MAE 290 A HW2 | Solutions

$\boxed{1}$ $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

LU Decomposition

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 2-2 & 4-6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} = U$$

1) subtract 2x row 1 from row 2

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 \\ +2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ i) subtract 2x row 1 from row 2
ii) subtract 3x row 1 from row 3

$$\downarrow$$

$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$ iii) subtract 2x row 2 from row 3

$$\downarrow$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 12 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Notice: My calculations use the convention $P^T A=LU$ rather than $PA=LU$. Similarly for $AQ=LU$ as opposed to $AQ^T=LU$. This follows the convention laid out in chapter two of Numerical Renaissance, but is different from that seen in class. Please remember to be consistent in your notation.

LU Decomposition w/ Partial Pivoting

$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $A_{21} > A_{11}$, swap rows 1 & 2

$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $P_1 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

i) Subtract $\frac{1}{2} \times$ row 1 from row 2

$\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = U$ $A = PLU$ $L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$

$PL = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ 1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ $A_{31} > A_{11}$, swap rows 1 & 3

$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $P_1 A = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}$

i) subtract $\frac{1}{3} \times$ row 1 from row 3
ii) subtract $\frac{2}{3} \times$ row 1 from row 2

$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix}$ $M_1 P_1 A = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

$A_{32} > A_{22}$ Swap rows 2 & 3

$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $P_2 M_1 P_1 A = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

iii) subtract $\frac{1}{2} \times$ row 2 from row 3

$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$

$$M_2 \underbrace{P_2 M_1}_{M_{1,2} P_2} P_1 A = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$M_{1,2} = P_2 M_1 P_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 0 & 1 \\ -\frac{1}{3} & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \end{bmatrix}$$

$$\underbrace{M_2 M_{1,2}}_{M} \underbrace{P_2 P_1}_{P^T} A =$$

$$M_2 M_{1,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix}}_{M^{-1} = L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PLU = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & 1/2 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

LU Decomposition w/ Complete Pivoting

$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $A_{21} > A_{11}$, swap rows

$P_1^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $P_1^T A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ $A_{11} < A_{12}$ swap columns

$Q_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $P_1 A Q_1 = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$

1) subtract $\frac{3}{4}$x row 1 from row 2

$M_1 = \begin{bmatrix} 1 & 0 \\ -\frac{3}{4} & 1 \end{bmatrix}$ $M_1 P_1^T A Q_1 = \begin{bmatrix} 1 & 0 \\ -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 2 \\ 0 & -\frac{1}{2} \end{bmatrix} = U$

$L = M^{-1} = \begin{bmatrix} 1 & 0 \\ +\frac{3}{4} & 1 \end{bmatrix}$ $A = PLUQ^T$

$PL = \begin{bmatrix} 1 & \frac{3}{4} \\ 0 & 1 \end{bmatrix}$ $UQ^T = \begin{bmatrix} 4 & 0 \\ 2 & -\frac{1}{2} \end{bmatrix}$

$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ $A_{31} > A_{11}$, swap rows 1 & 3

$P_1^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$P_1^T A = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}$ $A_{13} > A_{11}$, swap columns 1 & 3

$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$P^T A Q = \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{bmatrix} \qquad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -8/9 & 1 & 0 \\ -7/9 & 0 & 1 \end{bmatrix}$$

$$M_1 P_1^T A Q_1 = \begin{bmatrix} 9 & 6 & 3 \\ 0 & -2/3 & -4/3 \\ 0 & -1/3 & -2/3 \end{bmatrix}$$

$A_{22} < A_{23}$, swap columns 2&3

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_1 P_1^T A Q_1 Q_2 = \begin{bmatrix} 9 & 3 & 6 \\ 0 & -4/3 & -2/3 \\ 0 & -2/3 & -1/3 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \qquad M_2 M_1 P_1^T A Q_1 Q_2 = U = \begin{bmatrix} 9 & 3 & 6 \\ 0 & -4/3 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 8/9 & 1 & 0 \\ 7/9 & \frac{1}{2} & 1 \end{bmatrix} = L \qquad Q = Q_1 Q_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$PL = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 7/9 & 1 & 0 \\ 7/9 & 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 7/9 & 1/2 & 1 \\ 8/9 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$UQ^T = \begin{bmatrix} 9 & 3 & 6 \\ 0 & -4/3 & -2/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ -4/3 & -2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{2}$ The U matrix for the large A matrix from each decomposition has a row of zeroes. Thus, one column can be constructed by a linear combinatio of the others.

$\boxed{4}$ $A = LU \quad \det(A) = \det(L)\det(U)$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ x & \ddots & 0 \\ x & \cdots x & 1 \end{bmatrix} \quad \det(L) = \underline{1} \qquad U = \begin{bmatrix} x_{1,1} & \cdots & x_{1,N} \\ \emptyset & \ddots & \vdots \\ \emptyset & \emptyset & x_{N,N} \end{bmatrix} \quad \det(U) = \prod x_{ii}$$

$$x_{1,1}(\det U_{2:N})$$

$$= x_{1,1}\, x_{2,2}(\det U_{3:N})$$

$\boxed{5}$ $V = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & \emptyset \\ 3 & 6 & \emptyset \end{bmatrix}$

i) subtract 2x row 1 from row 2
ii) subtract 3x row 1 from row 3

$$\rightarrow \begin{bmatrix} 1 & 4 & 1 \\ \emptyset & -3 & -2 \\ \emptyset & -6 & -3 \end{bmatrix}$$

iii) subtract 2x row 2 from row 3

$$\rightarrow \begin{bmatrix} \underline{1} & 4 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & \underline{1} \end{bmatrix}$$

U has full rank, thus V_1, V_2 & V_3 are linearly independent.

MAE 290A HW2

Contents

problem 3

```
clear all
A1 = [1 3; 2 4];
A2 = [1 4 7; 2 5 8; 3 6 9];
[L1,U1] = lu(A1)
[L2,U2] = lu(A2)
%algorithm used is apparently partial pivoting
```

```
L1 =

    0.5000    1.0000
    1.0000         0


U1 =

     2     4
     0     1


L2 =

    0.3333    1.0000         0
    0.6667    0.5000    1.0000
    1.0000         0         0


U2 =

     3     6     9
     0     2     4
     0     0     0
```

problem 6

```
clear all
A3 = [1 4 1; 2 5 0; 3 6 0];
b1 = [1; 2; 2];
t1 = tic;
[L3, U3] = lu(A3)
y = L3\b1
x = U3\y
```

```
t2 = toc(t1)
t3 = tic;
x1 = A3\b1
t4 = toc(t3)
t5 = tic;
x2 = inv(A3)*b1
t6 = toc(t5)
```

```
L3 =

    0.3333    1.0000         0
    0.6667    0.5000    1.0000
    1.0000         0         0


U3 =

    3.0000    6.0000         0
         0    2.0000    1.0000
         0         0   -0.5000


y =

    2.0000
    0.3333
    0.5000


x =

   -0.6667
    0.6667
   -1.0000


t2 =

    0.0598


x1 =

   -0.6667
    0.6667
   -1.0000


t4 =

    0.0374


x2 =
```

```
   -0.6667
    0.6667
   -1.0000


t6 =

   2.6443e-04
```

problem 7

```
clear all
Ar = randn(500);
br = randn(500,1);
t1r = tic;
Cr = inv(Ar);
x1 = Cr * Cr * br;
t1a = toc(t1r)
t2r = tic;
x2 = Ar\Ar\br;
t2a = toc(t2r)
t3r = tic;
x3 = Ar\(Ar\br);
t3a = toc(t3r)
t4r = tic;
x4 = (Ar*Ar)\br;
t4a = toc(t4r)
t5r = tic;
[Lr, Ur] = lu(Ar);
x5 = Ur\(Lr\(Ur\(Lr\br)));
t5a = toc(t5r)
```

```
t1a =

    0.1289


t2a =

    0.0455


t3a =

    0.0287


t4a =

    0.0227


t5a =
```

```
0.0175
```