[\begin{array}{cc|cc} A+LC & LC-BK & -B & L \\ 0 & A+BK & B & -L \\ \hline K & K & I & 0 \\ C & C & 0 & I \end{array}\right] \\
&\quad = \left[\begin{array}{c|c} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} A+LC & 0 \\ 0 & A+BK \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} & \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & -L \end{bmatrix} \\ \hline \begin{bmatrix} K & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{array}\right] \\
&\quad = \left[\begin{array}{cc|cc} A+LC & 0 & 0 & 0 \\ 0 & A+BK & B & -L \\ \hline K & 0 & I & 0 \\ C & 0 & 0 & I \end{array}\right]
\end{aligned}
$$

What is left to prove is that all these transfer matrices are in RH_∞. But since (A, B) is stabilizable and (A, C) is detectable we can chose K and L such that they be all in RH_∞.
Then consider

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
$$

and the feedback law

$$
u = v + Kx
$$

The closed loop system is

$$
\begin{aligned}
\dot{x} &= (A+BK)x + Bv \\
u &= Kx + v \\
y &= (C+DK)x + Dv
\end{aligned}
$$

whose transfer function is

$$\left[\begin{array}{c|c} A+BK & B \\ \hline K & I \\ C+DK & D \end{array}\right] = \begin{bmatrix} M(s) \\ N(s) \end{bmatrix}$$

or

$$y = N(s)\, v, \qquad\qquad u = M(s)\, v$$

which imply

$$y = N(s)\, v = N(s)M^{-1}(s)\, u = H(s)\, u$$

The same argument can be used to show that $\hat{M}(s)^{-1}\hat{N}(s) = H(s)$.

Example (Colaneri, p. 77):

$$H(s) = \begin{bmatrix} \dfrac{s^2+s-1}{s^3-s} & \dfrac{s}{s^2-1} \\[2ex] \dfrac{2s^2-1}{s^3-s} & \dfrac{s^2}{s^2-1} \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Note that $\lambda(A) = \{-1, 0, 1\}$. Choose the stabilizing gains

$$K = \begin{bmatrix} 0 & 0 & -1 \\ -1 & -2 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} -4 & 0 \\ -4 & 0 \\ 1 & 0 \end{bmatrix}$$

and construct

$$\begin{bmatrix} M(s) & -Y(s) \\ N(s) & X(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+BK & B & -L \\ \hline K & I & 0 \\ C+DK & D & I \end{array}\right]$$

$$= \left[\begin{array}{ccc|cccc} -1 & -1 & 0 & 0 & 1 & 4 & 0 \\ 1 & 0 & -1 & 1 & 0 & 4 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$= \begin{bmatrix} \dfrac{s}{s+1} & 0 & \dfrac{1}{(s+1)} & 0 \\[2ex] \dfrac{-2s(s+0.5)}{(s+1)(s^2+s+1)} & \dfrac{(s+1)(s-1)}{(s^2+s+1)} & \dfrac{-12(s+1.384)(s+0.7829)}{(s+1)(s^2+s+1)} & 0 \\[2ex] \dfrac{(s^2+1)}{(s+1)(s^2+s+1)} & \dfrac{s}{(s^2+s+1)} & \dfrac{(s+4.537)(s+1.307)(s-0.8434)}{(s+1)(s^2+s+1)} & 0 \\[2ex] \dfrac{(s+1)}{(s+1)(s^2+s+1)} & \dfrac{s^2}{(s^2+s+1)} & \dfrac{-9(s+0.5556)(s+1)}{(s+1)(s^2+s+1)} & 1 \end{bmatrix}$$

Theorem: Consider the feedback system in Figure 1 where

$$P(s) = N(s)M^{-1}(s) = \hat{M}^{-1}(s)\hat{N}(s), \quad C(s) = U(s)V^{-1}(s) = \hat{V}^{-1}(s)\hat{U}(s)$$

are coprime factorizations in RH_∞. The following are equivalent:

1. It is internally stable.
2. $\begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix}$ is invertible in RH_∞.
3. $\begin{bmatrix} \hat{V}(s) & -\hat{U}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix}$ is invertible in RH_∞.
4. $\hat{V}(s)M(s) - \hat{U}(s)N(s)$ is invertible in RH_∞.
5. $\hat{M}(s)V(s) - \hat{N}(s)U(s)$ is invertible in RH_∞.

Proof: First note that

$$T_{wu}(s)^{-1} = \begin{bmatrix} I & -C(s) \\ -P(s) & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & C(s) \\ P(s) & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

so that the system in internally stable if and only if

$$\begin{aligned} \begin{bmatrix} I & C(s) \\ P(s) & I \end{bmatrix}^{-1} &= \begin{bmatrix} I & U(s)V^{-1}(s) \\ N(s)M^{-1}(s) & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} M(s) & 0 \\ 0 & V(s) \end{bmatrix} \begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix}^{-1} \end{aligned}$$

is in RH_∞. Then 1. is equivalent to 2. if we can prove that

$$\begin{bmatrix} M(s) & 0 \\ 0 & V(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix}$$

are coprime. Since $M(s)$, $N(s)$, $U(s)$ and $V(s)$ are coprime factorizations in RH_∞ there exists $\hat{X}_M(s)$, $\hat{Y}_N(s)$, $\hat{X}_U(s)$ and $\hat{Y}_V(s)$ all in RH_∞ such that

$$\begin{bmatrix} \hat{X}_M(s) & \hat{Y}_N(s) \end{bmatrix} \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = I, \qquad \begin{bmatrix} \hat{X}_U(s) & \hat{Y}_V(s) \end{bmatrix} \begin{bmatrix} U(s) \\ V(s) \end{bmatrix} = I$$

and hence

$$\begin{bmatrix} \hat{X}_M(s) & -\hat{Y}_N(s) & 0 & \hat{Y}_N(s) \\ -\hat{Y}_V(s) & \hat{X}_U(s) & \hat{Y}_V(s) & 0 \end{bmatrix} \begin{bmatrix} M(s) & 0 \\ 0 & V(s) \\ M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

proving that they are indeed right coprime. One can prove that 1. is equivalent to 3. in the same way.
Now note that 2. and 3. implies 4. and 5. since

$$\begin{aligned} &\begin{bmatrix} \hat{V}(s) & -\hat{U}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} \\ &\qquad = \begin{bmatrix} \hat{V}(s)M(s) - \hat{U}(s)N(s) & 0 \\ 0 & \hat{M}(s)V(s) - \hat{N}(s)U(s) \end{bmatrix} \end{aligned}$$

so that the right hand side must be invertible in RH_∞. To show the converse note that

$$\begin{aligned} \begin{bmatrix} I & C(s) \\ P(s) & I \end{bmatrix}^{-1} &= \begin{bmatrix} I & \hat{V}^{-1}(s)\hat{U}(s) \\ N(s)M^{-1}(s) & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} M(s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{V}(s)M(s) & \hat{U}(s) \\ N(s) & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{V}(s) & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

from which it follows that the system is internally stable if the matrix in the midle on the expression above is invertible in RH_∞. However, note that

$$\begin{bmatrix} \hat{V}(s)M(s) & \hat{U}(s) \\ N(s) & I \end{bmatrix} = \begin{bmatrix} I & \hat{U}(s) \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{V}(s)M(s) - \hat{U}(s)N(s) & 0 \\ N(s) & I \end{bmatrix}$$

which is invertible in RH_∞ if 4. holds. The proof of sufficiency of 5. follows the same pattern.

4.4 All Stabilizing Controllers

Corollary: Let

$$P(s) = N(s)M^{-1}(s) = \hat{M}^{-1}(s)\hat{N}(s)$$

be coprime factorizations in RH_∞ and $X(s)$, $Y(s)$, $\hat{X}(s)$, and $\hat{Y}(s)$ all in RH_∞ are such that

$$\begin{bmatrix} \hat{X}(s) & -\hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

The controller

$$C(s) = Y(s)X^{-1}(s) = \hat{X}^{-1}(s)\hat{Y}(s)$$

stabilizes the feedback system in Figure 1.
Proof: All we have to prove is stability of the closed loop system. Note that

$$\begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix}^{-1} = \begin{bmatrix} \hat{X}(s) & -\hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \qquad \in RH_\infty$$

which proves that $C(s) = Y(s)X^{-1}(s)$ is a stabilizing controller. The relation

$$\begin{bmatrix} \hat{X}(s) & -\hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix}^{-1} = \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} \qquad \in RH_\infty$$

proves that $C(s) = \hat{X}^{-1}(s)\hat{Y}(s)$ is also stabilizing.

Example: From the previous example pick

$$\begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{(s+1)} & 0 \\ \dfrac{-12(s+1.384)(s+0.7829)}{(s+1)(s^2+s+1)} & 0 \\ \dfrac{(s+4.537)(s+1.307)(s-0.8434)}{(s+1)(s^2+s+1)} & 0 \\ \dfrac{-9(s+0.5556)(s+1)}{(s+1)(s^2+s+1)} & 1 \end{bmatrix}$$

so that

$$\begin{aligned} C(s) &= Y(s)X^{-1}(s) \\ &= \begin{bmatrix} \frac{s^2+s+1}{(s+4.537)(s+1.307)(s-0.8434)} & 0 \\ \frac{-12(s+1.384)(s+0.7829)}{(s+4.537)(s+1.307)(s-0.8434)} & 0 \end{bmatrix} \end{aligned}$$

is a stabilizing controller

Theorem: Let

$$P(s) = N(s)M^{-1}(s) = \hat{M}^{-1}(s)\hat{N}(s)$$

be coprime factorizations in RH_∞ and $X(s)$, $Y(s)$, $\hat{X}(s)$, and $\hat{Y}(s)$ all in RH_∞ are such that

$$\begin{bmatrix} \hat{X}(s) & -\hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

All controllers that stabilize the feedback system in Figure 1 are of the form

$$\begin{aligned} C(s) &= [Y(s) - M(s)Q(s)]\,[X(s) - N(s)Q(s)]^{-1} \\ &= [\hat{X}(s) - Q(s)\hat{N}(s)]^{-1}\,[\hat{Y}(s) - Q(s)\hat{M}(s)] \end{aligned}$$

where $Q(s) \in RH_\infty$ and $\lim_{s\to\infty}[I - P(s)Q(s)]$ is nonsingular.

Proof: Start with

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} I & Q(s) \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -Q(s) \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & Q(s) \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{X}(s) & -\hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} \begin{bmatrix} I & -Q(s) \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \hat{X}(s) - Q(s)\hat{N}(s) & -[\hat{Y}(s) - Q(s)\hat{M}(s)] \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) - M(s)Q(s) \\ N(s) & X(s) - N(s)Q(s) \end{bmatrix} \end{aligned}$$

The $(1,2)$ block of the above expression is

$$[\hat{X}(s) - Q(s)\hat{N}(s)][Y(s) - M(s)Q(s)] = [\hat{Y}(s) - Q(s)\hat{M}(s)][X(s) - N(s)Q(s)]$$

which multiplied by the left and the right by $[\hat{X}(s) - Q(s)\hat{N}(s)]^{-1}$ and $[X(s) - N(s)Q(s)]^{-1}$ respectively yields

$$\begin{aligned} &[Y(s) - M(s)Q(s)]\,[X(s) - N(s)Q(s)]^{-1} \\ &\qquad = [\hat{X}(s) - Q(s)\hat{N}(s)]^{-1}[\hat{Y}(s) - Q(s)\hat{M}(s)] = C(s). \end{aligned}$$

Note that the inverses exist because

$$X(s) - N(s)Q(s) = \left[\begin{array}{c|c} A+BK & -L \\ \hline C+DK & I \end{array}\right] - \left[\begin{array}{c|c} A+BK & B \\ \hline C+DK & D \end{array}\right] \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]$$

$$= \left[\begin{array}{c|c} A+BK & -L \\ \hline C+DK & I \end{array}\right] - \left[\begin{array}{cc|c} A+BK & BC_Q & BD_Q \\ 0 & A_Q & B_Q \\ \hline C+DK & DC_Q & DD_Q \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} A+BK & 0 & 0 & -L \\ 0 & A+BK & BC_Q & -BD_Q \\ 0 & 0 & A_Q & -B_Q \\ \hline C+DK & C+DK & DC_Q & I - DD_Q \end{array}\right]$$

and $I - DD_Q = \lim_{s\to\infty}[I - P(s)Q(s)]$ is nonsingular. A similar manipulation can show that $\hat{X}(s) - Q(s)\hat{M}(s)$ is also invertible.
Now to prove that $C(s)$ stabilizes the feedaback system recall that

$$\begin{bmatrix} \hat{X}(s) - Q(s)\hat{N}(s) & -[\hat{Y}(s) - Q(s)\hat{M}(s)] \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) - M(s)Q(s) \\ N(s) & X(s) - N(s)Q(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

so that

$$\begin{bmatrix} \hat{X}(s) - Q(s)\hat{N}(s) & -[\hat{Y}(s) - Q(s)\hat{M}(s)] \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix}^{-1} = \begin{bmatrix} M(s) & Y(s) - M(s)Q(s) \\ N(s) & X(s) - N(s)Q(s) \end{bmatrix}$$

The feedback connection is stable because $M(s)$, $N(s)$, $X(s)$ and $Y(s)$ are in RH_∞ and

$$Y(s) - M(s)Q(s) \qquad \text{and} \qquad X(s) - N(s)Q(s)$$

are in RH_∞ if $Q(s)$ is in RH_∞.
We now need to prove the converse, that is, that a given stabilizing controller $C(s)$ admits a representation as a function of $Q(s) \in RH_\infty$. Let $U(s)$ and $V(s)$ be right coprime factors of the stabilizing controller $C(s) = U(s)V^{-1}(s)$. Therefore

$$\begin{bmatrix} \hat{X}(s) & -\hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} = \begin{bmatrix} I & \hat{X}(s)U(s) - \hat{Y}(s)V(s) \\ 0 & -\hat{N}(s)U(s) + \hat{M}(s)V(s) \end{bmatrix}.$$

Since the left hand side is invertible in RH_∞ (from the Theorem hypothesis and that $C(s)$ is stabilizing) the right hand side also is. Defining

$$P(s) := [\hat{M}(s)V(s) - \hat{N}(s)U(s)], \;\; Q(s) := -[\hat{X}(s)U(s) - \hat{Y}(s)V(s)]P^{-1}(s)$$

we have

$$\begin{bmatrix} I & \hat{X}(s)U(s) - \hat{Y}(s)V(s) \\ 0 & \hat{M}(s)V(s) - \hat{N}(s)U(s) \end{bmatrix}^{-1} = \begin{bmatrix} I & -Q(s)P(s) \\ 0 & P(s) \end{bmatrix}^{-1} = \begin{bmatrix} I & Q(s) \\ 0 & P^{-1}(s) \end{bmatrix} \in RH_\infty$$

so that $Q(s)$ is in RH_∞. One can prove that $\lim_{s\to\infty} I - P(s)Q(s)$ is nonsingular by building a realization as before. Now compute

$$\begin{aligned}
\begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} \begin{bmatrix} I & -Q(s)P(s) \\ 0 & P(s) \end{bmatrix} &= \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} \begin{bmatrix} \hat{X}(s) & -\hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} \\
&= \begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix}
\end{aligned}$$

which implies

$$U(s) = [Y(s) - M(s)Q(s)]P(s), \qquad V(s) = [X(s) - N(s)Q(s)]P(s)$$

so that

$$\begin{aligned}
C(s) &= U(s)V^{-1}(s) \\
&= [Y(s) - M(s)Q(s)]P(s)P^{-1}(s)[X(s) - N(s)Q(s)]^{-1} \\
&= [Y(s) - M(s)Q(s)]\,[X(s) - N(s)Q(s)]^{-1}
\end{aligned}$$

which is of the form given in the theorem. The other formula can be proved similarly using left coprime factors.

Example: From the previous example pick

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s}{s+1} & 0 \\ \dfrac{-2s(s+0.5)}{(s+1)(s^2+s+1)} & \dfrac{(s+1)(s-1)}{(s^2+s+1)} \\ \dfrac{(s^2+1)}{(s+1)(s^2+s+1)} & \dfrac{s}{(s^2+s+1)} \\ \dfrac{(s+1)}{(s+1)(s^2+s+1)} & \dfrac{s^2}{(s^2+s+1)} \end{bmatrix}$$

and

$$Q(s) = \begin{bmatrix} \dfrac{1}{s+3} & 0 \\ \dfrac{1}{s+2} & \dfrac{1}{s+1} \end{bmatrix}$$

so that

$$\begin{aligned} C(s) &= [Y(s) - M(s)Q(s)]\,[X(s) - N(s)Q(s)]^{-1} \\ &= \frac{1}{(s+5.026)(s+2.252)(s-1.399)(s+0.55)(s^2+3.571s+3.673)} \\ &\times \begin{bmatrix} 3(s+2)(s+0.5698)(s^2+0.4302s+1.755) & 0 \\ -13(s+2.56)(s+0.7159)(s+0.4773) & -(s+1.592)(s+2)(s+3.565) \\ (s^2+3.785s+3.782) & (s^2+1.843s+2.82) \end{bmatrix} \end{aligned}$$

is a stabilizing controller.

Lemma: Let

$$P(s) = N(s)M^{-1}(s) = \hat{M}^{-1}(s)\hat{N}(s)$$

be coprime factorizations in RH_∞ and $X(s)$, $Y(s)$, $\hat{X}(s)$, and $\hat{Y}(s)$ all in RH_∞ are such that

$$\begin{bmatrix} \hat{X}(s) & -\hat{Y}(s) \\ -\hat{N}(s) & \hat{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

All controllers that stabilize the feedback system in Figure 1 are of the form

$$C(s) = C_0(s) - \hat{X}^{-1}(s)Q(s)\left[I - X^{-1}(s)N(s)Q(s)\right]^{-1} X^{-1}(s)$$

where

$$C_0(s) = Y(s)X^{-1}(s) = \hat{X}^{-1}(s)\hat{Y}(s)$$

and $Q(s) \in RH_\infty$ and $\lim_{s\to\infty}[I - P(s)Q(s)]$ is nonsingular.

Proof: From the previous theorem

$$\begin{aligned} C(s) &= [Y(s) - M(s)Q(s)]\,[X(s) - N(s)Q(s)]^{-1} \\ &= [Y(s) - M(s)Q(s)]\left\{\left[I - N(s)Q(s)X^{-1}(s)\right]X(s)\right\}^{-1} \\ &= [Y(s) - M(s)Q(s)]\,X^{-1}\left[I - N(s)Q(s)X^{-1}(s)\right]^{-1} \\ &= \left[C_0(s) - M(s)Q(s)X^{-1}\right]\left[I - N(s)Q(s)X^{-1}(s)\right]^{-1}. \end{aligned}$$

Now note that

$$\begin{aligned} (I - AB)^{-1} &= (I - AB + AB)(I - AB)^{-1} \\ &= (I - AB)(I - AB)^{-1} + AB(I - AB)^{-1} \\ &= I + AB(I - AB)^{-1} \end{aligned}$$

and because $B - BAB = (I - BA)B = B(I - AB)$ we have

$$B(I - AB)^{-1} = (I - BA)^{-1}B$$

and

$$\begin{aligned} (I - AB)^{-1} &= I + AB(I - AB)^{-1} \\ &= I + A(I - BA)^{-1}B \end{aligned}$$

Applying the above formula with $A = N(s)Q(s)$ and $B = X^{-1}(s)$ we obtain

$$
\begin{aligned}
C(s) &= \left[C_0(s) - M(s)Q(s)X^{-1}\right]\left[I - N(s)Q(s)X^{-1}(s)\right]^{-1} \\
&= \left[C_0(s) - M(s)Q(s)X^{-1}\right]\left\{I + N(s)Q(s)\left[I - X^{-1}(s)N(s)Q(s)\right]^{-1}X^{-1}(s)\right\} \\
&= C_0(s) + K(s)
\end{aligned}
$$

where

$$
\begin{aligned}
K(s) &= C_0(s)N(s)Q(s)\left[I - X^{-1}(s)N(s)Q(s)\right]^{-1}X^{-1}(s) \\
&\qquad - M(s)Q(s)X^{-1}\left\{I + N(s)Q(s)\left[I - X^{-1}(s)N(s)Q(s)\right]^{-1}X^{-1}(s)\right\} \\
&= C_0(s)N(s)Q(s)\left[I - X^{-1}(s)N(s)Q(s)\right]^{-1}X^{-1}(s) \\
&\qquad - M(s)Q(s)X^{-1}\left[I - N(s)Q(s)X^{-1}(s)\right]^{-1}
\end{aligned}
$$

Using again the fact that $B(I - AB)^{-1} = (I - BA)^{-1}B$ with $A = N(s)Q(s)$ and $B = X^{-1}(s)$

$$
\begin{aligned}
K(s) &= C_0(s)N(s)Q(s)\left[I - X^{-1}(s)N(s)Q(s)\right]^{-1}X^{-1}(s) \\
&\qquad - M(s)Q(s)\left[I - X^{-1}(s)N(s)Q(s)\right]^{-1}X^{-1} \\
&= \left[C_0(s)N(s) - M(s)\right]Q(s)\left[I - X^{-1}(s)N(s)Q(s)\right]^{-1}X^{-1}(s).
\end{aligned}
$$

Finally

$$
\begin{aligned}
C_0(s)N(s) - M(s) &= \hat{X}^{-1}(s)\hat{Y}(s)N(s) - M(s) \\
&= \hat{X}^{-1}(s)\left[\hat{Y}(s)N(s) - \hat{X}(s)M(s)\right] \\
&= -\hat{X}^{-1}(s)
\end{aligned}
$$

so that

$$
K(s) = -\hat{X}^{-1}(s)Q(s)\left[I - X^{-1}(s)N(s)Q(s)\right]^{-1}X^{-1}(s).
$$

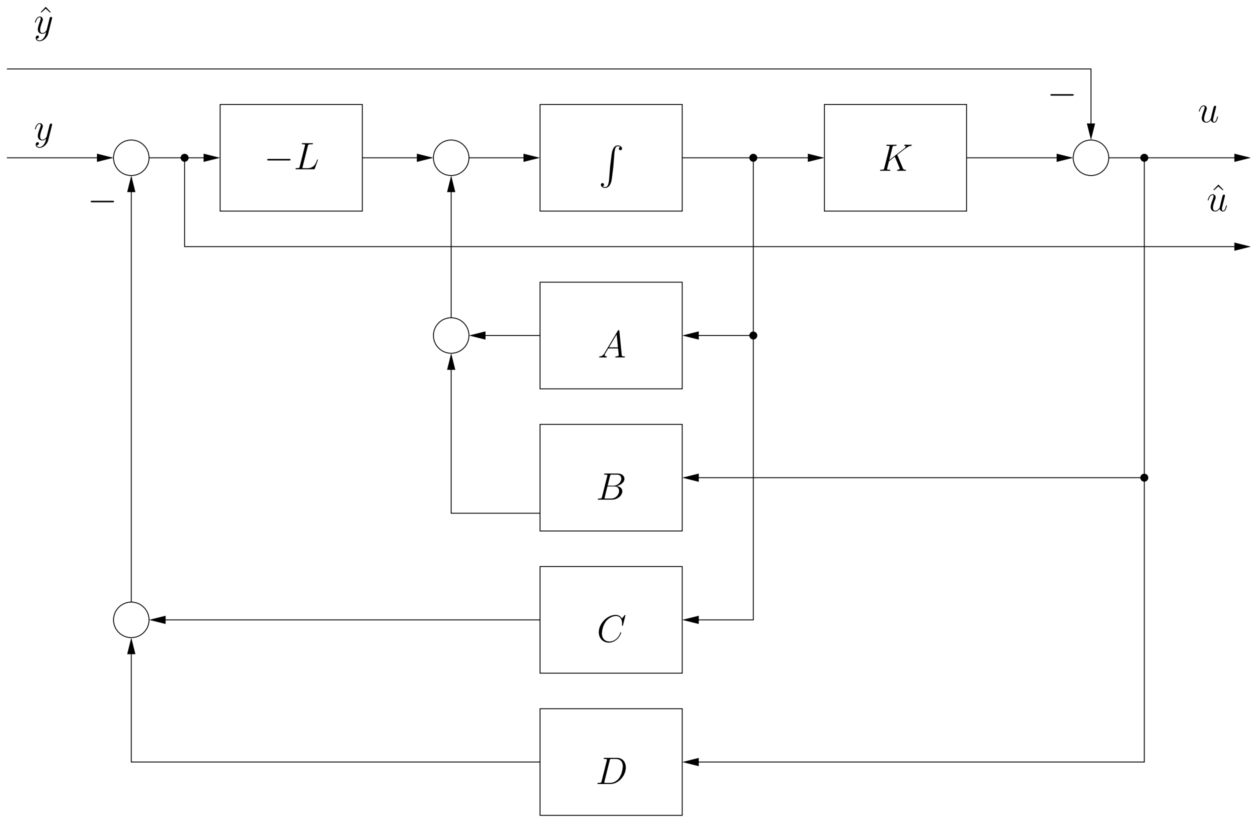


Figure 2: Observer-based controllers

Corollary: All controllers that stabilize the feedback system in Figure 1 are the result of the feedback connection of

$$\begin{pmatrix} U(s) \\ \hat{U}(s) \end{pmatrix} = R(s) \begin{pmatrix} Y(s) \\ \hat{Y}(s) \end{pmatrix} = \begin{bmatrix} C_0(s) & -\hat{X}^{-1}(s) \\ X^{-1}(s) & X^{-1}(s)N(s) \end{bmatrix} \begin{pmatrix} Y(s) \\ \hat{Y}(s) \end{pmatrix}$$

with

$$\hat{Y}(s) = Q(s)\hat{U}(s).$$

Furthermore $R(s)$ admits a realization

$$R(s) = \begin{bmatrix} C_0(s) & -\hat{X}^{-1}(s) \\ X^{-1}(s) & X^{-1}(s)N(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + BK + L(C + DK) & -L & -(B + LD) \\ \hline K & 0 & -I \\ -(C + DK) & I & D \end{array}\right],$$

for which one implementation is given in Figure 2.

Proof: For the first part compute

$$\begin{aligned}\hat{Y}(s) &= Q(s)\hat{U}(s)\\ &= Q(s)X^{-1}(s)\left[Y(s) + N(s)\hat{Y}(s)\right]\end{aligned}$$

so that

$$\left[I - Q(s)X^{-1}N(s)\right]\hat{Y}(s) = Q(s)X^{-1}(s)Y(s)$$

and

$$\begin{aligned}\hat{Y}(s) &= \left[I - Q(s)X^{-1}N(s)\right]^{-1} Q(s)X^{-1}(s)Y(s)\\ &= Q(s)\left[I - X^{-1}N(s)Q(s)\right]^{-1} X^{-1}(s)Y(s).\end{aligned}$$

Hence

$$U(s) = \left\{C_0(s) - \hat{X}^{-1}(s)Q(s)\left[I - X^{-1}N(s)Q(s)\right]^{-1} X^{-1}(s)\right\} Y(s) = C(s)Y(s).$$

For the second part start by constructing

$$X^{-1}(s) = \left[\begin{array}{c|c} A+BK & -L \\ \hline C+DK & I \end{array}\right]^{-1} = \left[\begin{array}{c|c} A+BK+L(C+DK) & L \\ \hline C+DK & I \end{array}\right]$$

and

$$\hat{X}^{-1}(s) = \left[\begin{array}{c|c} A+LC & -(B+LD) \\ \hline K & I \end{array}\right]^{-1} = \left[\begin{array}{c|c} A+LC+(B+LD)K & B+LD \\ \hline K & I \end{array}\right].$$

Then

$$\begin{aligned}
X^{-1}(s)N(s) &= \left[\begin{array}{c|c} A+BK+L(C+DK) & L \\ \hline C+DK & I \end{array}\right] \left[\begin{array}{c|c} A+BK & B \\ \hline C+DK & D \end{array}\right]\\
&= \left[\begin{array}{cc|c} A+BK & 0 & B \\ L(C+DK) & A+BK+L(C+DK) & LD \\ \hline C+DK & C+DK & D \end{array}\right]\\
&= \left[\begin{array}{c|c} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} A+BK+L(C+DK) & 0 \\ L(C+DK) & A+BK \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} & \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} B+LD \\ LD \end{bmatrix} \\ \hline \begin{bmatrix} C+DK & 0 \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} & D \end{array}\right]\\
&= \left[\begin{array}{cc|c} A+BK+L(C+DK) & 0 & B+LD \\ L(C+DK) & A+BK & LD \\ \hline C+DK & 0 & D \end{array}\right]\\
&= \left[\begin{array}{c|c} A+BK+L(C+DK) & B+LD \\ \hline C+DK & D \end{array}\right]
\end{aligned}$$

Similarly

$$
\begin{aligned}
C_0(s) &= Y(s)X^{-1}(s) \\
&= \left[\begin{array}{c|c} A + BK + L(C + DK) & L \\ \hline -K & 0 \end{array}\right].
\end{aligned}
$$

Therefore, since all submatrices share the same poles

$$
\begin{aligned}
R(s) = \begin{bmatrix} C_0(s) & -\hat{X}^{-1}(s) \\ X^{-1}(s) & X^{-1}(s)N(s) \end{bmatrix} &= \left[\begin{array}{cc|c} A + BK + L(C + DK) & L & B + LD \\ \hline -K & 0 & -I \\ C + DK & I & D \end{array}\right] \\
&= \left[\begin{array}{c|cc} A + BK + L(C + DK) & -L & -(B + LD) \\ \hline K & 0 & -I \\ -(C + DK) & I & D \end{array}\right].
\end{aligned}
$$

4.5 Matlab demo

```
>> % MAE 284
>> % Coprime factorizations
>> %
>> % MAE 284 - Robust and Multivariable Control
>> % Mauricio de Oliveira
>> % April 2006
>> %
>>
>> % Define state space matrices for plant
>> A = [
0  1  0
1  0  0
0  0  0
];
>>
>> B = [
0  1
1  0
1  0
];
>>
>> C = [
1  0  1
0  1  1
];
>>
>> D = [
0  0
0  1
];
>>
>> % construct plant
>> p = ss(A, B, C, D);
>>
>> % plant transfer function
>> zpk(p)

Zero/pole/gain from input 1 to output...
      (s+1.618) (s-0.618)
 #1:  -------------------
         s (s+1) (s-1)

      2 (s+0.7071) (s-0.7071)
 #2:  -----------------------
           s (s+1) (s-1)

Zero/pole/gain from input 2 to output...
           s
 #1:  -----------
      (s+1) (s-1)

          s^2
```

```
 #2:  -----------
      (s+1) (s-1)

>>
>> % define stabilizing gains
>> K = [
 0  0 -1
-1 -2 0
];
>>
>> L = [
-4 0
-4 0
 1 0
];
>>
>> % check stability
>> eig(A + B * K)
ans =
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
  -1.0000
>> eig(A + L * C)
ans =
  -1.0000 + 0.0000i
  -1.0000 - 0.0000i
  -1.0000
>>
>> % construct mnxy
>> mnxy = ss(A + B * K, [B -L], [K; C + D * K], [eye(2) zeros(2); D eye(2)])

a =
       x1  x2  x3
   x1  -1  -1   0
   x2   1   0  -1
   x3   0   0  -1


b =
       u1  u2  u3  u4
   x1   0   1   4  -0
   x2   1   0   4  -0
   x3   1   0  -1  -0


c =
       x1  x2  x3
   y1   0   0  -1
   y2  -1  -2   0
   y3   1   0   1
   y4  -1  -1   1


d =
```

```
       u1  u2  u3  u4
   y1   1   0   0   0
   y2   0   1   0   0
   y3   0   0   1   0
   y4   0   1   0   1

Continuous-time model.
>>
>> % construct mnxyhat
>> mnxyhat = ss(A + L*C, [-(B + L * D) L], [K; C], [eye(2) zeros(2); -D eye(2)])

a =
       x1  x2  x3
   x1  -4   1  -4
   x2  -3   0  -4
   x3   1   0   1


b =
       u1  u2  u3  u4
   x1  -0  -1  -4   0
   x2  -1  -0  -4   0
   x3  -1  -0   1   0


c =
       x1  x2  x3
   y1   0   0  -1
   y2  -1  -2   0
   y3   1   0   1
   y4   0   1   1


d =
       u1  u2  u3  u4
   y1   1   0   0   0
   y2   0   1   0   0
   y3  -0  -0   1   0
   y4  -0  -1   0   1

Continuous-time model.
>>
>> % check Bezout equation
>> minreal(mnxyhat * mnxy)
3 states removed.

a =
            x1       x2       x3
   x1  -0.2022   -1.095   0.3118
   x2    1.006   -1.103   0.3381
   x3   0.9026  -0.1481  -0.6943


b =
```

```
            u1       u2       u3       u4
   x1   -1.352  -0.5702   -7.393        0
   x2   -1.317   0.9995    3.319        0
   x3   0.6632   0.8222   0.5685        0


c =
               x1          x2          x3
   y1   5.172e-14   4.635e-14  -9.862e-14
   y2  -3.464e-13  -4.058e-13    7.55e-13
   y3   6.495e-14   9.483e-14  -1.615e-13
   y4   6.323e-14    8.57e-14  -1.489e-13


d =
       u1  u2  u3  u4
   y1   1   0   0   0
   y2   0   1   0   0
   y3   0   0   1   0
   y4   0   0   0   1

Continuous-time model.
>>
>> % obtain transfer function for mnxy
>> zpk(mnxy)

Zero/pole/gain from input 1 to output...
        s
 #1:  -----
      (s+1)

          -2 s (s+0.5)
 #2:  --------------------
      (s+1) (s^2  + s + 1)

           (s^2  + 1)
 #3:  --------------------
      (s+1) (s^2  + s + 1)

             (s+1)
 #4:  --------------------
      (s+1) (s^2  + s + 1)

Zero/pole/gain from input 2 to output...
 #1:  0

       (s+1) (s-1)
 #2:  --------------
      (s^2  + s + 1)

            s
 #3:  --------------
      (s^2  + s + 1)
```

```
           s^2
 #4:  --------------
      (s^2  + s + 1)

Zero/pole/gain from input 3 to output...
        1
 #1:  -----
      (s+1)

      -12 (s+1.384) (s+0.7829)
 #2:  ------------------------
        (s+1) (s^2  + s + 1)

      (s+4.537) (s+1.307) (s-0.8434)
 #3:  ------------------------------
           (s+1) (s^2  + s + 1)

      -9 (s+0.5556) (s+1)
 #4:  --------------------
      (s+1) (s^2  + s + 1)

Zero/pole/gain from input 4 to output...
 #1:  0

 #2:  0

 #3:  0

 #4:  1

>>
>> % extract x and y
>> y = mnxy(1:2, 3:4);
>> x = mnxy(3:4, 3:4);
>>
>> % compute stabilizing controller
>> c = y * inv(x)

a =
       x1  x2  x3  x4  x5  x6
   x1  -1  -1   0  -4   0  -4
   x2   1   0  -1  -4   0  -4
   x3   0   0  -1   1   0   1
   x4   0   0   0  -5  -1  -4
   x5   0   0   0  -3   0  -5
   x6   0   0   0   1   0   0


b =
       u1  u2
   x1   4   0
   x2   4   0
   x3  -1   0
   x4   4   0
```

```
   x5   4   0
   x6  -1   0


c =
       x1  x2  x3  x4  x5  x6
   y1   0   0  -1   0   0   0
   y2  -1  -2   0   0   0   0


d =
       u1  u2
   y1   0   0
   y2   0   0

Continuous-time model.
>>
>> % compute controller transfer function
>> zpk(minreal(c))
3 states removed.

Zero/pole/gain from input 1 to output...
              (s^2  + s + 1)
 #1:  ------------------------------
      (s+4.537) (s+1.307) (s-0.8434)

         -12 (s+1.384) (s+0.7829)
 #2:  ------------------------------
      (s+4.537) (s+1.307) (s-0.8434)

Zero/pole/gain from input 2 to output...
 #1:  0

 #2:  0

>>
>> % check closed loop stability
>> pole(inv([eye(2) -p; -c eye(2)]))
ans =
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
  -1.0001 + 0.0001i
  -1.0001 - 0.0001i
  -0.9999 + 0.0001i
  -0.9999 - 0.0001i
  -1.0000
>>
>> % extract x and y
>> m = mnxy(1:2, 1:2);
>> n = mnxy(3:4, 1:2);
>>
>> % define q
```

```
>> q = [tf(1,[1 3]) 0; tf(1,[1 2]) tf(1, [1 1])]

Transfer function from input 1 to output...
        1
 #1:  -----
      s + 3

        1
 #2:  -----
      s + 2

Transfer function from input 2 to output...
 #1:  0

        1
 #2:  -----
      s + 1

>>
>> % compute stabilizing controller
>> c = (y - m * q) * inv(x - n * q)

a =
          x1   x2   x3   x4   x5   x6   x7   x8   x9  x10  x11  x12  x13
   x1     -1   -1    0    0    0    0    0    0    0   -4    0   -4    4
   x2      1    0   -1    0    0    0    0    0    0   -4    0   -4    4
   x3      0    0   -1    0    0    0    0    0    0    1    0    1   -1
   x4      0    0    0   -1   -1    0    0    1    1    0    0    0    0
   x5      0    0    0    1    0   -1    1    0    0    0    0    0    0
   x6      0    0    0    0    0   -1    1    0    0    0    0    0    0
   x7      0    0    0    0    0    0   -3    0    0   -1    0   -1    1
   x8      0    0    0    0    0    0    0   -2    0   -1    0   -1    1
   x9      0    0    0    0    0    0    0    0   -1    1    1   -1   -1
   x10     0    0    0    0    0    0    0    0    0   -5   -1   -4    4
   x11     0    0    0    0    0    0    0    0    0   -3    0   -5    4
   x12     0    0    0    0    0    0    0    0    0    1    0    0   -1
   x13     0    0    0    0    0    0    0    0    0    0    0    0   -1
   x14     0    0    0    0    0    0    0    0    0    0    0    0    1
   x15     0    0    0    0    0    0    0    0    0    0    0    0    0
   x16     0    0    0    0    0    0    0    0    0   -1    0   -1    1
   x17     0    0    0    0    0    0    0    0    0   -1    0   -1    1
   x18     0    0    0    0    0    0    0    0    0    1    1   -1   -1

         x14  x15  x16  x17  x18
   x1      0    4    0    0    0
   x2      0    4    0    0    0
   x3      0   -1    0    0    0
   x4      0    0    0    0    0
   x5      0    0    0    0    0
   x6      0    0    0    0    0
   x7      0    1    0    0    0
   x8      0    1    0    0    0
   x9     -1    1    0    1    1
   x10     0    4    0    0    0
```

```
   x11    0    4    0    0    0
   x12    0   -1    0    0    0
   x13   -1    0    0    1    1
   x14    0   -1    1    0    0
   x15    0   -1    1    0    0
   x16    0    1   -3    0    0
   x17    0    1    0   -2    0
   x18   -1    1    0    1    0


b =
        u1  u2
   x1    4   0
   x2    4   0
   x3   -1   0
   x4    0   0
   x5    0   0
   x6    0   0
   x7    1   0
   x8    1   0
   x9    0   1
   x10   4   0
   x11   4   0
   x12  -1   0
   x13   0   0
   x14   0   0
   x15   0   0
   x16   1   0
   x17   1   0
   x18   0   1


c =
        x1   x2   x3   x4   x5   x6   x7   x8   x9  x10  x11  x12  x13
   y1    0    0   -1   -0   -0    1   -1   -0   -0    0    0    0    0
   y2   -1   -2    0    1    2   -0   -0   -1   -1    0    0    0    0

       x14  x15  x16  x17  x18
   y1    0    0    0    0    0
   y2    0    0    0    0    0


d =
       u1  u2
   y1   0   0
   y2   0   0

Continuous-time model.
>>
>> % compute controller transfer function
>> zpk(minreal(c))
12 states removed.

Zero/pole/gain from input 1 to output...
```

```
               3 (s+2) (s+0.5698) (s^2  + 0.4302s + 1.755)
 #1:  --------------------------------------------------------------
      (s+5.026) (s+2.252) (s-1.399) (s+0.55) (s^2  + 3.571s + 3.673)

        -13 (s+2.56) (s+0.7159) (s+0.4773) (s^2  + 3.785s + 3.782)
 #2:  --------------------------------------------------------------
      (s+5.026) (s+2.252) (s-1.399) (s+0.55) (s^2  + 3.571s + 3.673)

Zero/pole/gain from input 2 to output...
              4.9559e-15 s (s+2) (s^2  + 16.43s + 6.054e14)
 #1:  --------------------------------------------------------------
      (s+5.026) (s+2.252) (s-1.399) (s+0.55) (s^2  + 3.571s + 3.673)

            - (s+1.592) (s+2) (s+3.565) (s^2  + 1.843s + 2.82)
 #2:  --------------------------------------------------------------
      (s+5.026) (s+2.252) (s-1.399) (s+0.55) (s^2  + 3.571s + 3.673)

>>
>> % check closed loop stability
>> pole(inv([eye(2) -p; -c eye(2)]))
ans =
  -3.0000 + 0.0000i
  -3.0000 - 0.0000i
  -2.0000
  -2.0000
  -1.0004
  -1.0000 + 0.0004i
  -1.0000 - 0.0004i
  -0.9996
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
  -1.0000 + 0.0000i
  -1.0000 - 0.0000i
  -1.0000
  -1.0000
  -1.0000
>>
>> % find stabilizing gains through Riccati
>> R = B * B';
>> Q = eye(3);
>>
>> % find eigenvalues of Hamiltonean
>> eig([A R; -Q -A'])
ans =
        0 + 0.0000i
        0 - 0.0000i
  -0.0000 + 1.2720i
  -0.0000 - 1.2720i
```

```
  -0.7862
   0.7862
>>
>> % there exists eigenvalues on the imaginary axis, flip sign
>> R = -B * B';
>>
>> % find eigenvalues of Hamiltonean
>> [v, d] = eig([A R; -Q -A'])
v =
    0.7071    0.7071   -0.4082   -0.4082    0.4082    0.4082
    0.5000   -0.5000    0.6606   -0.6606   -0.2523    0.2523
         0         0    0.4082   -0.4082    0.4082   -0.4082
   -0.5000    0.5000   -0.0000    0.0000   -0.0000   -0.0000
         0         0    0.4082    0.4082   -0.4082   -0.4082
         0         0    0.2523    0.2523    0.6606    0.6606
d =
    1.4142         0         0         0         0         0
         0   -1.4142         0         0         0         0
         0         0   -1.6180         0         0         0
         0         0         0    1.6180         0         0
         0         0         0         0   -0.6180         0
         0         0         0         0         0    0.6180
>>
>> % go find eigenvectors of Hamiltonean
>> v1 = v(:, [2 3 5])
v1 =
    0.7071   -0.4082    0.4082
   -0.5000    0.6606   -0.2523
         0    0.4082    0.4082
    0.5000   -0.0000   -0.0000
         0    0.4082   -0.4082
         0    0.2523    0.6606
>> x1 = v1(1:3,:)
x1 =
    0.7071   -0.4082    0.4082
   -0.5000    0.6606   -0.2523
         0    0.4082    0.4082
>> x2 = v1(4:6,:)
x2 =
    0.5000   -0.0000   -0.0000
         0    0.4082   -0.4082
         0    0.2523    0.6606
>>
>> % compute are solution
>> X = x2 / x1
X =
    1.9239    1.7208   -0.8604
    1.7208    2.4335   -1.2168
   -0.8604   -1.2168    1.7264
>>
>> % stabilizing gain is
>> K = - B' * X
K =
   -0.8604   -1.2168   -0.5097
```

```
   -1.9239   -1.7208    0.8604
>>
>> eig(A + B * K)
ans =
   -0.6180
   -1.4142
   -1.6180
>>
>> % alternative solution
>> help are

 ARE  Algebraic Riccati Equation solution.

     X = ARE(A, B, C) returns the stablizing solution (if it
     exists) to the continuous-time Riccati equation:

            A'*X + X*A - X*B*X + C = 0

     assuming B is symmetric and nonnegative definite and C is
     symmetric.

     See also  RIC, CARE, DARE.

>>
>> X = are(A, B * B', Q)
X =
    1.9239    1.7208   -0.8604
    1.7208    2.4335   -1.2168
   -0.8604   -1.2168    1.7264
>>
>> % stabilizing filter gain
>> help are

 ARE  Algebraic Riccati Equation solution.

     X = ARE(A, B, C) returns the stablizing solution (if it
     exists) to the continuous-time Riccati equation:

            A'*X + X*A - X*B*X + C = 0

     assuming B is symmetric and nonnegative definite and C is
     symmetric.

     See also  RIC, CARE, DARE.

>>
>> Y = are(A', C' * C, Q)
Y =
    6.2756    5.8614   -3.5549
    5.8614    6.2756   -3.5549
   -3.5549   -3.5549    2.8478
>>
>> % stabilizing gain is
>> L = (- C * Y)'
```

```
L =
   -2.7208   -2.3066
   -2.3066   -2.7208
    0.7071    0.7071
>>
>> eig(A + L * C)
ans =
   -0.7654
   -1.8478
   -1.4142
>>
>> % construct mnxy
>> mnxy = ss(A + B * K, [B -L], [K; C + D * K], [eye(2) zeros(2); D eye(2)])

a =
            x1       x2       x3
   x1   -1.924  -0.7208   0.8604
   x2   0.1396   -1.217  -0.5097
   x3  -0.8604   -1.217  -0.5097


b =
            u1       u2       u3       u4
   x1        0        1    2.721    2.307
   x2        1        0    2.307    2.721
   x3        1        0  -0.7071  -0.7071


c =
            x1       x2       x3
   y1  -0.8604   -1.217  -0.5097
   y2   -1.924   -1.721   0.8604
   y3        1        0        1
   y4   -1.924  -0.7208     1.86


d =
       u1  u2  u3  u4
   y1   1   0   0   0
   y2   0   1   0   0
   y3   0   0   1   0
   y4   0   1   0   1

Continuous-time model.
>>
>> % construct mnxyhat
>> mnxyhat = ss(A + L*C, [-(B + L * D) L], [K; C], [eye(2) zeros(2); -D eye(2)])

a =
           x1      x2      x3
   x1  -2.721  -1.307  -5.027
   x2  -1.307  -2.721  -5.027
   x3  0.7071  0.7071   1.414
```

```
b =
             u1        u2        u3        u4
   x1        -0     1.307    -2.721    -2.307
   x2        -1     2.721    -2.307    -2.721
   x3        -1   -0.7071    0.7071    0.7071


c =
             x1        x2        x3
   y1   -0.8604    -1.217   -0.5097
   y2    -1.924    -1.721    0.8604
   y3         1         0         1
   y4         0         1         1


d =
        u1  u2  u3  u4
   y1    1   0   0   0
   y2    0   1   0   0
   y3   -0  -0   1   0
   y4   -0  -1   0   1

Continuous-time model.
>>
>> % compute stabilizing controller
>> c = y * inv(x)

a =
        x1  x2  x3  x4  x5  x6
   x1   -1  -1   0  -4   0  -4
   x2    1   0  -1  -4   0  -4
   x3    0   0  -1   1   0   1
   x4    0   0   0  -5  -1  -4
   x5    0   0   0  -3   0  -5
   x6    0   0   0   1   0   0


b =
        u1  u2
   x1    4   0
   x2    4   0
   x3   -1   0
   x4    4   0
   x5    4   0
   x6   -1   0


c =
        x1  x2  x3  x4  x5  x6
   y1    0   0  -1   0   0   0
   y2   -1  -2   0   0   0   0
```

```
d =
       u1  u2
   y1   0   0
   y2   0   0

Continuous-time model.
>>
>> % compute controller transfer function
>> zpk(minreal(c))
3 states removed.

Zero/pole/gain from input 1 to output...
              (s^2  + s + 1)
 #1:  ------------------------------
      (s+4.537) (s+1.307) (s-0.8434)

         -12 (s+1.384) (s+0.7829)
 #2:  ------------------------------
      (s+4.537) (s+1.307) (s-0.8434)

Zero/pole/gain from input 2 to output...
 #1:  0

 #2:  0

>>
>> % check closed loop stability
>> pole(inv([eye(2) -p; -c eye(2)]))
ans =
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
  -1.0001 + 0.0001i
  -1.0001 - 0.0001i
  -0.9999 + 0.0001i
  -0.9999 - 0.0001i
  -1.0000
>>
>> diary off
```

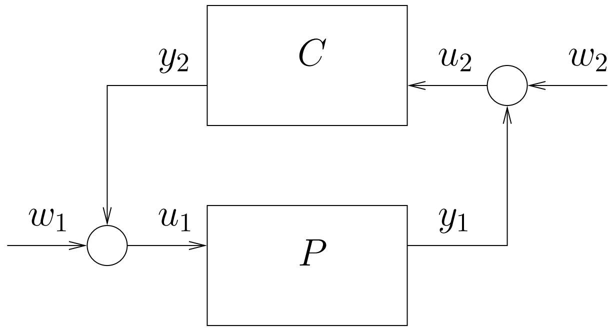


Figure 3: Feedback connection

Lemma: Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

where (A, B_u) is stabilizable and (A, C_y) is detectable. A controller $C(s)$ stabilizes the feedback system in Figure 3 if and only if it stabilizes $H_{yu}(s)$

Proof: Necessity is obvious. Sufficiency comes from stabilizability of (A, B_u) and detectability of (A, C_y).

Theorem: Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

where (A, B_u) is stabilizable and (A, C_y) is detectable. All stabilizing controllers $C(s)$ are parametrized as the feedback connection of

$$\begin{aligned} R(s) &= \begin{bmatrix} C_0(s) & -\hat{X}^{-1}(s) \\ X^{-1}(s) & X^{-1}(s)N(s) \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A + B_uK + L(C_y + D_{yu}K) & -L & -(B_u + LD_{yu}) \\ \hline K & 0 & -I \\ -(C_y + D_{yu}K) & I & D_{yu} \end{array}\right] \end{aligned}$$

with $Q(s) \in RH_\infty$ and $\lim_{s\to\infty}[I - H_{yu}(s)Q(s)]$ is nonsingular. Furthermore the closed loop transfer function from w to z is

$$G(s) = T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)$$

where

$$T(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A + B_uK & -B_uK & B_w & -B_u \\ 0 & A + LC_y & B_w + LD_{yw} & 0 \\ \hline C_z + D_{zu}K & -D_{zu}K & D_{zw} & -D_{zu} \\ 0 & C_y & D_{yw} & 0 \end{array}\right]$$

with $T_{ij} \in RH_\infty$, $i, j = \{1, 2\}$.

Proof: Because (A, B_u) is stabilizable and (A, C_y) is detectable, all stabilizing controllers are the ones that stabilize $H_{yu}(s)$, which are given in the theorem. The feedback connection of

$$\begin{pmatrix} \dot{x} \\ \hline z \\ y \end{pmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right] \begin{pmatrix} x \\ \hline w \\ u \end{pmatrix}$$

with

$$\begin{pmatrix} \dot{\hat{x}} \\ \hline u \\ \hat{u} \end{pmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_y & \hat{B}_{\hat{y}} \\ \hline \hat{C}_u & 0 & \hat{D}_{u\hat{y}} \\ \hat{C}_{\hat{u}} & \hat{D}_{\hat{u}y} & \hat{D}_{\hat{u}\hat{y}} \end{array}\right] \begin{pmatrix} \hat{x} \\ \hline y \\ \hat{y} \end{pmatrix}$$

is given by

$$
\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \\ \hline z \\ \hat{u} \end{pmatrix} = \left[\begin{array}{cc|cc} A & B_u \hat{C}_u & B_w & B_u \hat{D}_{u\hat{y}} \\ \hat{B}_y C_y & \hat{A} + \hat{B}_y D_{yu} \hat{C}_u & \hat{B}_y D_{yw} & \hat{B}_{\hat{y}} + \hat{B}_y D_{yu} \hat{D}_{u\hat{y}} \\ \hline C_z & D_{zu} \hat{C}_u & D_{zw} & D_{zu} \hat{D}_{u\hat{y}} \\ \hat{D}_{\hat{u}y} C_y & \hat{C}_{\hat{u}} + \hat{D}_{\hat{u}y} D_{yu} \hat{C}_u & \hat{D}_{\hat{u}y} D_{yw} & \hat{D}_{\hat{u}\hat{y}} + \hat{D}_{\hat{u}y} D_{yu} \hat{D}_{u\hat{y}} \end{array} \right] \begin{pmatrix} x \\ \hat{x} \\ \hline w \\ \hat{y} \end{pmatrix}
$$

Substituting

$$
\left[\begin{array}{c|cc} \hat{A} & \hat{B}_y & \hat{B}_{\hat{y}} \\ \hline \hat{C}_u & 0 & \hat{D}_{u\hat{y}} \\ \hat{C}_{\hat{u}} & \hat{D}_{\hat{u}y} & \hat{D}_{\hat{u}\hat{y}} \end{array} \right] = \left[\begin{array}{c|cc} A + B_u K + L(C_y + D_{yu} K) & -L & -(B_u + L D_{yu}) \\ \hline K & 0 & -I \\ -(C_y + D_{yu} K) & I & D_{yu} \end{array} \right]
$$

we obtain

$$
\begin{aligned}
T(s) &= \left[\begin{array}{cc|cc} A & B_u K & B_w & -B_u \\ -L C_y & A + B_u K + L C_y & -L D_{yw} & -B_u \\ \hline C_z & D_{zu} K & D_{zw} & -D_{zu} \\ C_y & -C_y & D_{yw} & 0 \end{array} \right] \\
&= \left[\begin{array}{c|c} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & B_u K \\ -L C_y & A + B_u K + L C_y \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} & \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} B_w & -B_u \\ -L D_{yw} & -B_u \end{bmatrix} \\ \hline \begin{bmatrix} C_z & D_{zu} K \\ C_y & -C_y \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} & \begin{matrix} D_{zw} & -D_{zu} \\ D_{yw} & 0 \end{matrix} \end{array} \right] \\
&= \left[\begin{array}{cc|cc} A + B_u K & -B_u K & B_w & -B_u \\ 0 & A + L C_y & B_w + L D_{yw} & 0 \\ \hline C_z + D_{zu} K & -D_{zu} K & D_{zw} & -D_{zu} \\ 0 & C_y & D_{yw} & 0 \end{array} \right].
\end{aligned}
$$

The closed loop transfer function $G(s)$ is the feedback connection of $T(s)$ with $Q(s)$, that is,

$$
\begin{aligned}
Z(s) &= T_{11}(s) W(s) + T_{12}(s) \hat{Y}(s) \\
\hat{U}(s) &= T_{21}(s) W(s)
\end{aligned}
$$

with

$$
\hat{Y}(s) = Q(s) \hat{U}(s)
$$

which produces

$$
Z(s) = G(s) W(s) = \left[T_{11}(s) + T_{12}(s) Q(s) T_{21}(s) \right] W(s).
$$

4.6 Linear Fractional Transformation

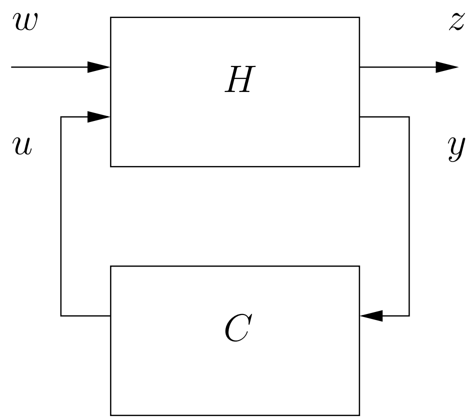


Figure 4: Feedback connection

Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix}$$

and the control

$$U(s) = C(s)Y(s).$$

Then

$$\begin{aligned} H_{yw}(s)W(s) &= Y(s) - H_{yu}(s)U(s) \\ &= \left[I - H_{yu}(s)C(s)\right] Y(s) \end{aligned}$$

and if the feeback loop is well-posed, i.e.,

$$\lim_{s\to\infty} I - H_{yu}(s)C(s) \qquad \text{is invertible}$$

then

$$Y(s) = \left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s)W(s)$$

and

$$\begin{aligned} Z(s) &= H_{zw}(s)W(s) + H_{zu}C(s)Y(s) \\ &= H_{zw}(s)W(s) + H_{zu}C(s)\left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s)W(s) \end{aligned}$$

That is

$$Z(s) = \mathcal{F}_l\left(H(s), C(s)\right)\, W(s)$$

where

$$\mathcal{F}_l\left(H(s), C(s)\right) := H_{zw}(s) + H_{zu}(s)C(s)\left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s)$$

is called a (Lower) Linear Fractional Transformation.

Simmilarly

$$Z(s) = \mathcal{F}_u\left(H(s), C(s)\right)\, W(s)$$

where

$$\mathcal{F}_u\left(H(s), C(s)\right) := H_{zw}(s) + H_{zu}(s)\left[I - C(s)H_{yu}(s)\right]^{-1} C(s)H_{yw}(s)$$

is called an (Upper) Linear Fractional Transformation.

For us, LFT is for convinience of notation!

A final remark on the parametrization of all stabilizing controllers. If $H(s) \in RH_\infty$ then

$$\begin{aligned} Z(s) &= \mathcal{F}_l\left(H(s), C(s)\right) W(s) \\ &= \left[H_{zw}(s) + H_{zu}\underbrace{C(s)\left[I - H_{yu}(s)C(s)\right]^{-1}}_{Q(s)} H_{yw}(s)\right] W(s) \end{aligned}$$

or

$$\begin{aligned} Z(s) &= \mathcal{F}_u\left(H(s), C(s)\right) W(s) \\ &= \left[H_{zw}(s) + H_{zu}(s)\underbrace{\left[I - C(s)H_{yu}(s)\right]^{-1} C(s)}_{Q(s)} H_{yw}(s)\right] W(s) \end{aligned}$$

and

$$Q(s) = C(s)\left[I - H_{yu}(s)C(s)\right]^{-1} = \left[I - C(s)H_{yu}(s)\right]^{-1} C(s) \in RH_\infty$$

(Recall the previously studied case $H_{yu}(s) \in RH_\infty$?)

5 The H_∞ Optimal Control Problem

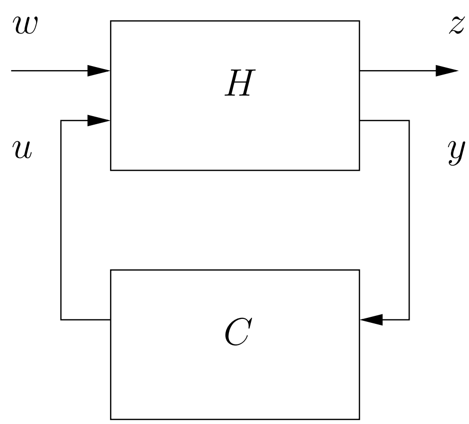


Figure 5: Feedback connection

5.1 Problem Statement

Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

where (A, B_u) is stabilizable and (A, C_y) is detectable.

H_∞ **Control Problem:** Find $C(s)$ such that

$$C(s) = \arg\min_{C(s)} \|\mathcal{F}_l\left(H(s), C(s)\right)\|_\infty$$

From the parametrization of all stabilizing controllers

$$\mathcal{F}_l\left(H(s), C(s)\right) = T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)$$

so that

$$\min_{C(s)} \|\mathcal{F}_l\left(H(s), C(s)\right)\|_\infty = \min_{Q(s) \in RH_\infty} \|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty$$

5.2 Equivalent Formulation as a Hankel Approximation Problem

Suppose there exist $T_{12}^{\perp}$ and $T_{21}^{\perp}$ such that

$$\begin{bmatrix} T_{12}(s) & T_{12}^{\perp}(s) \end{bmatrix}^{\sim} \begin{bmatrix} T_{12}(s) & T_{12}^{\perp}(s) \end{bmatrix} = I, \qquad \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix} \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix}^{\sim} = I$$

Note that

$$\begin{bmatrix} T_{12}(s) & T_{12}^{\perp}(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_{21}(s) \\ T_{21}^{\perp}(s) \end{bmatrix}$$

are inner and co-inner, respectively, and square. Once can chose K and L such that $T_{12}^{\perp}$ and $T_{21}^{\perp}$ with such properties always exist (not shown here!).

Lemma:
Let $G(s) \in RH_\infty$ be inner (i.e. $G^{\sim}(s)G(s) = I$), then for any $F(s) \in RH_\infty$

$$\|G(s)F(s)\|_\infty = \|F(s)\|_\infty$$

Proof:

$$\begin{aligned} \|F(s)G(s)\|_\infty &= \sup_\omega \overline{\sigma}(G(j\omega)F(j\omega)) \\ &= \sup_\omega \overline{\lambda}(F^{\sim}(j\omega)G^{\sim}(j\omega)G(j\omega)F(j\omega)) \\ &= \sup_\omega \overline{\lambda}(F^{\sim}(j\omega)F(j\omega)) \\ &= \|F(s)\|_\infty \end{aligned}$$

Lemma:
Let $G(s) \in RH_\infty$ be co-inner (i.e. $G(s)G^{\sim}(s) = I$), then for any $F(s) \in RH_\infty$

$$\|F(s)G(s)\|_\infty = \|F(s)\|_\infty$$

Proof:

$$\begin{aligned} \|F(s)G(s)\|_\infty &= \sup_\omega \overline{\sigma}(F(j\omega)G(j\omega)) \\ &= \sup_\omega \overline{\lambda}(F(j\omega)G(j\omega)G^{\sim}(j\omega)F^{\sim}(j\omega)) \\ &= \sup_\omega \overline{\lambda}(F(j\omega)F^{\sim}(j\omega)) \\ &= \|F(s)\|_\infty \end{aligned}$$

Therefore

$$
\begin{aligned}
&\|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty \\
&\qquad = \left\| T_{11}(s) + \begin{bmatrix} T_{12}(s) & T_{12}^\perp(s) \end{bmatrix} \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_{21}(s) \\ T_{21}^\perp(s) \end{bmatrix} \right\|_\infty \\
&\qquad = \left\| \begin{bmatrix} T_{12}(s) & T_{12}^\perp(s) \end{bmatrix} \left(\begin{bmatrix} T_{12}(s) & T_{12}(s)^\perp \end{bmatrix}^\sim T_{11}(s) \begin{bmatrix} T_{21}(s) \\ T_{21}^\perp(s) \end{bmatrix}^\sim \right.\right. \\
&\qquad\qquad\qquad \left.\left. + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} T_{21}(s) \\ T_{21}^\perp(s) \end{bmatrix} \right\|_\infty \\
&\qquad = \left\| \begin{bmatrix} T_{12}(s) & T_{12}^\perp(s) \end{bmatrix}^\sim T_{11}(s) \begin{bmatrix} T_{21}(s) \\ T_{21}^\perp(s) \end{bmatrix}^\sim + \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty \\
&\qquad = \left\| \begin{bmatrix} R_{11}(s) + Q(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{bmatrix} \right\|_\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
R_{11}(s) &= T_{12}^\sim(s)T_{11}(s)T_{21}^\sim(s), & R_{21}(s) &= T_{12}^\sim(s)T_{11}(s)T_{21}^{\perp\sim}(s), \\
R_{21}(s) &= T_{12}^{\perp\sim}(s)T_{11}(s)T_{21}^\sim(s), & R_{22}(s) &= T_{12}^{\perp\sim}(s)T_{11}(s)T_{21}^{\perp\sim}(s).
\end{aligned}
$$

5.2.1 The 1-Block Problem

If $T_{12}(s)$ and $T_{21}(s)$ are square and

$$T_{12}^{\sim}(s)T_{12}(s) = I, \qquad\qquad T_{21}(s)T_{21}^{\sim}(s) = I$$

then

$$\begin{aligned} \|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty &= \|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty \\ &= \|T_{12}(s)\left(T_{12}^{\sim}(s)T_{11}(s)T_{21}^{\sim}(s) + Q(s)\right)T_{21}(s)\|_\infty \\ &= \|T_{12}^{\sim}(s)T_{11}(s)T_{21}^{\sim}(s) + Q(s)\|_\infty \end{aligned}$$

Furthermore

$$\|T_{12}^{\sim}(s)T_{11}(s)T_{21}^{\sim}(s) + Q(s)\|_\infty = \|T_{21}(s)T_{11}^{\sim}(s)T_{12}(s) + Q^{\sim}(s)\|_\infty$$

where $T_{21}(s)T_{11}^{\sim}(s)T_{12}(s)$ can be shown to be in RH_∞.

Hankel Approximation Problem: Find $Q(s) \in RH_\infty$ such that

$$\min_{Q \in RH_\infty} \|T_{21}(s)T_{11}^{\sim}(s)T_{12}(s) + Q^{\sim}(s)\|_\infty$$

AKA as **Nehari Extension Problem**.

Solution is known (Glover, 1984)

5.3 Equivalent Formulation as a Model Matching Problem

Model Matching Problem: Find $Q(s) \in RH_\infty$ such that

$$Q(s) = \arg \min_{Q(s) \in RH_\infty} \|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_\infty$$

Model Matching Problem (SISO): Find $Q(s) \in RH_\infty$ such that

$$Q^*(s) = \arg \min_{Q(s) \in RH_\infty} \|T_1(s) + T_2(s)Q(s)\|_\infty$$

Trivial case: If $T_1(s)T_2^{-1}(s)$ is stable then

$$Q^*(s) = -T_1(s)T_2^{-1}(s), \qquad \|T_1(s) + T_2(s)Q^*(s)\|_\infty = 0$$

Simplest next case: If $T_2(s)$ has one zero s_0 such that $s_0 + s_0^* > 0$. Then

$$\begin{aligned}
\|T_1(s) + T_2(s)Q(s)\|_\infty &= \sup_{\mathrm{Re}(s)>0} |T_1(s) + T_2(s)Q(s)| \\
&\geq |T_1(s_0) + T_2(s_0)Q(s_0)| \\
&\geq |T_1(s_0)|
\end{aligned}$$

But for $Q^*(s) = [T_1(s) - T_1(s_0)]T_2^{-1}(s)$

$$\|T_1(s) + T_2(s)Q^*(s)\|_\infty = \|T_1(s_0)\|_\infty$$

Note that $Q^*(s)$ is stable because $T_1(s) - T_1(s_0)$ has a zero on s_0!

General case: **Nevanlina-Pick Problem**.

5.4 Super Simplified H_∞ Control Problem Solution

Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

Assumptions:

1. (A, B_u) is stabilizable and (A, C_y) is detectable
2. $D_{zw} = 0$
3. $D_{yu} = 0$
4. $D_{zu}^T C_z = 0$ and $D_{zu}^T D_{zu} = I$
5. $B_w D_{yw}^T = 0$ and $D_{yw} D_{yw}^T = I$
6. (A, B_w) is controllable and (A, C_z) is observable

Theorem: There exists a controller $C(s)$ such that $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ under the above assumptions if and only if all the following conditions are satisfied

1. There exists X symmetric and positive semidefinite stabilizing solution of the Riccati equation

$$A^T X + XA - X(B_u B_u^T - \gamma^{-2} B_w B_w^T)X + C_z^T C_z = 0$$

2. There exists Y symmetric and positive semidefinite stabilizing solution of the Riccati equation

$$AY + YA^T - Y(C_y^T C_y - \gamma^{-2} C_z^T C_z)Y + B_w B_w^T = 0$$

3. $\max_i |\lambda_i(XY)| < \gamma^2$

If so, all controllers such that $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ are parametrized as the feedback connection of

$$R(s) = \left[\begin{array}{c|cc} A_\infty & -Z_\infty L_\infty & Z_\infty B_u \\ \hline K_\infty & 0 & I \\ -C_y & I & 0 \end{array}\right],$$

where

$$\begin{aligned} A_\infty &:= A + \gamma^{-2} B_w B_w^T X + B_u K_\infty + Z_\infty L_\infty C_y \\ Z_\infty &:= (I - \gamma^{-2} YX)^{-1} \\ L_\infty &:= -Y C_y^T \\ K_\infty &:= -B_u^T X \end{aligned}$$

with $Q(s) \in RH_\infty$ such that $\|Q(s)\| < \gamma$.

5.5 Removing the Assumptions

AKA as *Loop Shifting*.

5.5.1 $D_{yu} = 0$

Define

$$\tilde{y} = y - D_{yw} u$$

so that

$$\tilde{H}(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

5.5.2 Normality of D_{zu} and D_{yw}

This assumption can be replaced by D_{zu} full column-rank and D_{yw} full row-rank. Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

If D_{zu} is full column rank, compute

$$D_{zu} = U \Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} 0 \\ \Sigma V^T \end{bmatrix} =: U_z \begin{bmatrix} 0 \\ S_u \end{bmatrix}, \qquad U_z^T U_z = I.$$

If D_{yw} is full row rank, compute

$$D_{yw} = U \Sigma V^T = \begin{bmatrix} 0 & U\Sigma \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} =: \begin{bmatrix} 0 & S_y \end{bmatrix} V_w^T, \qquad V_w^T V_w = I.$$

Now define

$$\tilde{w} = V_w^T w \qquad\qquad \tilde{u} = S_u u$$
$$\tilde{z} = U_z^T z \qquad\qquad \tilde{y} = S_y^{-1} y$$

so that

$$w = V_w \tilde{w} \qquad\qquad u = S_u^{-1} \tilde{u}$$

and

$$\tilde{H}(s) = \left[\begin{array}{c|cc} A & B_w V_w & B_u S_u^{-1} \\ \hline U_z^T C_z & U_z^T D_{zw} V_w & U_z^T D_{zu} S_u^{-1} \\ S_y^{-1} C_y & S_y^{-1} D_{yw} V_w & 0 \end{array}\right].$$

Note that

$$U_z^T D_{zu} S_u^{-1} = U_z^T U_z \begin{bmatrix} 0 \\ S_u \end{bmatrix} S_u^{-1} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

and

$$S_y^{-1} D_{yw} V_w = S_y^{-1} \begin{bmatrix} 0 & S_y \end{bmatrix} V_w^T V_w = \begin{bmatrix} 0 & I \end{bmatrix}$$

Also

$$\begin{aligned} \mathcal{F}_l\left(\tilde{H}(s), \tilde{C}(s)\right) &= \tilde{H}_{zw}(s) + \tilde{H}_{zu}(s)\tilde{C}(s)\left[I - \tilde{H}_{yu}(s)\tilde{C}(s)\right]^{-1} \tilde{H}_{yw}(s) \\ &= U_z^T H_{zw}(s) V_w + U_z^T H_{zu}(s) C(s) \left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s) V_w \\ &= U_z^T \mathcal{F}_l\left(H(s), C(s)\right) V_w \end{aligned}$$

and

$$\left\|\mathcal{F}_l\left(\tilde{H}(s), \tilde{C}(s)\right)\right\|_\infty = \left\|U_z^T \mathcal{F}_l\left(H(s), C(s)\right) V_w\right\|_\infty = \left\|\mathcal{F}_l\left(H(s), C(s)\right)\right\|_\infty$$

5.5.3 Orthogonality of C_z and D_{zu}

Define

$$\tilde{u} = u + D_{zu}^T C_z x, \qquad\qquad u = \tilde{u} - D_{zu}^T C_z x$$

so that

$$\tilde{H}(s) = \left[\begin{array}{c|cc} A - B_u D_{zu}^T C_z & B_w & B_u \\ \hline (I - D_{zu} D_{zu}^T) C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

Because $D_{zu}^T D_{zu} = I$ then

$$D_{zu}^T (I - D_{zu} D_{zu}^T) C_z = [D_{zu}^T - (D_{zu}^T D_{zu}) D_{zu}^T] C_z = 0$$

5.5.4 Orthogonality of B_w and D_{yw}

Note that

$$\begin{aligned}
\mathcal{F}_l\left(H(s), C(s)\right)^T &= \Big[H_{zw}(s) + H_{zu}(s)C(s)\left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s)\Big]^T \\
&= H_{zw}^T(s) + H_{yw}^T(s)\left[I - C^T(s)H_{yu}^T(s)\right]^{-1} C^T(s)H_{zu}^T(s) \\
&= \mathcal{F}_u\left(H^T(s), C^T(s)\right)
\end{aligned}$$

Note that

$$H^T(s) = \left[\begin{array}{c|cc} A^T & C_z^T & C_y^T \\ \hline B_w^T & 0 & D_{yw}^T \\ B_u^T & D_{zu}^T & 0 \end{array}\right]$$

and the definitions

$$\tilde{u} = u + D_{yw}B_w^T x, \qquad u = \tilde{u} - D_{yw}B_w^T x$$

are so that

$$\tilde{H}^T(s) = \left[\begin{array}{c|cc} A^T - C_y^T D_{yw} B_w^T & C_z^T & C_y^T \\ \hline (I - D_{yw}^T D_{yw})B_w^T & 0 & D_{yw}^T \\ B_u^T & D_{zu}^T & 0 \end{array}\right]$$

and because $D_{yw}D_{yw}^T = I$ then

$$D_{yw}(I - D_{yw}^T D_{yw})B_w^T = [D_{yw} - (D_{yw}D_{yw}^T)D_{yw}]B_w^T = 0$$

Of course

$$\begin{aligned}
\mathcal{F}_u\left(H^T(s), C^T(s)\right)^T &= \mathcal{F}_u\left(\tilde{H}^T(s), \tilde{C}^T(s)\right)^T \\
&= \mathcal{F}_l\left(\tilde{H}(s), \tilde{C}(s)\right)
\end{aligned}$$

where

$$H(s) = \left[\begin{array}{c|cc} A & B_w(I - D_{yw}^T D_{yw}) & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

and $B_w(I - D_{yw}^T D_{yw})$ and D_{yw} are orthogonal.

5.5.5 $D_{zw} = 0$ can be relaxed, but it is not so simple!

5.6 Not So Simplified H_∞ Control Problem Solution

Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

Definitions:

1. $A_c := A - B_u D_{zu}^T C_z$
2. $C_c := (I - D_{zu} D_{zu}^T) C_z$
3. $A_f := A - B_w D_{yw}^T C_y$
4. $B_f := B_w (I - D_{yw}^T D_{yw})$

Assumptions:

1. (A, B_u) is stabilizable and (A, C_y) is detectable
2. $D_{zw} = 0$
3. $D_{yu} = 0$
4. $D_{zu}^T D_{zu} = I$
5. $D_{yw} D_{yw}^T = I$
6. (A_c, C_c) is observable and (A_f, B_f) is controllable.

Theorem: There exists a controller $C(s)$ such that $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ under the above assumptions if and only if all the following conditions are satisfied

1. There exists X symmetric and positive semidefinite stabilizing solution of the Riccati equation

$$A_c^T X + X A_c - X(B_u B_u^T - \gamma^{-2} B_w B_w^T) X + C_c^T C_c = 0$$

2. There exists Y symmetric and positive semidefinite stabilizing solution of the Riccati equation

$$A_f Y + Y A_f^T - Y(C_y^T C_y - \gamma^{-2} C_z^T C_z) Y + B_f B_f^T = 0$$

3. $\max_i |\lambda_i(XY)| < \gamma^2$

If so, all controllers such that $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ are parametrized as the feedback connection of

$$R(s) = \left[\begin{array}{c|cc} A_\infty & -Z_\infty L_\infty & Z_\infty(B_u + \gamma^{-2} Y C_z^T D_{zu}) \\ \hline K_\infty & 0 & I \\ -C_y - \gamma D_{yw} B_w^T X & I & 0 \end{array}\right],$$

where

$$\begin{aligned}
A_\infty &:= A + \gamma^{-2} B_w B_w^T X + B_u K_\infty + Z_\infty L_\infty (C_y + \gamma^{-2} D_{yw} B_w^T X) \\
Z_\infty &:= (I - \gamma^{-2} Y X)^{-1} \\
L_\infty &:= -Y C_y^T - B_w D_{yw}^T \\
K_\infty &:= -B_u^T X - D_{zu}^T C_z
\end{aligned}$$

with $Q(s) \in RH_\infty$ such that $\|Q(s)\| < \gamma$.

5.7 A Proof Using Linear Matrix Inequalities

5.7.1 Preliminaries

Lemma [Schur Complement]: The following statements are equivalent:

1. $Z \succ 0, \quad X - Y^T Z^{-1} Y \succeq 0.$
2. $Z \succ 0, \quad \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succeq 0.$

Lemma [Schur Complement (strict version)]: The following statements are equivalent:

1. $Z \succ 0, \quad X - Y^T Z^{-1} Y \succ 0.$
2. $X \succ 0, \quad Z - Y X^{-1} Y^T \succ 0.$
3. $\begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succ 0.$

Proof: Assume $Z \succ 0$ and

$$\begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \succeq 0 \quad (\succ 0).$$

The nonsingular matrix

$$T = \begin{bmatrix} I & 0 \\ -Z^{-1} Y & I \end{bmatrix}$$

establishes the congruence transformation

$$T^T \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} T = \begin{bmatrix} X - Y^T Z^{-1} Y & 0 \\ 0 & Z \end{bmatrix} \succeq 0 \quad (\succ 0).$$

A similar argument proves 2.

Lemma [Projection]: The following are equivalent statements:

1. There exists X such that

$$Q + B^T X C + C^T X^T B \prec 0$$

2. The following two conditions hold

$$\begin{aligned} B_\perp^T Q B_\perp \prec 0 \quad &\text{or} \quad B^T B \succ 0 \\ C_\perp^T Q C_\perp \prec 0 \quad &\text{or} \quad C^T C \succ 0 \end{aligned}$$

Lemma [Bounded Real]: Let $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a realization of $H(s) \in RH_\infty$. Define

$$R_\gamma := \gamma^2 I - D^T D, \qquad\qquad F_\gamma := A + B R_\gamma^{-1} D^T C$$

The following statements are equivalent:

1. $\|H\| < \gamma$

2. $\overline{\sigma}(D) < \gamma$ and the Hamiltonean matrix

$$Z_\gamma := \begin{bmatrix} -F_\gamma^T & -C^T(I + D R_\gamma^{-1} D^T) C \\ B R_\gamma^{-1} B^T & F_\gamma \end{bmatrix}$$

   has no eigenvalue on the imaginary axis

3. $\overline{\sigma}(D) < \gamma$ and there exists $X \succeq 0$ stabilizing solution to the ARE

$$F_\gamma^T X + X F_\gamma + X B R_\gamma^{-1} B^T X + C^T (I + D R_\gamma^{-1} D^T) C = 0$$

4. $\overline{\sigma}(D) < \gamma$ and there exists $X \succ 0$ solution to the inequality

$$F_\gamma^T X + X F_\gamma + X B R_\gamma^{-1} B^T X + C^T (I + D R_\gamma^{-1} D^T) C \prec 0$$

5. There exists a symmetric positive definite solution X to the Linear Matrix Inequality

$$\begin{bmatrix} A^T X + X A & X B & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \prec 0$$

Proof: Equivalence of 1. to 3. has been shown before. For the equivalence of 3. to 4. define

$$H_\epsilon(s) = \begin{bmatrix} H(s) \\ \epsilon(sI - A)^{-1}B \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ \epsilon I & 0 \end{array}\right].$$

such that form some small enough $\epsilon > 0$

$$\|H(s)\| < \|H_\epsilon(s)\| \leq \|H(s)\| + \epsilon\|(sI - A)^{-1}B\| < \gamma.$$

Note that

$$\begin{bmatrix} C^T & \epsilon I \end{bmatrix} \left(I + \begin{bmatrix} D \\ 0 \end{bmatrix} R_\gamma^{-1} \begin{bmatrix} D^T & 0 \end{bmatrix} \right) \begin{bmatrix} C \\ \epsilon I \end{bmatrix} = C^T(I + DR_\gamma^{-1}D^T)C + \epsilon I.$$

That X is strictly positive is a consequence of the fact that

$$\left(A, \begin{bmatrix} C \\ \epsilon 0 \end{bmatrix} \right)$$

is observable for $\epsilon > 0$.

The equivalence of 4. and 5. comes from using Schur Complement. Indeed

$$\begin{aligned} 0 &\succ F_\gamma^T X + XF_\gamma + XBR_\gamma^{-1}B^TX + C^T(I + DR_\gamma^{-1}D^T)C \\ &= (A + BR_\gamma^{-1}D^TC)^TX + X(A + BR_\gamma^{-1}D^TC) \\ &\qquad + XBR_\gamma^{-1}B^TX + C^T(I + DR_\gamma^{-1}D^T)C \\ &= \underbrace{A^TX + XA + C^TC}_{X} + \underbrace{(XB + C^TD)}_{Y^T} \underbrace{R_\gamma^{-1}}_{Z^{-1}} \underbrace{(B^TX + D^TC)}_{Y} \end{aligned}$$

is equivalent to

$$\begin{aligned} 0 &\succ \begin{bmatrix} A^TX + XA + C^TC & XB + C^TD \\ B^TX + D^TC & -\gamma^2 I + D^TD \end{bmatrix} \\ &= \gamma \left(\underbrace{\begin{bmatrix} A^T(X/\gamma) + (X/\gamma)A & (X/\gamma)B \\ B^T(X/\gamma) & -\gamma I \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} C^T \\ D^T \end{bmatrix}}_{Y^T} \underbrace{\gamma^{-1} I}_{Z^{-1}} \underbrace{\begin{bmatrix} C & D \end{bmatrix}}_{Y} \right). \end{aligned}$$

Using Schur Complement once again after recalling that $\gamma > 0$ and defining $X_\gamma := X/\gamma$ we obtain

$$\begin{bmatrix} A^T X_\gamma + X_\gamma A & X_\gamma B & C^T \\ B^T X_\gamma & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \prec 0$$

which is the expression in 5.

5.7.2 Equivalence with Static Output Feedback

Lemma: Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

There exists a *dynamic output feedback controller*

$$C(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

that stabilizes $H(s)$ and $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exits a *static output feedback controller*

$$\mathcal{K} = \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix}$$

that stabilizes the system $\mathcal{H}(s)$ so that $\|\mathcal{F}_l\left(\mathcal{H}(s), \mathcal{K}\right)\| < \gamma$, where

$$\mathcal{H}(s) = \begin{bmatrix} \mathcal{H}_{zw}(s) & \mathcal{H}_{zu}(s) \\ \mathcal{H}_{yw}(s) & \mathcal{H}_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_w & \mathcal{B}_u \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} & \mathcal{D}_{zu} \\ \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{array}\right]$$

with

$$\left[\begin{array}{c|c|c} \mathcal{A} & \mathcal{B}_w & \mathcal{B}_u \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} & \mathcal{D}_{zu} \\ \hline \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{array}\right] := \left[\begin{array}{cc|c|cc} A & 0 & B_w & B_u & 0 \\ 0 & 0 & 0 & 0 & I \\ \hline C_z & 0 & D_{zw} & D_{zu} & 0 \\ \hline C_y & 0 & D_{yw} & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{array}\right]. \tag{2}$$

Proof: Follows from

$$\begin{aligned} \mathcal{F}_l\left(H(s), C(s)\right) &= \left[\begin{array}{cc|c} A + B_u\hat{D}C_y & B_u\hat{C} & B_w + B_u\hat{D}D_{yw} \\ \hat{B}C_y & \hat{A} & \hat{B}D_{yw} \\ \hline C_z + D_{zu}\hat{D}C_y & D_{zu}\hat{C} & D_{zw} + D_{zu}\hat{D}D_{yw} \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathcal{A} + \mathcal{B}_u\mathcal{K}\mathcal{C}_y & \mathcal{B}_w + \mathcal{B}_u\mathcal{K}\mathcal{D}_{yw} \\ \hline \mathcal{C}_z + \mathcal{D}_{zu}\mathcal{K}\mathcal{C}_y & \mathcal{D}_{zw} + \mathcal{D}_{zu}\mathcal{K}\mathcal{D}_{yw} \end{array}\right] \\ &= \mathcal{F}_l\left(\mathcal{H}(s), \mathcal{K}\right). \end{aligned}$$

Theorem: Let

$$\mathcal{H}(s) = \begin{bmatrix} \mathcal{H}_{zw}(s) & \mathcal{H}_{zu}(s) \\ \mathcal{H}_{yw}(s) & \mathcal{H}_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_w & \mathcal{B}_u \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} & \mathcal{D}_{zu} \\ \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{array}\right].$$

There exists a static output feedback controller $\mathcal{K}$ that stabilizes $\mathcal{H}(s)$ and keeps $\|\mathcal{F}_l(\mathcal{H}(s), \mathcal{K})\| < \gamma$ if and only if there exists $\mathcal{X} \succ 0$ such that

$$\begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T\mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp \prec 0$$

$$\begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{Y}\mathcal{A}^T + \mathcal{A}\mathcal{Y} & \mathcal{B}_w & \mathcal{Y}\mathcal{C}_z^T \\ \mathcal{B}_w^T & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z\mathcal{Y} & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp \prec 0$$

with $\mathcal{Y} = \mathcal{X}^{-1}$.

Proof: Define

$$\left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right] := \left[\begin{array}{c|c} \mathcal{A} + \mathcal{B}_u\mathcal{K}\mathcal{C}_y & \mathcal{B}_w + \mathcal{B}_u\mathcal{K}\mathcal{D}_{yw} \\ \hline \mathcal{C}_z + \mathcal{D}_{zu}\mathcal{K}\mathcal{C}_y & \mathcal{D}_{zw} + \mathcal{D}_{zu}\mathcal{K}\mathcal{D}_{yw} \end{array}\right]$$

Now use item 4. of the Bounded Real Lemma to establish that

$$\|\mathcal{F}_l(\mathcal{H}(s), \mathcal{K})\| < \gamma$$

if and only if there exits $\mathcal{X} \succ 0$ such that

$$\begin{aligned} 0 &\succ \begin{bmatrix} \mathsf{A}^T\mathcal{X} + \mathcal{X}\mathsf{A} & \mathcal{X}\mathsf{B} & \mathsf{C}^T \\ \mathsf{B}^T\mathcal{X} & -\gamma I & \mathsf{D}^T \\ \mathsf{C} & \mathsf{D} & -\gamma I \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T\mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix}}_{Q} \\ &\quad + \underbrace{\begin{bmatrix} \mathcal{X}\mathcal{B}_u \\ 0 \\ \mathcal{D}_{zu} \end{bmatrix}}_{B^T} \underbrace{\mathcal{K}}_{X} \underbrace{\begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}}_{C} + \underbrace{\begin{bmatrix} \mathcal{C}_y^T \\ \mathcal{D}_{yw}^T \\ 0 \end{bmatrix}}_{C^T} \underbrace{\mathcal{K}^T}_{X^T} \underbrace{\begin{bmatrix} \mathcal{B}_u^T\mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix}}_{B} \end{aligned}$$

The first inequality in the Theorem comes from direct application of the Projection Lemma to the above inequality. For the second inequality observe that

$$\begin{bmatrix} \mathcal{B}_u^T \mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix} = \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix} \begin{bmatrix} \mathcal{X} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

so that

$$\begin{bmatrix} \mathcal{B}_u^T \mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp = \begin{bmatrix} \mathcal{X}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp$$

The second inequality of the projection lemma yields

$$\begin{aligned}
0 \succ {} & \begin{bmatrix} \mathcal{B}_u^T \mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{A}^T \mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T \mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T \mathcal{X} & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp \\
= {} & \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \\
& \times \begin{bmatrix} \mathcal{X}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{A}^T \mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T \mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{X}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp \\
= {} & \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{X}^{-1}\mathcal{A}^T + \mathcal{A}\mathcal{X}^{-1} & \mathcal{B}_w & \mathcal{X}^{-1}\mathcal{C}_z^T \\ \mathcal{B}_w^T & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z \mathcal{X}^{-1} & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp
\end{aligned}$$

which is the second inequality listed in the theorem after defining

$$\mathcal{Y} := \mathcal{X}^{-1}.$$

Lemma [Matrix Completion]: Let $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$ be given such that $X \succ 0$ and $Y \succ 0$ and $X \succeq Y^{-1}$ with $\mathrm{rank}(X - Y^{-1}) = m \leq n$. There exists matrices $U, V \in \mathbb{R}^{n\times m}$ and $\hat{X}, \hat{Y} \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} X & U \\ U^T & \hat{X} \end{bmatrix}^{-1} = \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix} \succ 0.$$

Proof: The proof is constructive. From Schur Complement, for positiveness we need

$$\hat{X} \succ U^T X^{-1} U$$

since $X \succ 0$. One such $\hat{X}$ is

$$\hat{X} = U^T X^{-1} U + I.$$

Using the formula for the inverse of a 2-by-2 block matrix we have

$$\begin{bmatrix} X & U \\ U^T & \hat{X} \end{bmatrix}^{-1} = \begin{bmatrix} X^{-1} + X^{-1} U E^{-1} U^T X^{-1} & -X^{-1} U E^{-1} \\ -E^{-1} U^T X^{-1} & E^{-1} \end{bmatrix}$$

where

$$E = \hat{X} - U^T X^{-1} U = I.$$

That is,

$$\begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix} = \begin{bmatrix} X^{-1} + X^{-1} U U^T X^{-1} & -X^{-1} U \\ -U^T X^{-1} & I \end{bmatrix}$$

which implies

$$\hat{Y} = I, \qquad\qquad V = -X^{-1} U,$$

for any U satisfying

$$Y = X^{-1} + X^{-1} U U^T X^{-1}.$$

Note that this equation is always solvable since

$$U U^T = X(Y - X^{-1})X$$

and the right hand side has rank m.

Theorem: Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

There exists a dynamic output feedback controller $C(s)$ of order n that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exists symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$ satisfying the linear matrix inequalities

$$\begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_w & C_z^T \\ B_w^T X & -\gamma I & D_{zw}^T \\ C_z & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_z^T & B_w \\ C_z Y & -\gamma I & D_{zw} \\ B_w^T & D_{zw}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0.$$

Proof: Start by substituting $\mathcal{A}$, $\mathcal{B}_w$, $\mathcal{B}_u$, $\mathcal{C}_z$, $\mathcal{D}_{zw}$, $\mathcal{D}_{zu}$, $\mathcal{C}_y$, $\mathcal{D}_{yw}$ by the augmented system matrices from (2). Partition

$$\mathcal{X} = \begin{bmatrix} X & U \\ U^T & \hat{X} \end{bmatrix}, \qquad \mathcal{Y} = \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix},$$

and compute

$$\left[\begin{array}{c|c|c} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \hline \mathcal{B}_w^T\mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{array}\right] = \left[\begin{array}{cc|c|c} A^T X + XA & A^T U & XB_w & C_z^T \\ U^T A & 0 & U^T B_w & 0 \\ \hline B_w^T X & B_w U & -\gamma I & D_{zw}^T \\ \hline C_z & 0 & D_{zw} & -\gamma I \end{array}\right],$$

$$\left[\begin{array}{c|c|c} \mathcal{Y}\mathcal{A}^T + \mathcal{A}\mathcal{Y} & \mathcal{B}_w & \mathcal{Y}\mathcal{C}_z^T \\ \hline \mathcal{B}_w^T & -\gamma I & \mathcal{D}_{zw}^T \\ \hline \mathcal{C}_z\mathcal{Y} & \mathcal{D}_{zw} & -\gamma I \end{array}\right] = \left[\begin{array}{cc|c|c} AY + YA^T & AV & B_w & YC_z^T \\ V^T A & 0 & 0 & V^T C_z \\ \hline B_w^T & 0 & -\gamma I & D_{zw}^T \\ \hline C_z Y & C_z V & D_{zw} & -\gamma I \end{array}\right].$$

Note that

$$\begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix} = \begin{bmatrix} C_y & 0 & D_{yw} & 0 \\ 0 & I & 0 & 0 \end{bmatrix} = \begin{bmatrix} C_y & D_{yw} & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

so that

$$\begin{aligned} \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} C_y & D_{yw} & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}_\perp \\ &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

Using the above expressions, the first inequality of the previous theorem become

$$
\begin{aligned}
0 \succ & \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_w & \mathcal{C}_z^T \\ \mathcal{B}_w^T\mathcal{X} & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix}_\perp \\
&= \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}^T \\
&\qquad \times \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} A^TX + XA & A^TU & XB_w & C_z^T \\ U^TA & 0 & U^TB_w & 0 \\ B_w^TX & B_wU & -\gamma I & D_{zw}^T \\ C_z & 0 & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \\
&\qquad\qquad \times \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^TX + XA & XB_w & A^TU & C_z^T \\ B_w^TX & -\gamma I & B_wU & D_{zw}^T \\ U^TA & U^TB_w & 0 & 0 \\ C_z & D_{zw} & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^TX + XA & XB_w & C_z^T \\ B_w^TX & -\gamma I & D_{zw}^T \\ C_z & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}
\end{aligned}
$$

which is the first inequality in this theorem.
Similarly

$$
\begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp = \begin{bmatrix} B_u^T & 0 & 0 & D_{zu}^T \\ 0 & I & 0 & 0 \end{bmatrix}_\perp = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}
$$

and

$$
\begin{aligned}
0 \succ \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp^T \begin{bmatrix} \mathcal{A}\mathcal{Y} + \mathcal{Y}\mathcal{A}^T & \mathcal{B}_w & \mathcal{Y}\mathcal{C}_z^T \\ \mathcal{B}_w^T & -\gamma I & \mathcal{D}_{zw}^T \\ \mathcal{C}_z\mathcal{Y} & \mathcal{D}_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{B}_u^T & 0 & \mathcal{D}_{zu}^T \end{bmatrix}_\perp \\
= \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_z^T & B_w \\ C_zY & -\gamma I & D_{zw} \\ B_w^T & D_{zw}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}
\end{aligned}
$$

which is the second inequality in this theorem.
In order to complete the proof we have to establish that $\mathcal{Y}$ is the inverse of $\mathcal{X} \succ 0$. First note that from the formula of the inverse of a 2-by-2 block matrix we have

$$Y = (X - U\hat{X}^{-1}U^T)^{-1}$$

where $X \succ 0$, $\hat{X} \succ 0$ and $Y \succ 0$ since $\mathcal{X} \succ 0$ and $\mathcal{Y} \succ 0$. Therefore

$$X = Y^{-1} + U\hat{X}^{-1}U^T \succeq Y^{-1}$$

It is possible to show that if the controller has order equal to n then U can be assumed to be nonsingular so that the above inequality is strict since $\hat{X} \succ 0$.
The Schur Complement of the above strict inequality is

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0$$

which is the third inequality in this theorem, and for which we have established necessity.
Now for sufficiency, note that only the sub-blocks X and Y appear on the inequalities and that for some feasible $X \succ 0$ and $Y \succ 0$ we are free to choose U, V, $\hat{X}$ and $\hat{Y}$. We can therefore use the Matrix Completion Lemma with $\operatorname{rank}(X - Y^{-1}) = n$ to conclude on the existence of such matrices.

Corollary: Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

There exists a dynamic output feedback controller $C(s)$ of order $n_c \leq n$ that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\,(H(s), C(s))\,\| < \gamma$ if and only if there exists symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$ satisfying the linear matrix inequalities

$$\begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_w & C_z^T \\ B_w^T X & -\gamma I & D_{zw}^T \\ C_z & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_z^T & B_w \\ C_z Y & -\gamma I & D_{zw} \\ B_w^T & D_{zw}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0, \qquad X \succ 0, \qquad Y \succ 0.$$

and $\operatorname{rank}(X - Y^{-1}) = n_c$.

Proof: Necessity is as in the proof of the previous theorem, and sufficiency through the Matrix Completion Lemma with $\operatorname{rank}(X - Y^{-1}) = n_c$.

Theorem: Assume

1. $D_{zw} = 0$,
2. $D_{yu} = 0$,
3. $D_{zu}^T C_z = 0$ and $D_{zu}^T D_{zu} = I$,
4. $B_w D_{yw}^T = 0$ and $D_{yw} D_{yw}^T = I$.

There exists a controller $C(s)$ of order n that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exists symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$ satisfying the linear matrix inequalities

$$\begin{bmatrix} A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z & X B_w (D_{yw})_\perp \\ (D_{yw})_\perp^T B_w^T X & -\gamma (D_{yw})_\perp^T (D_{yw})_\perp \end{bmatrix} \prec 0,$$
$$\begin{bmatrix} AY + YA^T - \gamma C_y^T C_y + \gamma^{-1} B_w B_w^T & Y C_z^T (D_{zu}^T)_\perp \\ (D_{zu}^T)_\perp^T C_z Y & -\gamma (D_{zu}^T)_\perp^T (D_{zu}^T)_\perp \end{bmatrix} \prec 0,$$
$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0.$$

Proof: If $D_{yw} D_{yw}^T = I$ and $D_{zu}^T D_{zu} = I$ then

$$\begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp = \begin{bmatrix} I & 0 \\ -D_{yw}^T C_y & (D_{yw})_\perp \end{bmatrix}, \quad \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp = \begin{bmatrix} I & 0 \\ -D_{zu} B_u^T & (D_{zu}^T)_\perp \end{bmatrix}.$$

Consequently

$$\begin{aligned} 0 &\succ \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & X B_w & C_z^T \\ B_w^T X & -\gamma I & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} C_y & D_{yw} \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & 0 \\ -D_{yw}^T C_y & (D_{yw})_\perp & 0 \\ 0 & 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & X B_w & C_z^T \\ B_w^T X & -\gamma I & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -D_{yw}^T C_y & (D_{yw})_\perp & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A^T X + XA - \gamma C_y^T C_y & X B_w (D_{yw})_\perp & C_z^T \\ (D_{yw})_\perp^T B_w^T X & -\gamma (D_{yw})_\perp^T (D_{yw})_\perp & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} \end{aligned}$$

where we have used the fact that $D_{zw} = 0$, $B_w D_{yw}^T = 0$, and $D_{yw} D_{yw}^T = I$. Applying Schur Complement on the last inequality we obtain the equivalent inequality

$$\begin{bmatrix} A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z & X B_w (D_{yw})_\perp \\ (D_{yw})_\perp^T B_w^T X & -\gamma (D_{yw})_\perp^T (D_{yw})_\perp \end{bmatrix} \prec 0$$

which is the first inequality in the theorem.
Similarly for the second inequality

$$\begin{aligned} 0 &\prec \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_z^T & B_w \\ C_z Y & -\gamma I & 0 \\ B_w^T & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} AY + YA^T - \gamma C_y^T C_y & Y C_z^T (D_{zu}^T)_\perp & B_w \\ (D_{zu}^T)_\perp^T C_z Y & -\gamma (D_{zu}^T)_\perp^T (D_{zu}^T)_\perp & 0 \\ B_w^T & 0 & -\gamma I \end{bmatrix} \end{aligned}$$

Using Schur Complement

$$\begin{bmatrix} AY + YA^T - \gamma C_y^T C_y + \gamma^{-1} B_w B_w^T & Y C_z^T (D_{zu}^T)_\perp \\ (D_{zu}^T)_\perp^T C_z Y & -\gamma (D_{zu}^T)_\perp^T (D_{zu}^T)_\perp \end{bmatrix} \prec 0$$

which is the second inequality in the theorem.

Corollary: Assume

1. $D_{zw} = 0$,
2. $D_{yu} = 0$,
3. $D_{zu}^T C_z = 0$ and $D_{zu}^T D_{zu} = I$,
4. $B_w D_{yw}^T = 0$ and $D_{yw} D_{yw}^T = I$.

There exists a controller $C(s)$ of order n that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exists symmetric matrices $X, Y \in \mathbb{R}^{n \times n}$, $X \succ 0$, $Y \succ 0$ satisfying the Riccati inequalities

$$\begin{aligned} A^T X + XA - X\left(C_y^T C_y - \gamma^{-2} B_w B_w^T\right) X + C_z^T C_z \prec 0, \\ AY + YA^T - Y\left(C_y^T C_y - \gamma^{-2} C_z^T C_z\right) Y + B_w B_w^T \prec 0, \end{aligned}$$

with $\max_i |\lambda_i(XY)| < \gamma^2$.

Proof: We can always choose $(D_{yw})_\perp$ such that

$$(D_{yw})_\perp^T (D_{yw})_\perp = I$$

in which case the first inequality in the previous theorem simplifies to

$$\begin{bmatrix} A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z & XB_w (D_{yw})_\perp \\ (D_{yw})_\perp^T B_w^T X & -\gamma I \end{bmatrix} \prec 0$$

Applying Schur Complement and the fact that $B_w D_{yw}^T = 0$ we obtain the equivalent inequality

$$\begin{aligned} 0 &\succ A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z + \gamma^{-1} XB_w (D_{yw})_\perp (D_{yw})_\perp^T B_w^T X \\ &= A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z + \gamma^{-1} XB_w \begin{bmatrix} D_{yw}^T & (D_{yw})_\perp \end{bmatrix} \begin{bmatrix} D_{yw} \\ (D_{yw})_\perp^T \end{bmatrix} B_w^T X \\ &= A^T X + XA - \gamma C_y^T C_y + \gamma^{-1} C_z^T C_z + \gamma^{-1} XB_w B_w^T X \end{aligned}$$

The last inequality is a consequence of

$$\begin{bmatrix} D_{yw}^T & (D_{yw})_\perp \end{bmatrix} \begin{bmatrix} D_{yw} \\ (D_{yw})_\perp^T \end{bmatrix} = \begin{bmatrix} D_{yw} \\ (D_{yw})_\perp^T \end{bmatrix} \begin{bmatrix} D_{yw}^T & (D_{yw})_\perp \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Now defining $X_\gamma := \gamma^{-1} X$ we have

$$0 \succ \gamma \left(A^T X_\gamma + X_\gamma A - C_y^T C_y + \gamma^{-2} C_z^T C_z + X_\gamma B_w B_w^T X_\gamma \right).$$

Dividing by $\gamma > 0$ and multiplying on the left and on the right by $\tilde{Y} := X_\gamma^{-1}$ we obtain

$$A\tilde{Y} + \tilde{Y} A^T - \tilde{Y} \left(C_y^T C_y - \gamma^{-2} C_z^T C_z \right) \tilde{Y} + B_w B_w^T \prec 0$$

which is the second inequality in the corollary.
Similarly for the first inequality in the previous theorem we can choose

$$(D_{zu}^T)_\perp^T (D_{zu}^T)_\perp = I$$

so that it becomes equivalent to

$$AY + YA^T - \gamma C_y^T C_y + \gamma^{-1} B_w B_w^T + \gamma^{-1} Y C_z^T (D_{zu}^T)_\perp (D_{zu}^T)_\perp^T C_z Y \prec 0$$

Using the fact that $D_{zu}^T C_z = 0$ and defining $Y_\gamma := \gamma^{-1} Y$ we obtain

$$\begin{aligned} 0 &\succ AY + YA^T - \gamma C_y^T C_y + \gamma^{-1} B_w B_w^T + \gamma^{-1} Y C_z^T C_z Y, \\ &= \gamma \left(A Y_\gamma + Y_\gamma A^T - C_y^T C_y + \gamma^{-2} B_w B_w^T + Y_\gamma C_z^T C_z Y_\gamma \right). \end{aligned}$$

Dividing by $\gamma > 0$ and multiplying on the left and on the right by $\tilde{X} := Y_\gamma^{-1}$ we obtain

$$A^T \tilde{X} + \tilde{X} A - \tilde{X} \left(C_y^T C_y - \gamma^{-2} B_w B_w^T \right) \tilde{X} + C_z^T C_z \prec 0$$

which is the first inequality in the corollary.
The third inequality in the previous theorem is

$$\begin{bmatrix} \gamma \tilde{Y}^{-1} & I \\ I & \gamma \tilde{X}^{-1} \end{bmatrix} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0$$

which, using Schur Complement, is equivalent to

$$\gamma^2 \tilde{Y}^{-1} \succ \tilde{X}$$

or after multiplication by $\tilde{Y}^{1/2}$ on the left or on the right

$$\tilde{Y}^{1/2} \tilde{X} \tilde{Y}^{1/2} \prec \gamma^2 I$$

which means

$$\max_i \lambda_i(\tilde{Y}^{1/2}\tilde{X}\tilde{Y}^{1/2}) < \gamma^2.$$

After noting that

$$\tilde{X}\tilde{Y}^{1/2} = \tilde{Y}^{-1/2}\left(\tilde{Y}^{1/2}\tilde{X}\tilde{Y}^{1/2}\right)\tilde{Y}^{1/2} \qquad \sim \qquad \tilde{Y}^{1/2}\tilde{X}\tilde{Y}^{1/2}$$

we have

$$\max_i |\lambda_i(\tilde{X}\tilde{Y})| = \max_i \lambda_i(\tilde{Y}^{1/2}\tilde{X}\tilde{Y}^{1/2}) < \gamma^2.$$

Corollary: Assume

1. (A, B_u) is stabilizable and (A, C_y) is detectable.
2. $D_{zw} = 0$,
3. $D_{yu} = 0$,
4. $D_{zu}^T C_z = 0$ and $D_{zu}^T D_{zu} = I$,
5. $B_w D_{yw}^T = 0$ and $D_{yw} D_{yw}^T = I$.
6. (A, B_w) is controllable and (A, C_z) is observable.

There exists a controller $C(s)$ of order n that stabilizes $H(s)$ and keeps $\|\mathcal{F}_l\left(H(s), C(s)\right)\| < \gamma$ if and only if there exists $X \succeq 0$, $Y \succeq 0$ stabilizing solutions of the Riccati equations

$$
\begin{aligned}
A^T X + XA - X\left(C_y^T C_y - \gamma^{-2} B_w B_w^T\right) X + C_z^T C_z = 0,\\
AY + YA^T - Y\left(C_y^T C_y - \gamma^{-2} C_z^T C_z\right) Y + B_w B_w^T = 0,
\end{aligned}
$$

with $\max_i |\lambda_i(XY)| < \gamma^2$.

Proof: The extra assumptions are sufficient conditions for the existence of solutions to the Ricatti equations, which, through an argument similar to the one used in the proof of the Bounded Real Lemma, are equivalent to the Riccati inequalities of the previous corollary.

6 Applications of H_∞ Control

6.1 A Simple Frequency Domain Design

Let

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right].$$

Assume that the closed loop transfer $\mathcal{F}_l\left(H(s), C(s)\right)$ is SISO.

> **Frequency Domain Control Problem:** Find a controller $C(s)$ such that $\mathcal{F}_l\left(H(s), C(s)\right)$ is asymptotically stable and
>
> $$\left|\mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right| < \phi(\omega), \qquad \text{for all } \omega$$
>
> for some given frequency domain specification $\phi(\omega)$.

Let $W(s) \in RH_\infty$ be such that $W^{-1}(s) \in RH_\infty$ and

$$\left|W^{-1}(j\omega)\right| = \phi(\omega), \qquad \forall\, \omega$$

Then the above problem is solved if

$$\left|\mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right| < \left|W^{-1}(j\omega)\right| = \phi(\omega), \qquad \text{for all } \omega$$

or, equivalently,

$$\left|W(j\omega)\right| \left|\mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right| = \left|W(j\omega)\, \mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right| < 1 \qquad \text{for all } \omega.$$

The function $W(s)$ is called a *scaling* or *shapping* function. This specification can be translated as the following H_∞ control problem.

> H_∞ **Control Problem:** Given a scaling function $W(s)$ find a controller $C(s)$ such that be such that $\mathcal{F}_l\left(H(s), C(s)\right)$ is asymptotically stable and
>
> $$\left\|W(s)\, \mathcal{F}_l\left(H(j\omega), C(j\omega)\right)\right\| < 1.$$

Note that

$$
\begin{aligned}
&W(s)\mathcal{F}_l\left(H(s), C(s)\right) \\
&\qquad\qquad = W(s)\left\{H_{zw}(s) + H_{zu}(s)C(s)\left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s)\right\} \\
&\qquad\qquad = W(s)H_{zw}(s) + W(s)H_{zu}(s)C(s)\left[I - H_{yu}(s)C(s)\right]^{-1} H_{yw}(s) \\
&\qquad\qquad = \mathcal{F}_l\left(\begin{bmatrix} W(s) & 0 \\ 0 & I \end{bmatrix} H(s), C(s)\right)
\end{aligned}
$$

so that the scaling is applied in the open loop plant $H(s)$.

6.2 Example #1

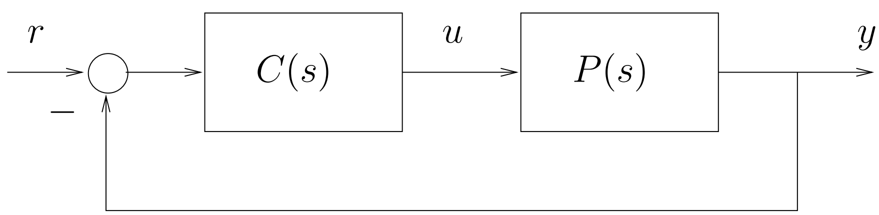


Figure 6: Block diagram

Plant:

$$P(s) = \frac{1}{(s+2)(s-1)}$$

Control objective:
Have a low pass transfer function from r to y. Say, below

$$\phi(\omega) = \frac{1}{\sqrt{(\omega/\omega_b)^2 + 1}}$$

for some $\omega_b > 0$.

Scaling design:
Set

$$W^{-1}(s) = \frac{1}{s/\omega_b + 1}, \qquad\qquad W(s) = s/\omega_b + 1.$$

Problem is $W(s) \notin RH_\infty$! General solution is to introduce a pole at 'minus infinity', that is

$$W^{-1}(s) = \frac{\epsilon s + 1}{s/\omega_b + 1}, \qquad\qquad W(s) = \frac{s/\omega_b + 1}{\epsilon s + 1}.$$

Then augment the plant with the scaling.

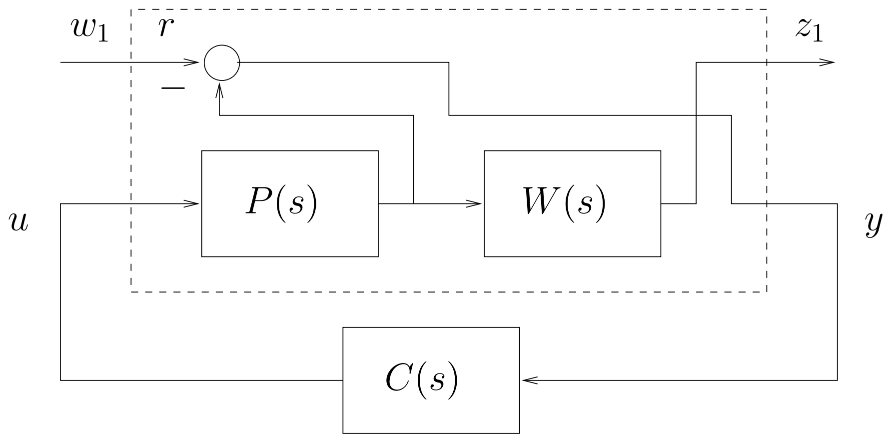


Figure 7: Plant augmented with scaling

The above system does not satisfy the assumptions on the H_∞ control problem for two reasons: $B_w = 0$[3] and $D_{zu} = 0$[4]. This is so because w_1 never gets into the plant before going through the controller and $P(s)$ is strictly proper. In order to overcome that, introduce an extra input w_2 and an extra output $z_2 = \epsilon u$, with $\epsilon > 0$ small, as in the diagram:

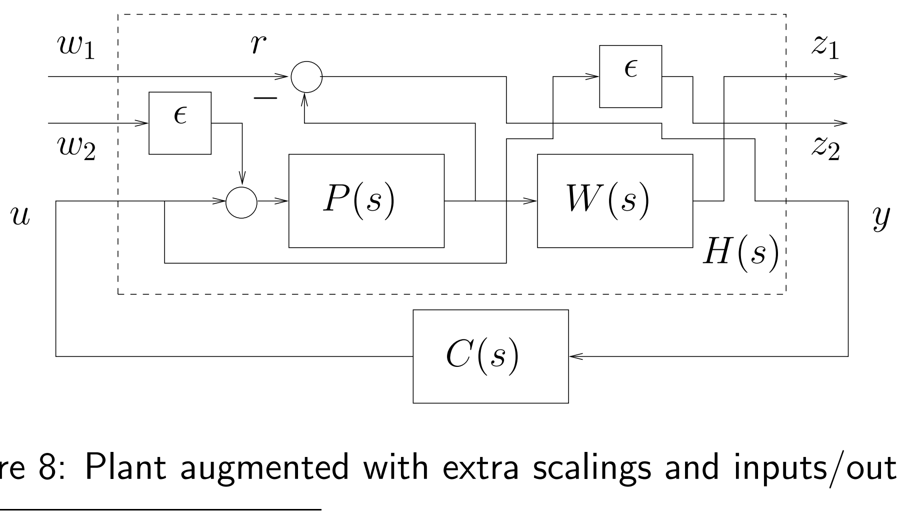


Figure 8: Plant augmented with extra scalings and inputs/outputs

[3] (A, B_w) is not controllable.
[4] $D_{zu}^T D_{zu} \neq 1$. Worse then that, D_{zu} is not full rank.

H_∞ **control problem:** Set $\omega_b = 1$ and find $C(s)$ such that

$$\|\mathcal{F}_l\,(H(s), C(s))\| < 1$$

State space realization:

```
% MAE 284 - Robust and Multivariable Control
% Mauricio de Oliveira
% April 2006
%
% Example #1
% Frequency domain design
% Low pass cutoff frequency
omega_b = 1;

% epsilon
epsilon = 1e-4;

% Plant:
P = tf(1, conv([1 2],[1 -1]))

Transfer function:
     1
-----------
s^2 + s - 2


% Scaling:
S = tf([1/omega_b 1],[epsilon 1])

Transfer function:
   s + 1
------------
0.0001 s + 1


% Convert to state-space
[Ap, Bp, Cp, Dp] = ssdata(P);
[As, Bs, Cs, Ds] = ssdata(S);

% Generalized plant H
A = [As Bs*Cp; zeros(size(Cp'*Bs')) Ap];
Bw1 = zeros(3,1);
Bu = [Bs*Dp; Bp];

Cy = -[zeros(1,1) Cp];
Dyw1 = 1;

Cz1 = [Cs Ds*Cp];
Dzu1 = [Ds*Dp];

% Extra input w2
```

```
Bw2 = epsilon*Bu;
Dyw2 = 0;

% Extra output penalizing control input
Cz2 = zeros(1,3);
Dzu2 = epsilon;

% Augmented plant
Cz = [Cz1; Cz2];
Dzu = [Dzu1; Dzu2];

Bw = [Bw1 epsilon*Bu];
Dyw = [Dyw1 Dyw2];

% Show matrices
A
A =
   1.0e+04 *
   -1.0000         0    0.4096
         0   -0.0001    0.0001
         0    0.0004         0
Bu
Bu =
         0
    0.5000
         0
Bw
Bw =
   1.0e-04 *
         0         0
         0    0.5000
         0         0
Cy
Cy =
         0         0   -0.5000
Dyw
Dyw =
     1     0
Cz
Cz =
   1.0e+04 *
   -1.2206         0    0.5000
         0         0         0
Dzu
Dzu =
   1.0e-04 *
         0
    1.0000

diary off
```

Assumptions:

```
% Checking assumptions
% Stabilizability and detectability
rank(ctrb(A,Bu))
ans =
     3
rank(obsv(A,Cy))  % not observable but detectable
ans =
     2

% Normality
Dzu' * Dzu
ans =
   1.0000e-08
Dyw * Dyw'
ans =
     1

% Enforcing Dyw * Dyw' = 1
Bu = Bu / epsilon
Bu =
           0
        5000
           0
Dzu = Dzu / epsilon
Dzu =
     0
     1

Dzu' * Dzu
ans =
     1
Dyw * Dyw'
ans =
     1

% Auxiliary matrices
Ac = A - Bu * Dzu' * Cz;
Cc = (eye(size(Dzu,1)) - Dzu*Dzu') * Cz;

Af = A - Bw * Dyw' * Cy;
Bf = Bw * (eye(size(Dyw,2)) - Dyw'*Dyw);

% Technical observability and controllability
rank(obsv(Ac,Cc))
ans =
     3
rank(ctrb(Af,Bf))
ans =
     3

diary off
```

Riccati equations:

```
% Set gamma
gamma = 1;

% Solve control ARE
X = are(Ac, Bu*Bu' - Bw*Bw'/gamma^2, Cc'*Cc);

% Solve filtering ARE
Y = are(Af', Cy'*Cy - Cz'*Cz/gamma^2, Bf*Bf');

% Positivity
mX = min(eig(X))
mX =
   2.5140e-09
mY = min(eig(Y))
mY =
   -3.2815

% Spectral condition
mXY = max(abs(eig(X*Y)))/gamma^2
mXY =
   4.6203e-04

% GAMMA = 1 IS NOT FEASIBLE
diary off
```

Conclusion: $\|\mathcal{F}_l\left(H(s), C(s)\right)\| \nless 1$. Raise γ and try again!

```
% Set gamma
gamma = 2.01;

% Solve control ARE
X = are(Ac, Bu*Bu' - Bw*Bw'/gamma^2, Cc'*Cc);

% Solve filtering ARE
Y = are(Af', Cy'*Cy - Cz'*Cz/gamma^2, Bf*Bf');

% Positivity
mX = min(eig(X))
mX =
   2.5180e-09
mY = min(eig(Y))
mY =
  -5.0831e-14

% Spectral condition
mXY = max(abs(eig(X*Y)))/gamma^2
mXY =
    0.0339

if ( mX > -epsilon^2 && mY > -epsilon^2 && mXY < 1 )

  % Constructing central controller
  Zoo = inv(eye(size(A)) - Y*X/gamma^2);
  Loo = -Y*Cy' - Bw*Dyw';
  Koo = -Bu'*X - Dzu'*Cz;
  Aoo = A + Bw*Bw'*X/gamma^2 + Bu*Koo + Zoo*Loo*(Cy + Dyw*Bw'*X/gamma^2);

  % Build controller (do not forget to scale Cc!)
  C = -ss(Aoo, -Zoo*Loo, Koo/epsilon, zeros(size(Koo*Loo))); zpk(C)

Zero/pole/gain:
   -4090077.8764 (s+2) (s+1e04)
-----------------------------------
(s+205.5) (s^2  + 1.732e04s + 1e08)


  % Build closed loop system
  Acl = [Ap, Bp*C.c; C.b*Cp C.a];
  Bcl = [zeros(2,1); C.b*1];
  Ccl = [Cp, zeros(1,3)];
  Dcl = zeros(size(Ccl*Bcl));
  CL = ss(Acl,Bcl,Ccl,Dcl);

  % Check closed loop stability
  max(real(eig(Acl)))
ans =
   -1.0000

  diary off
```

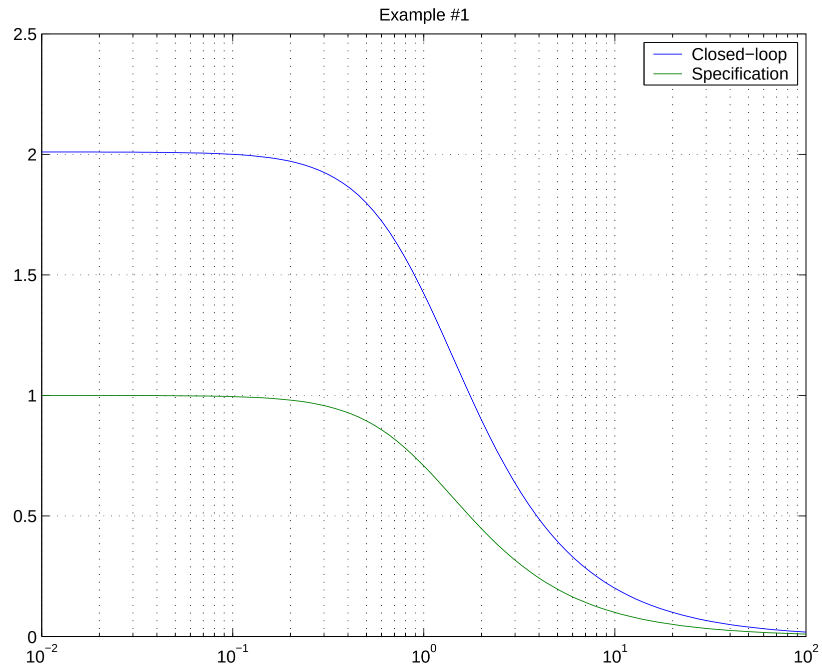


Figure 9: Example #1

6.3 Example #2

Same data as Example #1. Less stringent constraint.

H_∞ **control problem:** Set $\omega_b = 20$ and find $C(s)$ such that

$$\min\{\gamma : \quad \|\mathcal{F}_l\,(H(s), C(s))\| < \gamma\} = 1.06$$

Solution:

```
% MAE 284 - Robust and Multivariable Control
% Mauricio de Oliveira
% April 2006
%
% Example #2
% Frequency domain design
% Low pass cutoff frequency
omega_b = 20;

% epsilon
epsilon = 1e-4;

% Plant:
P = tf(1, conv([1 2],[1 -1]))
```

```
Transfer function:
     1
-----------
s^2 + s - 2


% Scaling:
S = tf([1/omega_b 1],[epsilon 1])

Transfer function:
 0.05 s + 1
------------
0.0001 s + 1


% Convert to state-space
[Ap, Bp, Cp, Dp] = ssdata(P);
[As, Bs, Cs, Ds] = ssdata(S);

% Generalized plant H
A = [As Bs*Cp; zeros(size(Cp'*Bs')) Ap];
Bw1 = zeros(3,1);
Bu = [Bs*Dp; Bp];

Cy = -[zeros(1,1) Cp];
Dyw1 = 1;

Cz1 = [Cs Ds*Cp];
Dzu1 = [Ds*Dp];

% Extra input w2
Bw2 = epsilon*Bu;
Dyw2 = 0;

% Extra output penalizing control input
Cz2 = zeros(1,3);
Dzu2 = epsilon;

% Augmented plant
Cz = [Cz1; Cz2];
Dzu = [Dzu1; Dzu2];

Bw = [Bw1 epsilon*Bu];
Dyw = [Dyw1 Dyw2];

% Show matrices
A
A =
   1.0e+04 *
   -1.0000         0    0.1024
         0   -0.0001    0.0001
         0    0.0004         0
Bu
Bu =
```

```
         0
    0.5000
         0
Bw
Bw =
   1.0e-04 *
         0         0
         0    0.5000
         0         0
Cy
Cy =
         0         0   -0.5000
Dyw
Dyw =
     1     0
Cz
Cz =
   1.0e+03 *
   -2.4365         0    0.2500
         0         0         0
Dzu
Dzu =
   1.0e-04 *
         0
    1.0000

% Checking assumptions
% Stabilizability and detectability
rank(ctrb(A,Bu))
ans =
     3
rank(obsv(A,Cy))  % not observable but detectable
ans =
     2

% Normality
Dzu' * Dzu
ans =
   1.0000e-08
Dyw * Dyw'
ans =
     1

% Enforcing Dyw * Dyw' = 1
Bu = Bu / epsilon
Bu =
           0
        5000
           0
Dzu = Dzu / epsilon
Dzu =
     0
     1
```

```
Dzu' * Dzu
ans =
     1
Dyw * Dyw'
ans =
     1

% Auxiliary matrices
Ac = A - Bu * Dzu' * Cz;
Cc = (eye(size(Dzu,1)) - Dzu*Dzu') * Cz;

Af = A - Bw * Dyw' * Cy;
Bf = Bw * (eye(size(Dyw,2)) - Dyw'*Dyw);

% Technical observability and controllability
rank(obsv(Ac,Cc))
ans =
     3
rank(ctrb(Af,Bf))
ans =
     3

% Set gamma
gamma = 1.06;

% Solve control ARE
X = are(Ac, Bu*Bu' - Bw*Bw'/gamma^2, Cc'*Cc);

% Solve filtering ARE
Y = are(Af', Cy'*Cy - Cz'*Cz/gamma^2, Bf*Bf');

% Positivity
mX = min(eig(X))
mX =
   3.3507e-07
mY = min(eig(Y))
mY =
  -1.3188e-15

% Spectral condition
mXY = max(abs(eig(X*Y)))/gamma^2
mXY =
    0.2134

if ( mX > -epsilon^2 && mY > -epsilon^2 && mXY < 1 )

  % Constructing central controller
  Zoo = inv(eye(size(A)) - Y*X/gamma^2);
  Loo = -Y*Cy' - Bw*Dyw';
  Koo = -Bu'*X - Dzu'*Cz;
  Aoo = A + Bw*Bw'*X/gamma^2 + Bu*Koo + Zoo*Loo*(Cy + Dyw*Bw'*X/gamma^2);

  % Build controller (do not forget to scale Cc!)
  C = -ss(Aoo, -Zoo*Loo, Koo/epsilon, zeros(size(Koo*Loo))); zpk(C)
```

```
Zero/pole/gain:
-1409708.2925 (s+2) (s+1e04)
----------------------------
(s+9987) (s+491.9) (s+162.3)


  % Build closed loop system
  Acl = [Ap, Bp*C.c; C.b*Cp C.a];
  Bcl = [zeros(2,1); C.b*1];
  Ccl = [Cp, zeros(1,3)];
  Dcl = zeros(size(Ccl*Bcl));
  CL = ss(Acl,Bcl,Ccl,Dcl);

  % Check closed loop stability
  max(real(eig(Acl)))
ans =
   -2.0000

  diary off
```

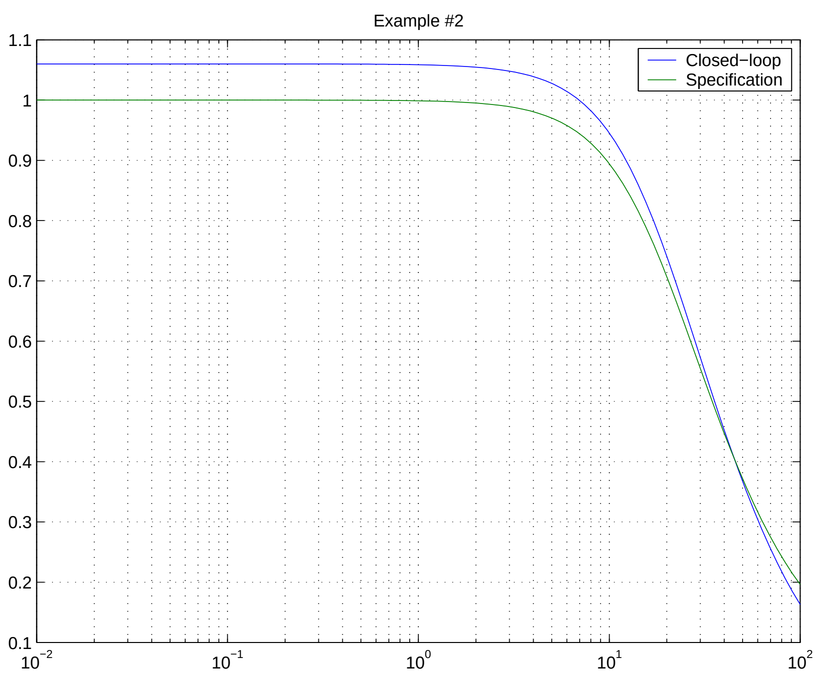


Figure 10: Example #2

6.4 Example #3

Same plant and problem data as Example #1.

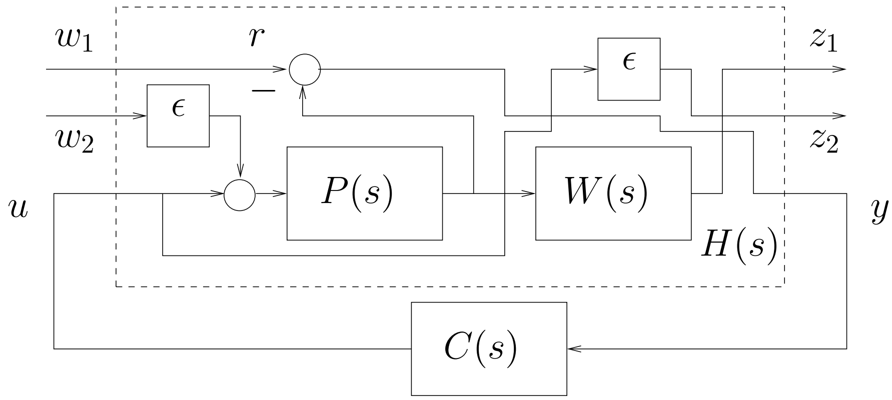


Figure 11: Plant augmented with extra scalings and inputs/outputs

Scaling design:
Set

$$W^{-1}(s) = \frac{1}{s/\omega_b + 1}, \qquad W(s) = s/\omega_b + 1.$$

$W(s) \notin RH_\infty$ but $P(s)$ is strictly proper! That means $W(s)P(s)$ is realizable!

State space realization:
Let

$$P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array}\right].$$

Then

$$W(s)P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p(A_p/\omega_b + I) & C_p B_p/\omega_b \end{array}\right] = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p(A_p/\omega_b + I) & 0 \end{array}\right]$$

Note that $C_p B_p = 0$ because the original system has relative degree 2!

ADVANTAGE: smaller plant, smaller controller!

```
% MAE 284 - Robust and Multivariable Control
% Mauricio de Oliveira
% April 2006
%
% Example #3
% Frequency domain design
% Low pass cutoff frequency
omega_b = 1;

% epsilon
epsilon = 1e-4;

% Plant:
P = tf(1, conv([1 2],[1 -1]))

Transfer function:
     1
-----------
s^2 + s - 2


% Scaling:
S = tf([1/omega_b 1],1)

Transfer function:
s + 1


% Convert to state-space
[Ap, Bp, Cp, Dp] = ssdata(P);

% Generalized plant H
A = Ap;
Bw1 = zeros(2,1);
Bu = Bp;

Cy = -Cp;
Dyw1 = 1;

Cz1 = Cp*(Ap/omega_b + eye(2));
Dzu1 = Cp*Bu
Dzu1 =
     0

% Extra input w2
Bw2 = epsilon*Bu;
Dyw2 = 0;

% Extra output penalizing control input
Cz2 = zeros(1,2);
Dzu2 = epsilon;

% Augmented plant
```

```
Cz = [Cz1; Cz2];
Dzu = [Dzu1; Dzu2];

Bw = [Bw1 epsilon*Bu];
Dyw = [Dyw1 Dyw2];

% Show matrices
A
A =
   -1.0000    0.5000
    4.0000         0
Bu
Bu =
    0.5000
         0
Bw
Bw =
   1.0e-04 *
         0    0.5000
         0         0
Cy
Cy =
         0   -0.5000
Dyw
Dyw =
     1     0
Cz
Cz =
    2.0000    0.5000
         0         0
Dzu
Dzu =
   1.0e-04 *
         0
    1.0000

diary off
```

H_∞ **control problem:** Set $\omega_b = 1$ and find $C(s)$ such that

$$\min\{\gamma : \quad \|\mathcal{F}_l\,(H(s), C(s))\| < \gamma\} = 2.01$$

Assumptions:

```
% Checking assumptions
% Stabilizability and detectability
rank(ctrb(A,Bu))
ans =
     2
rank(obsv(A,Cy))  % This time it is observable!
ans =
     2

% Normality
Dzu' * Dzu
ans =
   1.0000e-08
Dyw * Dyw'
ans =
     1

% Enforcing Dyw * Dyw' = 1
Bu = Bu / epsilon;
Dzu = Dzu / epsilon;

Dzu' * Dzu
ans =
     1
Dyw * Dyw'
ans =
     1

% Auxiliary matrices
Ac = A - Bu * Dzu' * Cz;
Cc = (eye(size(Dzu,1)) - Dzu*Dzu') * Cz;

Af = A - Bw * Dyw' * Cy;
Bf = Bw * (eye(size(Dyw,2)) - Dyw'*Dyw);

% Technical observability and controllability
rank(obsv(Ac,Cc))
ans =
     2
rank(ctrb(Af,Bf))
ans =
     2

diary off
```

Solution:

```
% Set gamma
gamma = 2.01;

% Solve control ARE
X = are(Ac, Bu*Bu' - Bw*Bw'/gamma^2, Cc'*Cc);

% Solve filtering ARE
Y = are(Af', Cy'*Cy - Cz'*Cz/gamma^2, Bf*Bf');

% Positivity
mX = min(eig(X))
mX =
   4.7048e-09
mY = min(eig(Y))
mY =
   5.8822e-10

% Spectral condition
mXY = max(abs(eig(X*Y)))/gamma^2
mXY =
    0.0200

if ( mX > -epsilon^2 && mY > -epsilon^2 && mXY < 1 )

  % Constructing central controller
  Zoo = inv(eye(size(A)) - Y*X/gamma^2);
  Loo = -Y*Cy' - Bw*Dyw';
  Koo = -Bu'*X - Dzu'*Cz;
  Aoo = A + Bw*Bw'*X/gamma^2 + Bu*Koo + Zoo*Loo*(Cy + Dyw*Bw'*X/gamma^2);

  % Build controller (do not forget to scale Cc!)
  C = -ss(Aoo, -Zoo*Loo, Koo/epsilon, zeros(size(Koo*Loo))); zpk(C)

Zero/pole/gain:
-4112493.4645 (s+2)
-------------------
(s+1e04) (s+206.6)


  % Build closed loop system
  Acl = [Ap, Bp*C.c; C.b*Cp C.a];
  Bcl = [zeros(2,1); C.b*1];
  Ccl = [Cp, zeros(1,2)];
  Dcl = zeros(size(Ccl*Bcl));
  CL = ss(Acl,Bcl,Ccl,Dcl);

  % Check closed loop stability
  max(real(eig(Acl)))
ans =
   -1.0000

  diary off
```

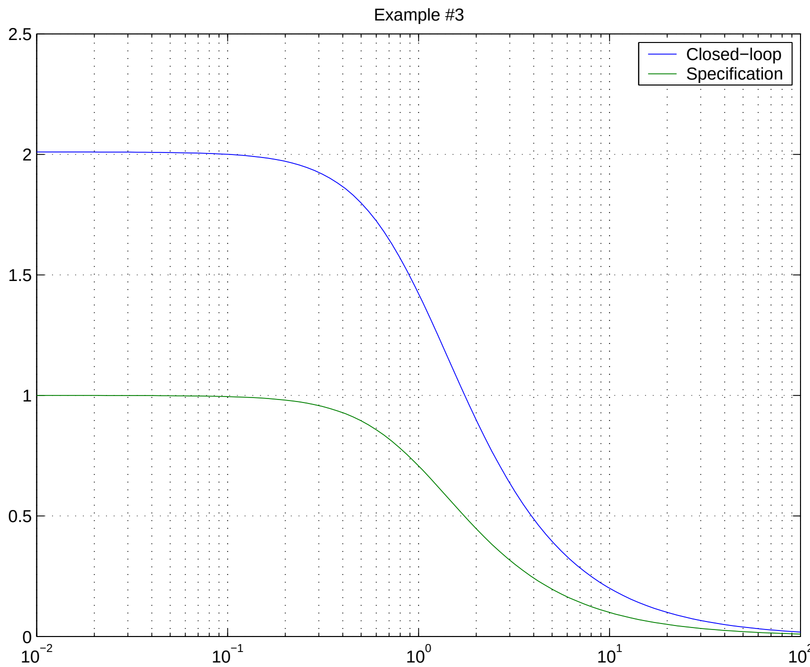


Figure 12: Example #3

6.5 Robust Analysis

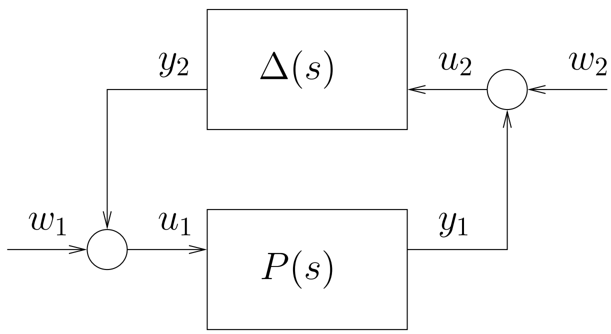


Figure 13: Feedback connection

Theorem [Small Gain]: Let $P(s) \in RH_\infty$. Then the feedback connection if Figure 13 is well posed and internally stable for all $\Delta(s) \in RH_\infty$ such that $\|\Delta(s)\| < 1$ if and only if $\|P(s)\| \leq 1$.

In the proof we will make use of the following technical lemma.

Lemma: $\underline{\sigma}(A + B) \geq \underline{\sigma}(A) - \overline{\sigma}(B)$.

Proof: First note that

$$\frac{\|Ax\| - \|Bx\|}{\|x\|} \geq \frac{\|Ax\|}{\|x\|} - \max_{\|x\| \neq 0} \frac{\|Bx\|}{\|x\|} = \frac{\|Ax\|}{\|x\|} - \overline{\sigma}(B).$$

Consequently

$$\min_{\|x\| \neq 0} \frac{\|Ax\| - \|Bx\|}{\|x\|} \geq \min_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} - \overline{\sigma}(B) = \underline{\sigma}(A) - \overline{\sigma}(B)$$

and

$$\begin{aligned} \underline{\sigma}(A + B) &= \min_{\|x\| \neq 0} \frac{\|Ax + Bx\|}{\|x\|} \\ &\geq \min_{\|x\| \neq 0} \frac{\|Ax\| - \|Bx\|}{\|x\|} \\ &\geq \underline{\sigma}(A) - \overline{\sigma}(B) \end{aligned}$$

Proof of Small Gain Theorem: We prove only sufficiency. Since $P(s) \in RH_\infty$, the feedback connection is asymptotically stable if and only if

$$\Delta(s)[I - P(s)\Delta(s)]^{-1} \in RH_\infty.$$

But as $\Delta(s) \in RH_\infty$ the above condition is equivalent to

$$[I - P(s)\Delta(s)]^{-1} \in RH_\infty,$$

or in other words, $\det[I - P(s)\Delta(s)]$ has no poles in the closed right-half plane. That is

$$\inf_{\mathrm{Re}(s)\geq 0} \underline{\sigma}[I - P(s)\Delta(s)] > 0$$

Using the previous lemma

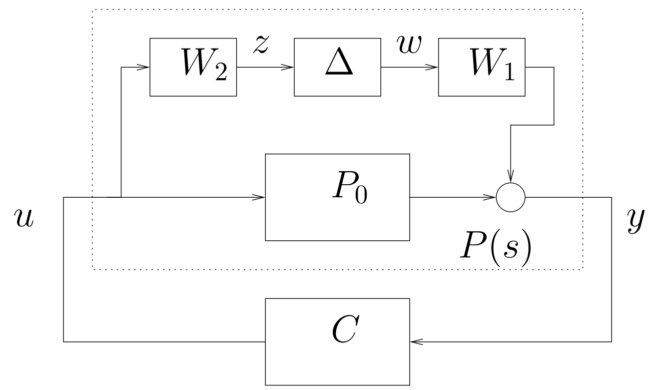
$$\begin{aligned}
\inf_{\mathrm{Re}(s)\geq 0} \underline{\sigma}[I - P(s)\Delta(s)] &\geq \inf_{\mathrm{Re}(s)\geq 0} \{\underline{\sigma}(I) - \overline{\sigma}[P(s)\Delta(s)]\} \\
&= 1 + \inf_{\mathrm{Re}(s)\geq 0} \{-\overline{\sigma}[P(s)\Delta(s)]\} \\
&= 1 - \sup_{\mathrm{Re}(s)\geq 0} \overline{\sigma}[P(s)\Delta(s)] \\
&= 1 - \|P(s)\Delta(s)\|
\end{aligned}$$

Now if $\|\Delta(s)\| < 1$

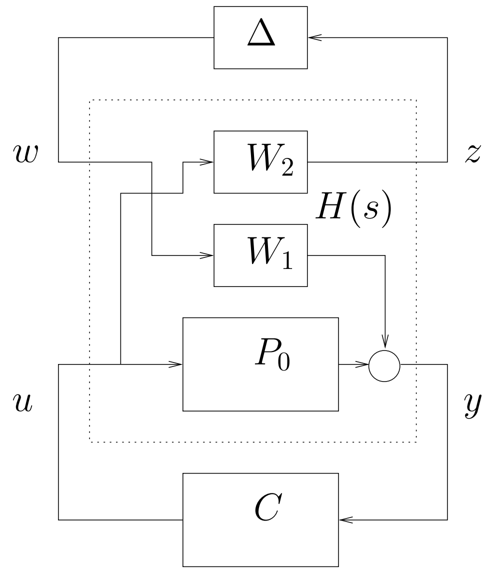
$$\begin{aligned}
\inf_{\mathrm{Re}(s)\geq 0} \underline{\sigma}[I - P(s)\Delta(s)] &\geq 1 - \|P(s)\Delta(s)\| \\
&\geq 1 - \|P(s)\|\,\|\Delta(s)\| \\
&> 1 - \|P(s)\| \geq 0
\end{aligned}$$

A similar argument can show that the connection is well-posed. For a proof of necessity see Zhou p. 138.

6.5.1 Unstructured Additive Uncertainty



(a) Uncertain System



(b) Generalized Plant

Figure 14: Additive Uncertainty

Theorem: The feedback connection in Figure 14(a) where

$$P(s) = P_0(s) + W_1(s)\Delta(s)W_2(s)$$

and the uncertainty $\Delta(s) \in RH_\infty$ is such that $\|\Delta(s)\| < 1$ is robustly stable if and only if

$$\|W_2(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s)\| \leq 1.$$

Proof: Define the input w and output z as in Fig.14(b), where the generalized plant $H(s)$ is given by

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \begin{bmatrix} 0 & W_2(s) \\ W_1(s) & P_0(s) \end{bmatrix}$$

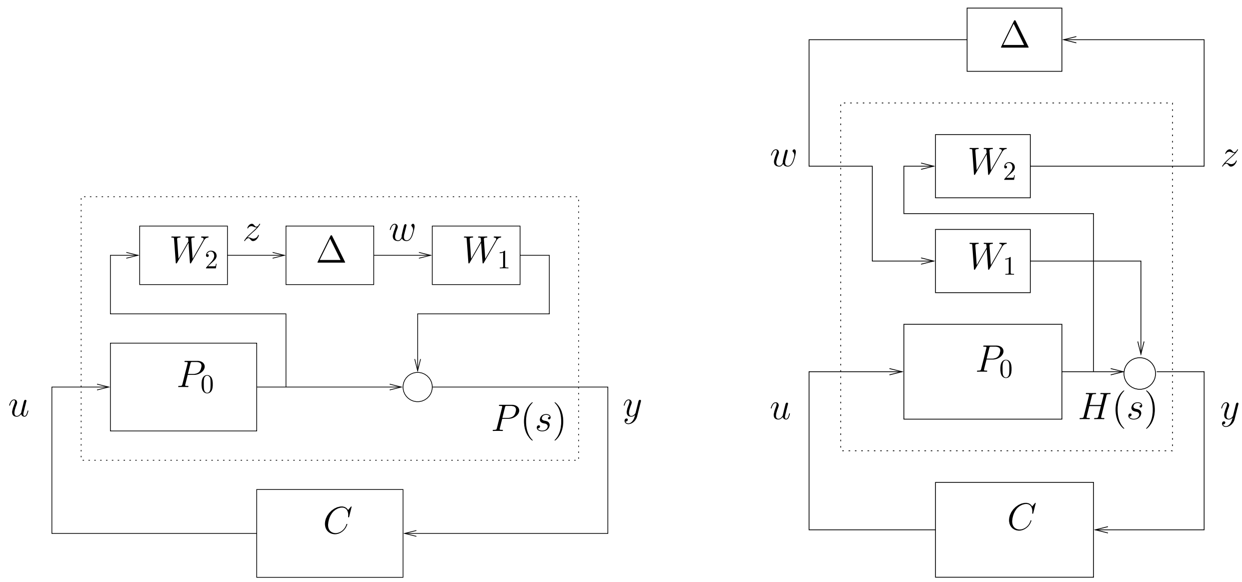
Use the small gain theorem to conclude that the system is robustly stable if and only if

$$\|\mathcal{F}_l\left(H(s), C(s)\right)\| \leq 1$$

where

$$\mathcal{F}_l\left(H(s), C(s)\right) = W_2(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s).$$

6.5.2 Unstructured Multiplicative Uncertainty



(a) Uncertain System

(b) Generalized Plant

Figure 15: Multiplicative Uncertainty

Theorem: The feedback connection in Figure 15(a) where

$$P(s) = [I + W_1(s)\Delta(s)W_2(s)]P_0(s)$$

and the uncertainty $\Delta(s) \in RH_\infty$ is such that $\|\Delta(s)\| < 1$ is robustly stable if and only if

$$\|W_2(s)P_0(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s)\| \leq 1.$$

Proof: Define the input w and output z as in Fig.15(b), where the generalized plant $H(s)$ is given by

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \begin{bmatrix} 0 & W_2(s)P_0(s) \\ W_1(s) & P_0(s) \end{bmatrix}$$

Use the small gain theorem to conclude that the system is robustly stable if and only if

$$\|\mathcal{F}_l\left(H(s), C(s)\right)\| \leq 1$$

where

$$\mathcal{F}_l\left(H(s), C(s)\right) = W_2(s)P_0(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s).$$

6.5.3 Unstructured Fractional Uncertainty

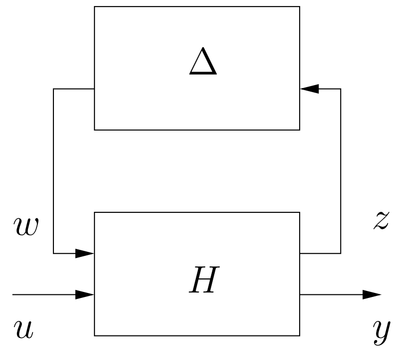


Figure 16: Uncertainty System

Lemma: The uncertain system

$$P(s) = [I + W_1(s)\Delta(s)W_2(s)]^{-1}P_0(s)$$

can be represented as the feedback connection of the plant

$$H(s) = \begin{bmatrix} H_{zw}(s) & H_{zu}(s) \\ H_{yw}(s) & H_{yu}(s) \end{bmatrix} = \begin{bmatrix} -W_2(s)W_1(s) & W_2(s)P_0(s) \\ -W_1(s) & P_0(s) \end{bmatrix}$$

as in Figure 16.

Proof: Use the identity

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

to rewrite

$$\begin{aligned} &[I + W_1(s)\Delta(s)W_2(s)]^{-1}P_0(s) \\ &\qquad = P_0(s) - W_1(s)\Delta(s)[I + W_2(s)W_1(s)\Delta(s)]^{-1}W_2(s)P_0(s). \end{aligned}$$

Equivalently

$$\begin{aligned} &\mathcal{F}_l\left(\begin{bmatrix} H_{yu}(s) & H_{yw}(s) \\ H_{zu}(s) & H_{zw}(s) \end{bmatrix}, \Delta(s)\right) \\ &\qquad = \underbrace{P_0(s)}_{H_{yu}(s)} + \underbrace{[-W_1(s)]}_{H_{yw}(s)}\Delta(s)\{I - \underbrace{[-W_2(s)W_1(s)]}_{H_{zw}(s)}\Delta(s)\}^{-1}\underbrace{W_2(s)P_0(s)}_{H_{zu}(s)} \end{aligned}$$

from which the blocks of $H(s)$ are defined.

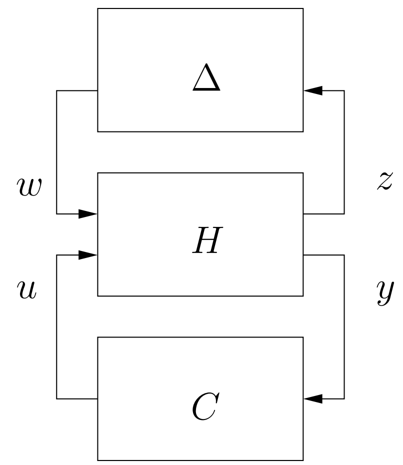


Figure 17: Uncertainty System

Theorem: The feedback connection in Figure 17 where

$$P(s) = [I + W_1(s)\Delta(s)W_2(s)]^{-1}P_0(s)$$

and the uncertainty $\Delta(s) \in RH_\infty$ is such that $\|\Delta(s)\| < 1$ is robustly stable if and only if

$$\|W_2(s)[I - P_0(s)C(s)]^{-1}W_1(s)\| \leq 1.$$

Proof: From the previous lemma and the small gain theorem the system is robustly stable if and only if

$$\|\mathcal{F}_l\left(H(s), C(s)\right)\| \leq 1$$

where

$$\begin{aligned}
\mathcal{F}_l\left(H(s), C(s)\right) &= -W_2(s)W_1(s) - W_2(s)P_0(s)C(s)[I - P_0(s)C(s)]^{-1}W_1(s) \\
&= -W_2(s)\left\{I + P_0(s)C(s)[I - P_0(s)C(s)]^{-1}\right\}W_1(s) \\
&= -W_2(s)[I - P_0(s)C(s)]^{-1}W_1(s),
\end{aligned}$$

after using the same identity used in the proof of the previous lemma.

6.5.4 Structured Uncertainty

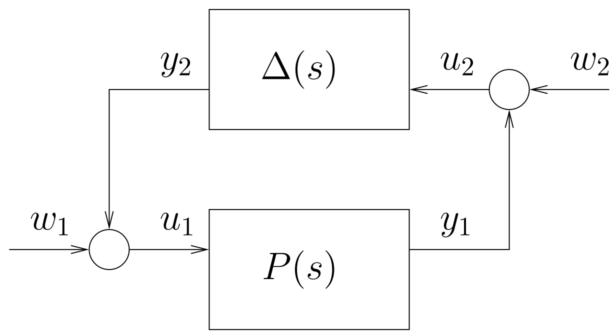


Figure 18: Uncertain System

Structured Uncertainty Robust Analysis: Is the feedback system in Figure 18 robustly stable for all $\Delta(s) \in RH_\infty$ such that $\|\Delta(s)\| < 1$ in the presence of some structural constraint $\Delta(s) \in \mathbf{\Delta}_S(s)$?

Without loss of generality

$$\mathbf{\Delta}_S(s) := \{\Delta(s) : \quad \Delta(s) = \mathrm{diag}[\delta_1(s)I, \ldots, \delta_s(s)I, \quad \Delta_1(s), \ldots, \Delta_F(s)] \in RH_\infty\}$$

or

$$\mathbf{\Delta}_S := \{\Delta : \quad \Delta = \mathrm{diag}[\delta_1 I, \ldots, \delta_s I, \quad \Delta_1, \ldots, \Delta_F], \quad \delta_i \in \mathbb{C}, \quad \Delta_j \in \mathbb{C}^{m_j \times m_j}\}.$$

6.5.5 μ Analysis

In this section we focus on uncertainty in $\mathbf{\Delta}_S$. Results are presented without proof. See Zhou, Chapter 10.

Definition: For $M \in \mathbb{C}^{n \times n}$, $\Delta \in \mathbf{\Delta_S}$, the structured singular value of M is the scalar

$$\mu_{\mathbf{\Delta}_S}(M) := \frac{1}{\min\{\|\Delta\| : \Delta \in \mathbf{\Delta}_S, \quad \det(I - M\Delta) = 0\}}$$

If $\det(I - M\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}_S$ then $\mu_{\mathbf{\Delta}_S}(M) := 0$.

Lemma: $\max_i |\lambda_i(M)| \leq \mu_{\mathbf{\Delta}_S}(M) \leq \overline{\sigma}(M)$.

Scalings:

$$\mathbf{D}_S := \{\Delta : \quad \Delta = \mathrm{diag}[D_1, \ldots, D_s, \quad d_1 I, \ldots, d_F I], \\ D_i \in \mathbb{C}^{r_i \times r_i}, D_i = D_i^* > 0, \quad d_j \in \mathbb{R}, d_j > 0\}.$$

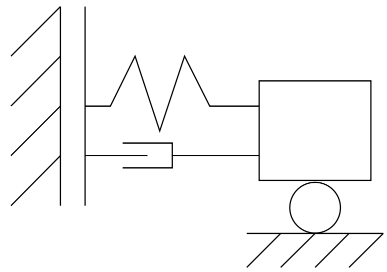
Lemma: $\max_{i, UU^* = I} |\lambda_i(UM)| = \mu_{\mathbf{\Delta}_S}(M) \leq \inf_{D \in \mathbf{D}_S} \overline{\sigma}(DMD^{-1})$.

Structured Small Gain Theorem: Let $P(s) \in RH_\infty$. Then the feedback connection if Figure 18 is well posed and internally stable for all $\Delta \in \mathbf{\Delta}_S$ such that $\|\Delta\| < 1$ if and only if

$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}_S}[P(j\omega)] \leq 1.$$

6.6 Example #4 [Parametric Uncertainty]

Mass/Spring/Damper system from Zhou, p. 171.



Uncertain System:

$$m\ddot{x} + c\dot{x} + kx = f$$

where $m > 0, \quad c > 0, \quad k > 0.$

Uncertainty:

$$c = \bar{c}(1 + \delta_c) \qquad\qquad k = \bar{k}(1 + \delta_k)$$

where

$$|c - \bar{c}| < \delta_c \bar{c} \qquad\qquad |k - \bar{k}| < \delta_k \bar{k}$$

and $|\delta_c| < \bar{\delta}_c \leq 1, \quad |\delta_k| < \bar{\delta}_k \leq 1.$

Pulling out uncertainty:
Write system equation

$$\begin{aligned}\ddot{x} &= -m^{-1}c\dot{x} - m^{-1}kx + m^{-1}f \\ &= -m^{-1}\bar{c}(1 + \delta_c)\dot{x} - m^{-1}\bar{k}(1 + \delta_k)x + m^{-1}f \\ &= -m^{-1}\bar{c}\dot{x} - m^{-1}\bar{k}x - m^{-1}\delta_c\bar{c}\dot{x} - m^{-1}\delta_k\bar{k}x + m^{-1}f\end{aligned}$$

Define auxiliary outputs

$$\begin{aligned}z_c &= \bar{c}\dot{x}, \\ z_k &= \bar{k}x\end{aligned}$$

System with auxiliary outputs

$$
\begin{aligned}
\ddot{x} &= -m^{-1}\bar{c}\dot{x} - m^{-1}\bar{k}x - m^{-1}\delta_c z_c - m^{-1}\delta_k z_k + m^{-1}f \\
z_c &= \bar{c}\dot{x} \\
z_k &= \bar{k}x
\end{aligned}
$$

Define auxiliary inputs in feedback

$$
\begin{aligned}
w_c &= \delta_c z_c \\
w_k &= \delta_k z_k
\end{aligned}
$$

System with auxiliary inputs and outputs

$$
\begin{aligned}
\ddot{x} &= -m^{-1}\bar{c}\dot{x} - m^{-1}\bar{k}x - m^{-1}w_c - m^{-1}w_k + m^{-1}f \\
z_c &= \bar{c}\dot{x} \\
z_k &= \bar{k}x
\end{aligned}
$$

Uncertain System in State Space: $(x_2 = \dot{x},\ x_1 = x)$

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \hline z_c \\ z_k \end{pmatrix} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -m^{-1}\bar{k} & -m^{-1}\bar{c} & -m^{-1} & -m^{-1} & m^{-1} \\ \hline 0 & \bar{c} & 0 & 0 & 0 \\ \bar{k} & 0 & 0 & 0 & 0 \end{array}\right] \begin{pmatrix} x_1 \\ x_2 \\ \hline w_c \\ w_k \\ f \end{pmatrix}
$$

$$
\begin{pmatrix} w_c \\ w_k \end{pmatrix} = \begin{bmatrix} \delta_c & 0 \\ 0 & \delta_k \end{bmatrix} \begin{pmatrix} z_c \\ z_k \end{pmatrix}
$$

6.7 Example #5 [Parametric Uncertainty]

Mass/Spring/Damper system from Zhou, p. 171.

Uncertain System:

$$m\ddot{x} + c\dot{x} + kx = f$$

where $m > 0, \quad c > 0, \quad k > 0.$

Uncertainty:

$$m = \bar{m}(1 + \delta_m) \qquad c = \bar{c}(1 + \delta_c) \qquad k = \bar{k}(1 + \delta_k)$$

where

$$|m - \bar{m}| < \delta_m \bar{m} \qquad |c - \bar{c}| < \delta_c \bar{c} \qquad |k - \bar{k}| < \delta_k \bar{k}$$

and $|\delta_m| < \bar{\delta}_m \leq 1, \quad |\delta_c| < \bar{\delta}_c \leq 1, \quad |\delta_k| < \bar{\delta}_k \leq 1.$

Auxiliary inputs/outputs:
Define auxiliary outputs

$$\begin{aligned} z_c &= \bar{c}\dot{x}, \\ z_k &= \bar{k}x \end{aligned}$$

Define auxiliary inputs in feedback

$$\begin{aligned} w_c &= \delta_c z_c \\ w_k &= \delta_k z_k \end{aligned}$$

System with auxiliary inputs and outputs

$$\begin{aligned} \ddot{x} &= m^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f\right) \\ z_c &= \bar{c}\dot{x} \\ z_k &= \bar{k}x \end{aligned}$$

The fractional uncertainty:

Use the lemma of the inverse

$$\begin{aligned} m^{-1} &= \bar{m}^{-1}(1+\delta_m)^{-1} \\ &= \bar{m}^{-1}[1-\delta_m(1+\delta_m)^{-1}] \\ &= \bar{m}^{-1}\mathcal{F}_l\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \delta_m\right) \end{aligned}$$

Since for any u

$$y = \mathcal{F}_l\left(M, \delta\right) u$$

is the feedback connection of

$$\begin{pmatrix} y \\ z \end{pmatrix} = M \begin{pmatrix} u \\ w \end{pmatrix}$$

with

$$w = \delta z$$

we have

$$\begin{aligned} \begin{pmatrix} m^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f\right) \\ z_m \end{pmatrix} &= \begin{bmatrix} \bar{m}^{-1} & -\bar{m}^{-1} \\ 1 & -1 \end{bmatrix} \begin{pmatrix} -\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f \\ w_m \end{pmatrix} \\ &= \begin{pmatrix} \bar{m}^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k - w_m + f\right) \\ -\bar{c}\dot{x} - \bar{k}x - w_c - w_k - w_m + f \end{pmatrix} \end{aligned}$$

System with auxiliary inputs and outputs

$$\begin{aligned} \ddot{x} &= \bar{m}^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k - w_m + f\right) \\ z_c &= \bar{c}\dot{x} \\ z_k &= \bar{k}x \\ z_m &= -\bar{c}\dot{x} - \bar{k}x - w_c - w_k - w_m + f \end{aligned}$$

Uncertain System in State Space: $(x_2 = \dot{x},\, x_1 = x)$

$$
\left(\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \hline z_c \\ z_k \\ z_m \end{array}\right) = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\bar{m}^{-1}\bar{k} & -\bar{m}^{-1}\bar{c} & -\bar{m}^{-1} & -\bar{m}^{-1} & -\bar{m}^{-1} & \bar{m}^{-1} \\ \hline 0 & \bar{c} & 0 & 0 & 0 & 0 \\ \bar{k} & 0 & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & -1 & -1 & -1 & 1 \end{array}\right] \left(\begin{array}{c} x_1 \\ x_2 \\ \hline w_c \\ w_k \\ w_m \\ f \end{array}\right)
$$

$$
\begin{pmatrix} w_c \\ w_k \\ w_m \end{pmatrix} = \begin{bmatrix} \delta_c & 0 & 0 \\ 0 & \delta_k & 0 \\ 0 & 0 & \delta_m \end{bmatrix} \begin{pmatrix} z_c \\ z_k \\ z_m \end{pmatrix}
$$

6.8 Example #6 [Parametric Uncertainty]

Mass/Spring/Damper system from Zhou, p. 171.

Uncertain System:

$$\ddot{x} + m^{-1}c\dot{x} + m^{-1}kx = m^{-1}f$$

where $m > 0, \quad c > 0, \quad k > 0.$

Uncertainty:

$$m^{-1} = \bar{m}^{-1}(1 + \delta_m) \qquad c = \bar{c}(1 + \delta_c) \qquad k = \bar{k}(1 + \delta_k)$$

where

$$|m^{-1} - \bar{m}^{-1}| < \delta_m \bar{m}^{-1} \qquad |c - \bar{c}| < \delta_c \bar{c} \qquad |k - \bar{k}| < \delta_k \bar{k}$$

and $|\delta_m| < \bar{\delta}_m \leq 1, \quad |\delta_c| < \bar{\delta}_c \leq 1, \quad |\delta_k| < \bar{\delta}_k \leq 1.$

Auxiliary inputs/outputs:

Define auxiliary outputs

$$\begin{aligned} z_c &= \bar{c}\dot{x}, \\ z_k &= \bar{k}x \end{aligned}$$

Define auxiliary inputs in feedback

$$\begin{aligned} w_c &= \delta_c z_c \\ w_k &= \delta_k z_k \end{aligned}$$

System with auxiliary inputs and outputs

$$\begin{aligned} \ddot{x} &= m^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f\right) \\ z_c &= \bar{c}\dot{x} \\ z_k &= \bar{k}x \end{aligned}$$

The multiplicative uncertainty:
Since

$$m^{-1} = \bar{m}^{-1}(1 + \delta_m)$$

we have

$$m^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f\right) = \bar{m}^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + w_m + f\right)$$

where

$$\begin{aligned} w_m &= \delta_m z_m \\ z_m &= -\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f \end{aligned}$$

System with auxiliary inputs and outputs

$$\begin{aligned} \ddot{x} &= \bar{m}^{-1}\left(-\bar{c}\dot{x} - \bar{k}x - w_c - w_k + w_m + f\right) \\ z_c &= \bar{c}\dot{x} \\ z_k &= \bar{k}x \\ z_m &= -\bar{c}\dot{x} - \bar{k}x - w_c - w_k + f \end{aligned}$$

Uncertain System in State Space: ($x_2 = \dot{x}$, $x_1 = x$)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \hline z_c \\ z_k \\ z_m \end{pmatrix} = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\bar{m}^{-1}\bar{k} & -\bar{m}^{-1}\bar{c} & -\bar{m}^{-1} & -\bar{m}^{-1} & \bar{m}^{-1} & \bar{m}^{-1} \\ \hline 0 & \bar{c} & 0 & 0 & 0 & 0 \\ \bar{k} & 0 & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & -1 & -1 & 0 & 1 \end{array}\right] \begin{pmatrix} x_1 \\ x_2 \\ \hline w_c \\ w_k \\ w_m \\ f \end{pmatrix}$$

$$\begin{pmatrix} w_c \\ w_k \\ w_m \end{pmatrix} = \begin{bmatrix} \delta_c & 0 & 0 \\ 0 & \delta_k & 0 \\ 0 & 0 & \delta_m \end{bmatrix} \begin{pmatrix} z_c \\ z_k \\ z_m \end{pmatrix}$$