FINAL

Instructions:

- You have 150 minutes
- This exam is open notes, books
- No computers, calculators, phones, etc
- There are 7 questions for a total of 60 points plus 5 bonus points
- Good luck!

Questions:

1. [10 points] Consider the matrix and vectors

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Answer the following questions:

a) [1 point] Verify that x solves the equation $Ax = b$.

Multiply to verify that $Ax = b$.	(+1 point)

b) [5 points] Compute the LU Decomposition $PA = LU$ with partial pivoting.

First iteration is:

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad P_1 A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}, \qquad (+1 \text{ point})$$

$$M_1 P_1 A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1/2 \end{bmatrix} \qquad (+1 \text{ point})$$

Second iteration is:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_2 M_1 P_1 A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1/2 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}, \quad (+1 \text{ point})$$

$$\tilde{M}_1 = P_2 M_1 P_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = M_2 P_2 M_1 P_1 A_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1/2 \\ 0 & 0 & 1/4 \end{bmatrix} \quad (+1 \text{ point})$$

Extracting the factors

$$P = P_2 P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1/2 \\ 0 & 0 & 1/4 \end{bmatrix} \quad (+1 \text{ point})$$

c) [4 points] Use the computed LU decomposition to solve the system of equations $Ax = b$.

To solve $Ax = b$ we note that $PAx = LUx = Pb$ and solve

$$Ly = Pb \qquad\qquad Ux = y$$

Computing y

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = Ly = Pb = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad (+1 \text{ point})$$

$$y_1 = 1, \qquad y_2 = 1 - (1/2)y_1 = 1/2, \qquad y_3 = 0 - (1/2)y_2 = -1/4 \qquad (+1 \text{ point})$$

then computing x

$$Ux = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1/2 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ -1/4 \end{pmatrix} \qquad (+1 \text{ point})$$

$$x_3 = 4(-1/4) = -1, \quad x_2 = (1/2 + (1/2)x_3)/2 = 0, \quad x_1 = (1 - x_3)/2 = 1 \quad (+1 \text{ point})$$

2. [10 points] Consider a matrix $A \in \mathbb{R}^{m\times n}$ that has the reduced SVD decomposition

$$A = U\Sigma V^*, \qquad U^*U = I, \qquad V^*V = I$$

and $\Sigma > 0$ is square and diagonal. Let $A^\dagger = V\Sigma^{-1}U^*$.

a) [2 points] Show that $AA^\dagger A = A$.

$$AA^\dagger A = U\Sigma V^*(V\Sigma^{-1}U^*)U\Sigma V^* = U\Sigma\Sigma^{-1}\Sigma V^* = U\Sigma V^* = A. \qquad \text{(+2 points)}$$

b) [4 points] Show that if $(I - AA^\dagger)b = 0$ then $x = A^\dagger b + (I - A^\dagger A)z$ solves $Ax = b$.

Substitute

$$x = A^\dagger b + (I - A^\dagger A)z$$

into

$$\begin{aligned} Ax &= AA^\dagger b + A(I - A^\dagger A)z \\ &= AA^\dagger b + (A - AA^\dagger A)z \\ &= AA^\dagger b && \text{(from a))} \\ &= b && \text{(from } AA^\dagger b = b\text{)} \end{aligned}$$

c) [4 points] Show that if $Ax = b$ for some x then $(I - AA^\dagger)b = 0$ and $z = x$ is such that $A^\dagger b + (I - A^\dagger A)z = x$.

Multiply

$$b = Ax$$

by $AA^\dagger$ on both sides to obtain

$$\begin{aligned} AA^\dagger b &= AA^\dagger Ax \\ &= Ax \\ &= b \end{aligned}$$

Furthermore, with $z = x$

$$\begin{aligned} A^\dagger b + (I - A^\dagger A)z &= A^\dagger b - A^\dagger Ax + x \\ &= A^\dagger b - A^\dagger b + x \\ &= x \end{aligned}$$

3. [10 points] Consider the matrix and vectors

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

a) [3 points] Calculate the reduced SVD decomposition of A and ~~use it to compute $A^\dagger$~~ show that

$$A^\dagger = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Note that

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{U} \underbrace{\ 2\ }_{\Sigma} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}}_{V^*} \end{aligned} \qquad (+2 \text{ points})$$

then

$$A^\dagger = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{V} \underbrace{(1/2)}_{\Sigma} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}}_{U^*} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \qquad (+1 \text{ point})$$

b) [5 points] Calculate $(I - AA^\dagger)b_1$. Does a solution to equation $Ax = b_1$ exist? If so, calculate $x = A^\dagger b_1$ and matrix $(I - A^\dagger A)$. Is the solution x unique or not? Why?

Calculate

$$AA^\dagger = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$I - AA^\dagger = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad (I - AA^\dagger)b_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad \text{(+1 point)}$$

Because $(I - AA^\dagger)b_1 = 0$ a solution exists. (+1 point)
Calculating

$$x = A^\dagger b_1 = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{(+1 point)}$$

is one such solution.
Because

$$A^\dagger A = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad I - A^\dagger A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq 0 \quad \text{(+1 point)}$$

the solution is not unique. In fact, any

$$x = A^\dagger b_1 + (I - A^\dagger A)z = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} z \qquad \text{(+1 point)}$$

is also a solution.

c) [2 points] Repeat item b) for b_2.

Borrowing some calculations from b)

$$(I - AA^\dagger)b_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \qquad \text{(+1 point)}$$

Because $(I - AA^\dagger)b_2 \neq 0$ no solution exists. (+1 point)

4. [10 points] Consider the matrix

$$N = \begin{bmatrix} \rho & 2 & 0 \\ 0 & \rho & 0 \\ 2 & 0 & \rho \end{bmatrix}$$

a) [2 points] Show that the characteristic polynomial of the matrix N is $p_N(\lambda) = (\lambda - \rho)^3$.

Hint: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (ceg + bdi + afh)$

The characteristic polynomial is

$$\begin{aligned} p_N(\lambda) &= |\lambda I - N| && (+1 \text{ point}) \\ &= \begin{vmatrix} \lambda-\rho & -2 & 0 \\ 0 & \lambda-\rho & 0 \\ -2 & 0 & \lambda-\rho \end{vmatrix} = (\lambda-\rho)^3 && (+1 \text{ point}) \end{aligned}$$

after using the determinant formula.

b) [4 points] Show that the singular values of matrix N when $\rho = 0$ are $\{2, 2, 0\}$.

When $\rho = 0$ matrix N is simply

$$N_0 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \qquad (+1 \text{ point})$$

and

$$N_0^* N_0 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad (+1 \text{ point})$$

which is diagonal and therefore has as eigenvalues $\{4, 4, 0\}$.
Because for any matrix A

$$A^* A = V \Sigma^2 V^* \qquad (+1 \text{ point})$$

the singular values of N_0 are the square roots of the eigenvalues of $N_0^* N_0$. That is the singular values of N_0 are $\{2, 2, 0\}$. (+1 point)

c) [4 points] Show that N is always defective for any value of ρ.

The eigenvalues of N are the roots of the characteristic polynomial $p_N(\lambda) = (\lambda - \rho)^3$, which are all equal to ρ. (+1 point)
Therefore, $x \neq 0$ is an eigenvector of N if

$$0 = (N - \rho I)x = N_0 x = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \Longrightarrow \quad x = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}. \qquad (+2 \text{ points})$$

Because such x span a subspace of dimension 1 and ρ is a triple root, N must be defective for all values of ρ. (+1 point)

5. [4 bonus points] Prove or disprove with a counter-example the following statements:

a) [2 bonus points] A square matrix with all singular values equal to zero has all eigenvalues equal to zero as well.

True. Since $\Sigma = 0$ implies that $A = U\Sigma V^* = U 0 V^* = 0$ is the zero matrix, and therefore has all eigenvalues equal to zero.

b) [2 bonus points] A square matrix with all eigenvalues equal to zero has all singular values equal to zero as well.

False. The matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all eigenvalues equal to zero but

$$A^* A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which implies that A has one singular value equal to 1.

6. [10 points] Split matrix A given in Q.1 as

$$A = M + N, \qquad M = (1 - \rho) I, \qquad N = A - M$$

and consider the following iterative algorithm:

$$M\, x_{k+1} = -N\, x_k + b.$$

As shown in class, if $\bar{\lambda}(H)$, that is the eigenvalue with largest absolute value of the matrix

$$H = M^{-1} N,$$

has magnitude less than one, then the algorithm converges to the solution of the equation $Ax = b$. The smaller the $|\bar{\lambda}(H)|$ the faster the convergence.

a) [2 points] Show that $\bar{\lambda}(H) = \rho\,(1-\rho)^{-1}$. *Hint: Use Q.4*

Because

$$H = M^{-1} N = \frac{1}{1-\rho} N$$

matrix H has all eigenvalues

$$\lambda_i(H) = \frac{1}{1-\rho} \lambda_i(N) = \frac{\rho}{1-\rho}, \qquad i = 1, 2, 3. \qquad \text{(+1 point)}$$

where $\lambda_i(N)$ come from Q.4. Therefore $\lambda(\bar{H}) = \rho\,(1-\rho)^{-1}$. (+1 point)

b) [2 points] Find the range of ρ's for which the algorithm converges. Which value of ρ ensures the fastest possible convergence?

The function $\rho\,(1-\rho)^{-1}$ is singular at $\rho = 1$. If $\rho > 1$ then $\rho\,(1-\rho)^{-1} = -(1-\rho^{-1})^{-1} < -1$, therefore ρ must be less than one. Solving

$$\rho\,(1-\rho)^{-1} = 1 \qquad\qquad \Longrightarrow \qquad\qquad \rho = 1/2$$

we conclude that $0 \leq \rho\,(1-\rho)^{-1} < 1$ if $\rho \in [0, 1)$. Because $-1 < \rho\,(1-\rho)^{-1} < 0$ when $\rho < 0$ we conclude that the algorithm converges if

$$\rho \in (-\infty, 1/2). \qquad\qquad \text{(+1 point)}$$

The fastest possible convergence is achieved for $\rho = 0$ which is the point for which $\rho\,(1-\rho)^{-1}$ is zero (+1 point)

c) [6 points] Starting at $x_0 = 0$, evaluate x_k, $k = 1, 2, \cancel{3}$, for $\rho = 0$, $\rho = 0.1$ and $\rho = 0.9$ and b as given in Q.1. Compare x_k with the solution obtained in Q.1 and explain the behaviour of the algorithm in each case.

For $\rho = 0$, $M = I$ and $H = M^{-1}N = N_0$. Iterating

$$x_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{bmatrix}0 & -2 & 0\\0 & 0 & 0\\-2 & 0 & 0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad (+0.5 \text{ point})$$

$$x_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{bmatrix}0 & -2 & 0\\0 & 0 & 0\\-2 & 0 & 0\end{bmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \qquad (+0.5 \text{ point})$$

which converged to the exact solution x already in x_2. Convergence was expected since $|\bar{\lambda}(H)| = 0 < 1$. (+1 point)

For $\rho = 1/10$, $M = 9/10\ I$ and

$$H = M^{-1}N = \frac{10}{9}N_{1/10} = \frac{1}{9}\begin{bmatrix}1 & 20 & 0\\0 & 1 & 0\\20 & 0 & 1\end{bmatrix}$$

Iterating

$$x_1 = \frac{1}{9}\begin{pmatrix}10\\0\\10\end{pmatrix} - \frac{1}{9}\begin{bmatrix}1 & 20 & 0\\0 & 1 & 0\\20 & 0 & 1\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix} = \frac{1}{9}\begin{pmatrix}10\\0\\10\end{pmatrix} \qquad (+0.5 \text{ point})$$

$$x_2 = \frac{1}{81}\begin{pmatrix}90\\0\\90\end{pmatrix} - \frac{1}{81}\begin{bmatrix}1 & 20 & 0\\0 & 1 & 0\\20 & 0 & 1\end{bmatrix}\begin{pmatrix}10\\0\\10\end{pmatrix} = \frac{1}{81}\begin{pmatrix}80\\0\\-120\end{pmatrix} \qquad (+0.5 \text{ point})$$

which is converging slower to the solution. Convergence is expected since $|\bar{\lambda}(H)| = 1/9 < 1$. (+1 point)

For $\rho = 9/10$, $M = 1/10\ I$ and

$$H = M^{-1}N = 10N_{9/10} = \begin{bmatrix}9 & 20 & 0\\0 & 9 & 0\\20 & 0 & 9\end{bmatrix}$$

Iterating

$$x_1 = \begin{pmatrix}10\\0\\10\end{pmatrix} - \begin{bmatrix}9 & 20 & 0\\0 & 9 & 0\\20 & 0 & 9\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}10\\0\\10\end{pmatrix} \qquad (+0.5 \text{ point})$$

$$x_2 = \begin{pmatrix}10\\0\\10\end{pmatrix} - \begin{bmatrix}9 & 20 & 0\\0 & 9 & 0\\20 & 0 & 9\end{bmatrix}\begin{pmatrix}10\\0\\10\end{pmatrix} = \begin{pmatrix}-80\\0\\-280\end{pmatrix} \qquad (+0.5 \text{ point})$$

which is quickly diverging from the solution. Convergence is not expected since $|\bar{\lambda}(H)| = 9 > 1$. (+1 point)

7. [10 points] Newton's second law for a "free" particle of mass $m > 0$ in the plane (x, y) is

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix},$$

where $f = (f_x, f_y)$ is a force and $\ddot{x}$, $\ddot{y}$ denotes the second time-derivative of x, y. Suppose that the particle is now constrained to lie in the "line" parametrized by the equation

$$y = \alpha\, x \tag{1}$$

Note that $\dot{y} = \alpha\, \dot{x}$ as well as $\ddot{y} = \alpha\, \ddot{x}$. In order for that to happen, the "line" must exert an additional force $\xi = (\xi_x, \xi_y)$ so that

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} - \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}, \qquad \ddot{y} = \alpha\, \ddot{x}. \tag{2}$$

In this question you will calculate the force $\xi = (\xi_x, \xi_y)$.

a) [1 point] Show that if (x, y) is in (1) then $(x + \delta_x, y + \delta_y)$ is also in (1) if

$$\begin{pmatrix} \alpha & -1 \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = 0.$$

Since $y = \alpha x$, for $(x + \delta_x, y + \delta_y)$ to be in (1) it is necessary that

$$y + \delta_y = \alpha(x + \delta_x) \qquad \Longrightarrow \qquad \alpha\delta_x - \delta_y = 0 \qquad \text{(+1 point)}$$

which is the given equation.

b) [1 point] The work produced by the force $\xi = (\xi_x, \xi_y)$ during a displacement $\delta = (\delta_x, \delta_y)$ is

$$W = \left\langle \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}, \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} \right\rangle = \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}^T \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix}.$$

The condition that $\xi = (\xi_x, \xi_y)$ "does no work in (1)" is therefore

$$\begin{bmatrix} \alpha & -1 \\ \xi_x & \xi_y \end{bmatrix} \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = 0.$$

Show that the above equation admits a non-zero solution $\delta = (\delta_x, \delta_y)$ only when

$$\begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ -1 \end{pmatrix}$$

for some scalar λ. This scalar is known as a *Lagrange multiplier*.

The equation

$$\begin{bmatrix} \alpha & -1 \\ \xi_x & \xi_y \end{bmatrix} \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = 0$$

admits a non-zero solution only if the coefficient matrix is singular. Since there are only two rows, this happen only if the rows are linearly dependent, that is that is, if

$$\begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ -1 \end{pmatrix}. \qquad \text{(+1 point)}$$

c) [1 point] Show that the equations

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix}=\begin{pmatrix}f_x\\ f_y\end{pmatrix}-\begin{pmatrix}\xi_x\\ \xi_y\end{pmatrix},\qquad \ddot{y}=\alpha\,\ddot{x},\qquad \begin{pmatrix}\xi_x\\ \xi_y\end{pmatrix}=\lambda\begin{pmatrix}\alpha\\ -1\end{pmatrix},$$

are equivalent to

$$M\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \lambda\end{pmatrix}=\begin{pmatrix}f_x\\ f_y\\ 0\end{pmatrix},\qquad M=\begin{bmatrix}m & 0 & \alpha\\ 0 & m & -1\\ \alpha & -1 & 0\end{bmatrix}. \tag{3}$$

Simply substitute ξ and adjoin the third equation to obtain

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix}+\lambda\begin{pmatrix}\alpha\\ -1\end{pmatrix}=\begin{pmatrix}f_x\\ f_y\end{pmatrix}$$
$$\alpha\,\ddot{x}-\ddot{y}=0$$

which is the one in (3). (+1 point)

d) [1 bonus point] Show that

$$M=m\begin{bmatrix}I & b\\ b^T & 0\end{bmatrix},\qquad M^{-1}=\frac{1}{m\,b^Tb}\begin{bmatrix}(b^Tb)I-bb^T & b\\ b^T & -1\end{bmatrix}.$$

Multiply out to obtain

$$\begin{aligned}MM^{-1}&=\frac{1}{b^Tb}\begin{bmatrix}I & b\\ b^T & 0\end{bmatrix}\begin{bmatrix}(b^Tb)I-bb^T & b\\ b^T & -1\end{bmatrix}\\ &=\frac{1}{b^Tb}\begin{bmatrix}(b^Tb)I & 0\\ 0 & b^Tb\end{bmatrix}=I\end{aligned}$$

(+1 point)

e) [1 point] Use d) to compute:

$$\lambda=\frac{\alpha f_x-f_y}{1+\alpha^2}.$$

In order to use d) define

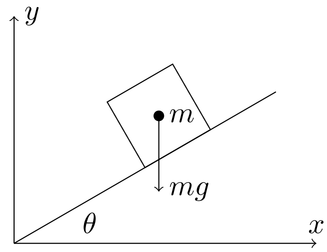
$$b=\frac{1}{m}\begin{pmatrix}\alpha\\ -1\end{pmatrix}$$

and calculate

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \lambda\end{pmatrix}=M^{-1}\begin{pmatrix}f_x\\ f_y\\ 0\end{pmatrix}\qquad\Longrightarrow\qquad \begin{aligned}\lambda&=\frac{1}{m\,b^Tb}\begin{bmatrix}b^T & -1\end{bmatrix}\begin{pmatrix}f_x\\ f_y\\ 0\end{pmatrix}\\ &=\frac{1}{m^2\,(1+\alpha^2)/m^2}\begin{bmatrix}\alpha & -1\end{bmatrix}\begin{pmatrix}f_x\\ f_y\end{pmatrix}\\ &=\frac{\alpha f_x-f_y}{(1+\alpha^2)}\end{aligned}$$

(+1 point)

f) [6 points] Construct a model using equation (3) to describe the motion of the point mass m in the "frictionless" inclined plane subject only to gravity as shown in the following diagram:



Calculate f_x, f_y, ξ_x, ξ_y and λ in terms of the angle θ, m and g. Represent the force $\xi = (\xi_x, \xi_y)$ in the diagram and calculate $\|\xi\|$.

Hint: $1 + \tan(\theta) = 1/\cos(\theta)^2$.

First the inclined plane equation is

$$y = \alpha\, x, \qquad \alpha = \tan(\theta). \qquad \text{(+1 point)}$$

The forces are

$$f_x = 0, \qquad f_y = -m\, g. \qquad \text{(+1 point)}$$

The Lagrange multiplier follows from e)

$$\lambda = \frac{\alpha f_x - f_y}{1 + \alpha^2} = \frac{m\, g}{1 + \tan(\theta)^2} = m\, g \cos(\theta)^2 \qquad \text{(+1 point)}$$

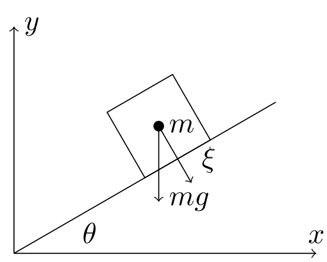
The non-working force is therefore

$$\begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ -1 \end{pmatrix} = m\, g \cos(\theta)^2 \begin{pmatrix} \tan(\theta) \\ -1 \end{pmatrix} = m\, g \cos(\theta) \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \end{pmatrix} \qquad \text{(+1 point)}$$

with magnitude

$$\|\xi\| = m\, g |\cos(\theta)| \qquad \text{(+1 point)}$$

In the diagram



(+1 point)