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MIDTERM

Instructions:

- You have 75 minutes
- This exam is open notes, books
- No computers, calculators, phones, etc
- There are 3 questions for a total of 45 points and
  1 bonus question with 5 bonus points
- Good luck!

Questions:

1. [20 points] Consider the matrix and vectors

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad x = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}.$$

Answer the following questions:

a) [2 points] Verify that x solves the equation $Ax = b$.

Multiply to verify that $Ax = b$.	(+2 points)

b) [2 points] Compute the LU Decomposition $A = LU$ or explain why one does not exist.

The LU Decomposition does not exist	(+1 point)
Because the first pivot, the a_{11} entry is zero.	(+1 point)

c) [8 points] Compute the LU Decomposition $PA = LU$ with partial pivoting.
Use the computed decomposition to solve the system of equations $Ax = b$.

First iteration is:

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_1A_1 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_1 = I, \quad M_1P_1A_1 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(+1.5 points)

Second iteration is:

$$P_2 = I, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1 \end{bmatrix}, \quad U = M_2P_2M_1P_1A_1 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & -1/4 \end{bmatrix} \quad \text{(+1.5 points)}$$

Extracting the factors

$$P = P_2P_1 = P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & -1/4 \end{bmatrix} \quad \text{(+1 point)}$$

To solve $Ax = b$ we note that $PAx = LUx = Pb$ and solve

$$Ly = Pb \qquad\qquad Ux = y$$

Computing y

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/4 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = Ly = Pb = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad \text{(+1 point)}$$

$$y_1 = 0, \qquad y_2 = 2, \qquad y_3 = 1 - (1/4)y_2 = 1 - 1/2 = 1/2 \quad \text{(+1 point)}$$

then computing x

$$Ux = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & -1/4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1/2 \end{pmatrix} \quad \text{(+1 point)}$$

$$x_3 = -4(1/2) = -2, \quad x_2 = (2 - x_3)/4 = (2+2)/4 = 1, \quad x_1 = x_3/2 = -1 \quad \text{(+1 point)}$$

d) [8 points] Compute the LU Decomposition $PAQ = LU$ with complete pivoting.
Use the computed decomposition to solve the system of equations $Ax = b$.

First iteration is:

$$P_1 = I, \qquad Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_1 A_1 Q_1 = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix}, \qquad M_1 P_1 A_1 Q_1 = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1/4 \end{bmatrix} \qquad \text{(+1.5 points)}$$

Second iteration is:

$$P_2 = Q_2 = M_2 = I, \qquad U = M_2 P_2 M_1 P_1 A_1 Q_1 Q_2 = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1/4 \end{bmatrix} \qquad \text{(+1 point)}$$

Extracting the factors

$$P = P_2 P_1 = I, \quad Q = Q_1 Q_2 = Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1/4 \end{bmatrix} \qquad \text{(+1 point)}$$

To solve $Ax = b$ we note that $PAQx = LUQx = Pb$ and solve

$$Ly = Pb \qquad Uz = y, \qquad x = Q^T z$$

Computing y

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = Ly = Pb = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{(+1 point)}$$

$$y_1 = 2, \qquad y_2 = 0, \qquad y_3 = 1 - (1/4)y_1 = 1 - 1/2 = 1/2 \qquad \text{(+1 point)}$$

then computing z

$$Ux = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1/4 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1/2 \end{pmatrix} \qquad \text{(+1 point)}$$

$$z_3 = -4(1/2) = -2, \quad z_2 = x_3/2 = -2/2 = -1, \quad z_1 = (2 - z_3)/4 = (2+2)/4 = 1 \qquad \text{(+1 point)}$$

and finally computing x

$$x = Q^T z \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \qquad \text{(+.5 point)}$$

2. [10 points] Consider the matrices

$$A = \begin{bmatrix} \rho & 1 \\ 0 & 1 \end{bmatrix}, \qquad\qquad B = \begin{bmatrix} \rho^{-1} & 1 \\ 0 & \rho^{-1} \end{bmatrix}.$$

Answer the following questions:

a) [5 points] Using as matrix norm the $p = \infty$ induced norm show that

$$\kappa(A) = 2 + \frac{2}{\rho}, \qquad\qquad \kappa(B) = (1+\rho)^2$$

when $0 < \rho \ll 1$.

Hint: Use the fact that $\begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix}^{-1} = \begin{bmatrix} x^{-1} & -x^{-1}y^{-1} \\ 0 & y^{-1} \end{bmatrix}$. Recall that $\kappa(X) = \|X\| \|X^{-1}\|$.

Compute

$$\|A\| = \max(1+\rho, 1) = 1+\rho, \qquad \|B\| = \max(1+\rho^{-1}, \rho^{-1}) = 1+\rho^{-1}. \qquad \text{(+1 point)}$$

Using the formula provided

$$A^{-1} = \begin{bmatrix} \rho^{-1} & -\rho^{-1} \\ 0 & 1 \end{bmatrix}, \qquad\qquad B^{-1} = \begin{bmatrix} \rho & -\rho^2 \\ 0 & \rho \end{bmatrix} \qquad \text{(+1 point)}$$

and

$$\|A^{-1}\| = \max(2\rho^{-1}, 1) = 2\rho^{-1}, \qquad \|B^{-1}\| = \max(\rho+\rho^2, \rho) = \rho + \rho^2. \qquad \text{(+1 point)}$$

Therefore

$$\kappa(A) = \|A\|\|A^{-1}\| = 2\rho^{-1}(1+\rho) = 2 + \frac{2}{\rho} \qquad \text{(+1 point)}$$

$$\kappa(B) = \|B\|\|B^{-1}\| = (1+\rho^{-1})(\rho+\rho^2) = (1+\rho)^2 \qquad \text{(+1 point)}$$

b) [5 points] Provide a *short* explanation about the implications of the above calculations on the accuracy of solutions to the matrix equations $Ax = b$ and $Bx = b$ computed by a backward stable linear algebra algorithm when $\rho > 0$ is small.

Hint: Recall that a backward stable algorithm is one which calculates the exact solution of the perturbed problem $(A + \Delta A)y = b + \Delta b$, where $\|\Delta A\| \leq \epsilon \|A\|$, $\|\Delta b\| \leq \epsilon \|b\|$.

Based on sensitivity analysis we know the calculated solution y will have a relative error

$$\frac{\|y-x\|}{\|x\|} = O(\epsilon\kappa(X)) \qquad \text{(+1 point)}$$

where x is the solution to the linear equations $Xx = b$ when $X = A$ or $X = B$.
For $X = A$ we have

$$\frac{\|y-x\|}{\|x\|} = O(\epsilon\kappa(A)) = O(\epsilon(2+2/\rho)) \qquad \text{(+1 point)}$$

therefore we should expect large relative errors if ρ is small since $\epsilon(2+2/\rho) \to \infty$ for a fixed ϵ. (+1 point)
For $X = B$ we have

$$\frac{\|y-x\|}{\|x\|} = O(\epsilon\kappa(A)) = O(\epsilon(2+2/\rho)) \qquad \text{(+1 point)}$$

therefore we should expect small relative errors if ρ is small since $\epsilon(1+\rho)^2 \to \epsilon$ for a fixed ϵ. (+1 point)

3. [15 points] The matrix

$$P(v) = I - \frac{1}{v^T v} v v^T$$

is called a *projector*.

a) [2 points] Show that $y = P(v)\,x = x - \alpha\,v$, where $\alpha = v^T x/(v^T v)$.

$$\begin{aligned} y &= P(v)x \\ &= \left(I - \frac{1}{v^T v} v v^T\right) x \\ &= x - \frac{v^T x}{v^T v} v = x - \alpha v \end{aligned} \qquad \text{(+2 points)}$$

b) [2 points] Show that $P(v)\,v = 0$.

With $x = v$ we have

$$\alpha = (v^T v)/(v^T v) = 1 \qquad \text{(+1 point)}$$

and therefore

$$P(v)v = v - \alpha v = 0. \qquad \text{(+1 point)}$$

c) [2 points] Show that if $y = P(v)\,x$ then $v \perp y$.

Calculate the inner product

$$v^T y = v^T P(v) v = v^T x - \frac{v^T x}{v^T v} v^T v = 0 \qquad \text{(+2 points)}$$

d) [2 points] Show that if $v \perp x$ then $P(v)\,x = x$.

When $v \perp x$ then

$$v^T x = 0 \quad \Longrightarrow \quad \alpha = 0 \qquad \text{(+1 point)}$$

so that

$$P(v)x = x - \alpha v = x \qquad \text{(+1 point)}$$

e) [2 points] Prove that $P(v)$ is not full-rank unless $v = 0$.

For any $v \neq 0$ from item b) $P(v)v = 0$ hence $P(v)$ is not full-rank. (+1 point)
With $v = 0$ then $P(v) = I$ and full-rank. (+1 point)

f) [5 points] For

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

compute $P(v)$, $y_1 = P(v)\,x_1$, and $y_2 = P(v)\,x_2$. Draw a cartesian plan representing the vectors v, x_1, x_2, y_1 and y_2. What is the geometric interpretation of the operation $P(v)\,x$?

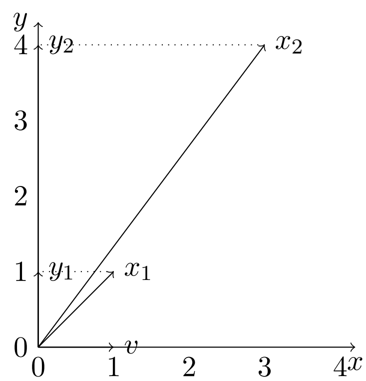
Calculating

$$P(v) = I - \frac{1}{v^T v} v v^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(+1 point)}$$

$$y_1 = P(v)x_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{(+1 point)}$$

$$y_2 = P(v)x_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \qquad \text{(+1 point)}$$

Graphically:



which reveals that $P(v)x$ is an orthogonal projection onto axis y. (+2 points)

4. [5 bonus points] Every complex matrix $A \in \mathbb{C}^{m \times n}$, $m \leq n$, admits the complex QR Decomposition $A = QR$ where $Q \in \mathbb{C}^{m \times n}$ is such that $Q^* Q = I$ and $R \in \mathbb{C}^{n \times n}$ is upper triangular.

a) [2 bonus points] Show that matrices

$$\hat{Q} = Q\,\Theta, \qquad \hat{R} = \Theta^* R,$$

where $\Theta \in \mathbb{C}^{n \times n}$ is a diagonal matrix with diagonal entries equal to

$$\Theta_{ii} = e^{j\theta_i}, \quad \theta_i \in [0, 2\pi), \quad i = 1, \ldots, n,$$

and $j = \sqrt{-1}$, also constitute a QR Decomposition of A.

Note that $\Theta\Theta^*$ is a diagonal $n \times n$ matrix with diagonal entries

$$\Theta_{ii}\overline{\Theta_{ii}} = e^{j\theta_i}e^{-j\theta_i} = 1, \qquad i = 1, \ldots, n, \qquad \text{(+1 point)}$$

That is $\Theta\Theta^* = \Theta^*\Theta = I$ is unitary. Therefore

$$\hat{Q}\hat{R} = Q\Theta\Theta^*R = QR = A, \qquad \hat{Q}^*\hat{Q} = \Theta^*Q^*Q\Theta = \Theta^*\Theta = I \qquad \text{(+1 point)}$$

is a QR decomposition because $\hat{R} = \Theta^*R$ is still upper-triangular.

b) [2 bonus points] Explain how item a) can be used to construct a matrix $\hat{R}$ which has real and positive diagonal entries, such as the one produced by the Gram-Schmidt Algorithm.

In general r_{ii} will be a complex number. Representing r_{ii} in polar form

$$r_{ii} = |r_{ii}|e^{j\angle r_{ii}}$$

With the choice

$$\Theta_{ii} = e^{j\angle r_{ii}} \qquad \text{(+1 point)}$$

The product Θ^*R is such that the diagonal entries of $\hat{R}$ are

$$\hat{r}_{ii} = e^{-j\angle r_{ii}}r_{ii} = |r_{ii}|e^{-j\angle r_{ii}}e^{j\angle r_{ii}} = |r_{ii}| \qquad \text{(+1 point)}$$

c) [1 bonus point] What are the possible values Θ can assume when A, Q, R, $\hat{Q}$ and $\hat{R}$ are real matrices?

In order for Q and $Q\Theta$ be real we will need Θ to be real. This is only possible if

$$\Theta_{ii} = e^{j\theta_i}, \quad \theta_i = \{0, \pi\}, \quad i = 1, \ldots, n, \qquad \text{(+1 point)}$$

in which case Θ_{ii} is either equal to '1' or -1'.