1 Linear Multivariable Systems

1.1 Nomenclature

Inputs (u), outputs (y), and state (x).

1.2 SISO

Single-Input-Single-Output
Several special results
"Classic" control theory (until the 60's)

1.3 MIMO

Multiple-Input-Multiple-Output
"Modern" control theory (from the 60's on)

1.4 Linear Time-Invariant (LTI)

Linear and ... $u(t) \to y(t) \quad \Rightarrow \quad u(t-\tau) \to y(t-\tau)$ for any τ.

1.5 State Space

LTI

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx + Du. \end{aligned}$$

Dimensions:

$$x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m, \qquad y \in \mathbb{R}^p.$$

Notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Transfer function:

$$H(s) = C(sI - A)^{-1}B + D$$

Note that $H(s)$ is always proper!

Similarity transformation:

$$\left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Similarity does not change transfer function

$$H(s) = CT(sI - T^{-1}AT)^{-1}T^{-1}B + D = C(sI - A)^{-1}B + D$$

System response:

$$Y(s) = \underbrace{H(s)U(s)}_{\text{Input response}} + \underbrace{C(sI - A)^{-1}x(0)}_{\text{Initial conditions response}}$$

MIMO comes for free!

1.6 Controllability

Controllability Matrix:

$$\mathcal{C}(A,B) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

Controllability Gramian:

$$X(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau,$$

Theorem: The following are equivalent:

1. The pair (A,B) is controllable
2. The controllability Gramian $X(t) \succ 0$ for any $0 < t < \infty$.
3. The controllability matrix $\mathcal{C}(A,B)$ has full-row rank.
4. There exists no $\lambda \in \mathbb{C}$ and $z \neq 0$ such that $z^* A = \lambda z^*, \quad z^* B = 0$.
5. There exists $K \in \mathbb{R}^{m \times n}$ so that the eigenvalues of the real matrix $A + BK$ can be placed anywhere in the complex plane. (recall that complex poles must appear as complex conjugates!)
6. The pair (A^T, B^T) is observable.

Lemma: If $X = \lim_{t \to \infty} X(t)$ exists then

$$AX + XA^T + BB^T = 0.$$

1.7 Observability

Observability Matrix:

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Controllability Gramian:

$$Y(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau.$$

Theorem: The following are equivalent:

1. The pair (A, C) is observable.
2. The observability Gramian $Y(t) \succ 0$ for any $0 < t < \infty$.
3. The observability matrix $\mathcal{O}(A, C)$ has full-column rank.
4. There exists no $\lambda \in \mathbb{C}$ and $x \neq 0$ such that $Ax = \lambda x, \quad Cx = 0$.
5. There exists $L \in \mathbb{R}^{m \times n}$ so that the eigenvalues of the real matrix $A + LC$ can be placed anywhere in the complex plane.
6. The pair (A^T, C^T) is controllable.

Lemma: If $Y = \lim_{t \to \infty} Y(t)$ exists then

$$A^T Y + YA + C^T C = 0.$$

1.8 Stabilizability

Theorem: The following are equivalent:

1. The pair (A, B) is stabilizable.
2. If there exists λ and $z \neq 0$ such that $z^* A = \lambda z^*$ with $\lambda + \lambda^* \geq 0$ then $z^* B \neq 0$.
3. There exists $K \in \mathbb{R}^{m \times n}$ so that the real matrix $A + BK$ is Hurwitz.
4. The pair (A^T, B^T) is detectable.

1.9 Detectability

Theorem: The following are equivalent:

1. The pair (A, C) is detectable.
2. If there exists λ and $x \neq 0$ such that $Ax = \lambda x$ with $\lambda + \lambda^* \geq 0$ then $Cx \neq 0$.
3. There exists $L \in \mathbb{R}^{m \times n}$ so that the real matrix $A + LC$ is Hurwitz.
4. The pair (A^T, C^T) is stabilizable.

1.10 Operations

1. Transpose: $H^T(s)$
2. Conjugate: $H^{\sim}(s) = H^T(-s)$
3. Inverse: $H^{-1}(s)H(s) = H(s)H^{-1}(s) = I$
4. Series: $H(s) = H_1(s)H_2(s)$
5. Parallel: $H(s) = H_1(s) + H_2(s)$

WARNING: MIMO systems do not commute $H_1(s)H_2(s) \neq H_2(s)H_1(s)$!

State space formulas make it easy...

1. Transpose: $H^T(s) = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$
2. Conjugate: $H^{\sim}(s) = H^T(-s) = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right]$
3. Inverse: $H^{-1}(s) = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$
4. Series: $H(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$
5. Parallel: $H(s) = H_1(s) + H_2(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$

but usually lead to non minimal realizations.

1.11 Polynomial and Rational Matrices

Definition: Let $Q(s)$ be a matrix with polynomial entries on $s \in \mathbb{C}$.

$$\text{normal rank}(Q(s)) = \max_{s\in\mathbb{C}} \operatorname{rank} Q(s)$$

Example:

$$Q(s) = \begin{bmatrix} s & s+1 \\ 0 & 1 \end{bmatrix}$$

Normal rank $Q(s) = 2$ since rank $Q(1) = 2$. Note that rank $Q(0) = 1$.

Definition: A polynomial square matrix $U(s)$ is called *unimodular* if its inverse is also a polynomial matrix.

Theorem: A polynomial matrix $U(s)$ is *unimodular* if and only if $\det U(s)$ is independent of s, i.e., a constant (not zero).

Proof:

$$U^{-1}(s) = \frac{1}{\det U(s)} \operatorname{Adj} U(s)$$

and $\operatorname{Adj} U(s)$ is always a polynomial matrix.

Example:

$$U(s) = \begin{bmatrix} s+1 & s \\ s & s-1 \end{bmatrix}$$

is unimodular since

$$|U(s)| = (s+1)(s-1) - s^2 = s^2 - 1 - s^2 = -1.$$

Definition: The square polynomial matrix $R(s)$ is a *right divisor* of the polynimial matrix $N(s)$ if

$$N(s) = \bar{N}(s)R(s)$$

for some polynomial matrix $\bar{N}(s)$.

Definition: The square polynomial matrix $R(s)$ is a *greatest common right divisor* of the polynimial matrices $D(s)$ and $N(s)$ if $R(s)$ is a right divisor of $D(s)$ and $N(s)$ and $R(s) = W(s)\bar{R}(s)$ with $W(s)$ polynomial for any other right divisor $\bar{R}(s)$ of $D(s)$ and $N(s)$.

Example:

$$D(s) = \begin{bmatrix} s^2 - s & s^2 \\ 2s^2 + 9s + 5 & 2s^2 + 5s + 5 \end{bmatrix}, \qquad N(s) = \begin{bmatrix} s^2 + 1 & s^2 + 2s + 1 \end{bmatrix}$$

Verify that

$$\begin{aligned}
\bar{D}(s) &= \begin{bmatrix} s^2 - s & \frac{1}{6}(17s^3 - 23s^2 + 6s + 6) \\ 2s^2 + 9s + 5 & \frac{1}{6}(34s^3 + 141s^2 + 31s - 54) \end{bmatrix}, \\
\bar{N}(s) &= \begin{bmatrix} s^2 + 1 & \frac{1}{6}(17s^3 - 6s^2 + 17s + 6) \end{bmatrix}, \\
R(s) &= \begin{bmatrix} 1 & \frac{1}{6}(-17s^2 + 6s + 6) \\ 0 & s \end{bmatrix}
\end{aligned}$$

are such that

$$N(s) = \bar{N}(s)R(s), \qquad D(s) = \bar{D}(s)R(s)$$

In this case $R(s)$ is also a greatest common right divisor.

Matrix divisors can be computed using a combination of the Gauss elimination algorithm with the Euclidian division of scalar polynomials. Usually messy!

Definition: Two polynomial matrices $N(s)$ and $D(s)$ are *right coprime* if the equations

$$N(s) = \bar{N}(s)U(s), \qquad D(s) = \bar{D}(s)U(s)$$

can be solved only with $U(s)$ unimodular. In other words, if all greatest common divisors are unimodulars.

Theorem [Bezout identity]: Two polynomial matrices $N(s)$ and $D(s)$ are *right coprime* if and only if there exists polynomial matrices $X(s)$ and $Y(s)$ such that

$$X(s)N(s) + Y(s)D(s) = I.$$

Proof: Sufficiency: Find the greatest common right divisor $R(s)$

$$N(s) = \bar{N}(s)R(s), \qquad D(s) = \bar{D}(s)R(s).$$

If the Bezout identity is satisfied for some $X(s)$ and $Y(s)$ then

$$\begin{aligned} I &= X(s)N(s) + Y(s)D(s) \\ &= X(s)\bar{N}(s)R(s) + Y(s)\bar{D}(s)R(s) \\ &= \left[X(s)\bar{N}(s) + Y(s)\bar{D}(s)\right] R(s) \end{aligned}$$

so that

$$R^{-1}(s) = X(s)\bar{N}(s) + Y(s)\bar{D}(s)$$

Since the right hand side of the above equation is a polynomial $R(s)$ must be unimodular, i.e., $N(s)$ and $D(s)$ are coprime.
Necessity: By construction. It is possible to show that any greatest common right divisor of $N(s)$ and $D(s)$ can be written as

$$R(s) = \hat{X}(s)N(s) + \hat{Y}(s)D(s)$$

for some polynomial $\hat{X}(s)$ and $\hat{Y}(s)$. Therefore

$$\begin{aligned} I &= R^{-1}(s)R(s) \\ &= R^{-1}(s)\hat{X}(s)N(s) + R^{-1}(s)\hat{Y}(s)D(s) \\ &= X(s)N(s) + Y(s)D(s) \end{aligned}$$

for $X(s) = R^{-1}(s)\hat{X}(s)$, $Y(s) = R^{-1}(s)\hat{Y}(s)$.

Theorem [Smith form]: Let $N(s)$ be a polynomial matrix on s with normal rank r. There exists two polynomial and unimodular matrices $L(s)$ and $R(s)$ such that $N(s) = L(s)S(s)R(s)$ and

$$S(s) = \begin{bmatrix} \sigma_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \sigma_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

where

1. $\sigma_i(s)$ is monic,
2. $\sigma_i(s)$ divides $\sigma_{i+1}(s)$,

for all $i = 1, \cdots, r$.

Corresponding definitions and properties hold for *left* divisor, coprime, etc.

Similar definitions and properties hold for *left* divisor, coprime, etc, for *rational* matrices. We will see some of that next!

Theorem [Bezout identity]: Two rational matrices $F(s)$ and $G(s)$ are *right coprime* if and only if there exists rational matrices $X(s)$ and $Y(s)$ such that

$$X(s)F(s) + Y(s)G(s) = I.$$

Theorem [Smith-McMillian form]: Let $H(s)$ be a proper rational matrix function of s with normal rank r. There exists two polynomial and unimodular matrices $L(s)$ and $R(s)$ such that $H(s) = L(s)M(s)R(s)$ and

$$M(s) = \begin{bmatrix} \epsilon_1(s)/\psi_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \epsilon_r(s)/\psi_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

where

1. $\epsilon_i(s), \psi_i(s)$ are monic,
2. $\epsilon_i(s), \psi_i(s)$ are coprime,

for all $i = 1, \cdots, r$ and

3. $\epsilon_i(s)$ divides $\epsilon_{i+1}(s)$,
4. $\psi_{i+1}(s)$ divides $\psi_i(s)$,

for all $i = 1, \cdots, r-1$.

1.12 Poles and Zeros

Definition: Let $H(s)$ be a rational transfer matrix and $S(s)$ its associated Smith-McMillian form. The poles of $H(s)$ are the roots of $\prod_{i=1}^{r} \psi_i(s)$ and the zeros (also known as transmission zeros) are the roots of $\prod_{i=1}^{r} \epsilon_i(s)$.

Example: Let

$$H(s) = \begin{bmatrix} \frac{(s-3)(s+1)}{(s-5)(s-2)} \\ \frac{(s-3)}{(s-5)} \end{bmatrix} = \begin{bmatrix} s+1 & 1 \\ s-2 & 1 \end{bmatrix} \begin{bmatrix} \frac{(s-3)}{(s-5)(s-2)} \\ 0 \end{bmatrix}$$

$H(s)$ has two poles $\{2, 5\}$ and one zero $\{-3\}$. Note that $\{-1\}$ is not a zero!

Theorem: If $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a state space realization of $H(s)$ then the eigenvalues of A are the poles of $H(s)$.

Theorem: If $s_0 \in \mathbb{C}$ is such that

$$\operatorname{rank} H(s_0) < \text{normal rank}\, H(s)$$

then s_0 is a *transmission zero* of $H(s)$.

Example: Note that normal rank $H(s) = 1$, rank $H(3) = 0$ but rank $H(-1) = 1$!

Theorem: If $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a state space realization of $H(s)$ and $s_0 \in \mathbb{C}$ is such that

$$\operatorname{rank} \begin{bmatrix} A - s_0 I & B \\ C & D \end{bmatrix} < \text{normal rank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$$

then s_0 is an *invariant zero* of $H(s)$.

Example: A state space realization for $H(s)$ on the previous example is

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -10 & 7 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 & 2 \\ 1 & -1 & 1 & 6 & 0 \\ \hline -13 & 5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Therefore

$$P(s) = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} -s & 1 & 0 & 0 & 0 \\ -10 & 7-s & 0 & 0 & 1 \\ 0 & 0 & 5-s & 0 & 2 \\ 1 & -1 & 1 & 6-s & 0 \\ -13 & 5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

so that normal rank $P(s) = 5$, e.g. rank $P(0) = 5$, while rank $P(3) = 4$ and rank $P(6) = 4$. Invariant zeros are then $\{3, 6\}$. Note that 6 is not a transmission zero!

Theorem: A transmission zero is also an invariant zero.

Lemma: Invariant zeros are not affected by state feedback or similarity transformations.
Proof: Note that

$$\begin{bmatrix} A + BK - s_0 I & B \\ C + DK & D \end{bmatrix} = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ K & I \end{bmatrix}$$

and $\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}$ has full rank.
For the same reason

$$\begin{bmatrix} T^{-1}AT - sI & T^{-1}B \\ CT & D \end{bmatrix} = \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}.$$

1.13 Minimal Realizations

Theorem: A state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is minimal if and only if (A, B) is controllable and (A, C) is observable.

Check out material from MAE 280A on how to construct minimal realizations.

WARNING: Realizations for MIMO transfer functions are not trivial! Check out Kailath, Chapter 6 and beyond.

Theorem: A state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is minimal if and only if $n = r$, where r is the degree of the Smith-McMillan form associated with $H(s) = C(sI - A)^{-1}B + D$.

Theorem: If $H(s)$ is minimal, then all invariant zeros are transmission zeros.