HOMEWORK #4

1. Consider the matrix

$$A_0 = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

Use the power method

$$v_k = Au_k, \qquad u_{k+1} = \frac{v_k}{\|v_k\|}$$

to compute the eigenvalue and eigenvectors associated with the largest eigenvalue of A_0. Use the shifted power method

$$v_k = (A - \mu I)u_k, \qquad u_{k+1} = \frac{v_k}{\|v_k\|}$$

to compute the eigenvalue and eigenvectors associated with the smallest eigenvalue of A_0. Explain how to choose μ.

2. Use the QR iteration

$$Q_{k+1}R_{k+1} = R_k Q_k, \qquad Q_k^* Q_k = I, \qquad R_k \text{ upper triangular}$$

to compute an approximate Schur Decomposition of A_0. Starting from $Q_0 R_0 = A_0$, where A_0 is from P.1, how many iterations did it take to achieve an approximately upper triangular matrix $R_k Q_k$ where the entries below the diagonal are less that 10^{-6} in absolute value? Compute $T = R_k Q_k$, $U = Q_0 Q_1 \cdots Q_k$ and verify that $A_0 \approx UTU^*$ and $U^*U \approx I$. Enumerate all eigenvalues of A_0.

3. Repeat P.2 but first use Matlab to transform A_0 into Hessemberg form. Explain the necessary modifications to compute the Schur form matrices T and U.

4. Use the Rayleigh iteration

$$(A - \mu_k I)v_k = u_k, \qquad u_{k+1} = \frac{v_k}{\|v_k\|}, \qquad \mu_{k+1} = u_{k+1}^* A u_{k+1},$$

to compute all eigenvectors and eigenvalues of A_0. Use the approximate Schur decomposition from P.2 to initialize μ_0 for each eigenvalue. Use the computed eigenvectors to diagonalize A_0.

5. Use Matlab to compute the SVD Decomposition $A = U\Sigma V^*$, $U^*U = I$, $V^*V = I$, $\Sigma \geq 0$, for the matrices A_0 from P.1 and

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

6. Show that $\|A\|_2 = \max_{x \neq 0} \|Ax\|_2/\|x\|_2 = \sigma_1$ and $\|A\|_F^2 = \mathrm{tr}(A^*A) = \sum_i \sigma_i^2$, where σ_i's are the singular values of matrix A. Show that $\|A\|_2 = \|A^*\|_2$ and $\|A\|_F = \|A^*\|_F$. Calculate the $\|\cdot\|_2$ and $\|\cdot\|_F$ norms of A_1, A_2 and A_3 in P.5.